Algebraic cycles on K3 and derived equivalences

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The canonical class in $CH^2(S)$

S K3 surface over $\mathbb{C}$. Chow ring $CH(S)$:

$$CH^0(S) = \mathbb{Z}\ ,\ \ CH^1(S) = \mathrm{Pic}(S)\ ,\ \ CH^2(S) \text{ very large.}$$

Theorem (Voisin, AB)

$\exists\ c_S \in CH^2(S)$, $\deg c_S = 1$, *such that:*

(i) $\mathrm{Pic}(S) \otimes \mathrm{Pic}(S) \longrightarrow \mathbb{Z} \cdot c_S \subset CH^2(S)$;

(ii) $c_2(S) = 24 c_S$.

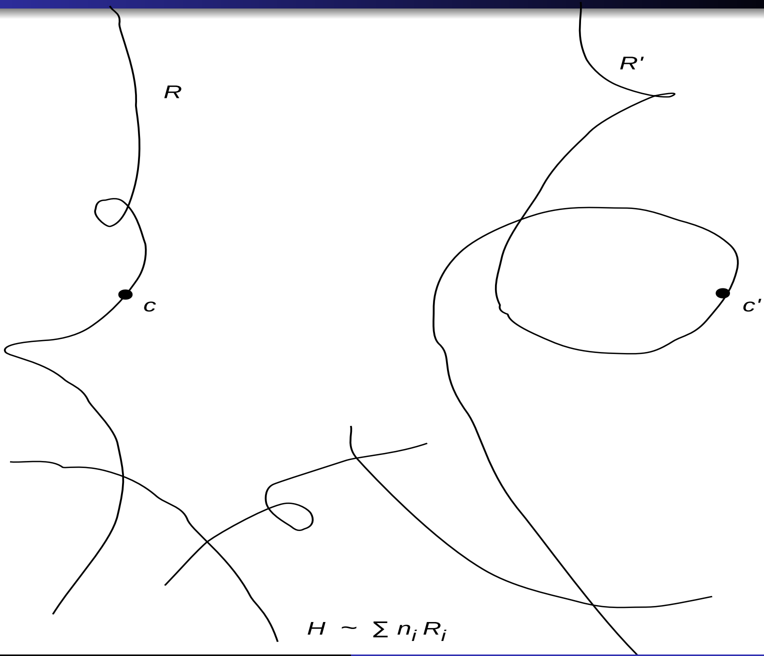
Proof of (i):

Key point : Any effective divisor on S is a sum of rational curves (Mumford; the curves are singular in general).

Pick up R rational curve and $c \in R$.

Claim : $[c] \in CH^2(S)$ independant of choices.

Proof of claim:



$H \sim \sum n_i R_i$

End of proof:

We want to prove: $D \cdot D' = \deg(D \cdot D')\, c_S$ for D, D' divisors on S.

By linearity, suffices to do it for D, D' rational curves: obvious. ∎

Proof of (ii) much more involved: use 1-dimensional family of elliptic curves on S to get relations in $CH(S \times S)$. ∎

Define $R(S) := \mathbb{Z} \oplus \mathrm{Pic}(S) \oplus \mathbb{Z} \cdot c_S$.

Equivalent formulation

There exists a subring $R(S)$ of $CH(S)$, containing $c_i(S)$, such that the cycle class map $CH(S) \to H^*_{\mathrm{alg}}(S, \mathbb{Z})$ maps $R(S)$ isomorphically onto $H^*_{\mathrm{alg}}(S, \mathbb{Z})$.

Digression

I suspect this holds for every holomorphic symplectic manifold X.

Out of reach, but the following consequence is easier to check :

Conjecture

Any polynomial relation

$$P(\ell_1, \ldots, \ell_k; c_2(S), \ldots, c_n(S)) = 0 \quad \text{in } H^*(X, \mathbb{Z})$$

with $\ell_1, \ldots, \ell_k \in \mathrm{Pic}(X)$, already holds in $CH(X)$.

This has been checked by C. Voisin for $X = S^{[n]}$ (the Hilbert scheme of length n subschemes of a K3 S) for $n \leq 8$, and for $X=$ variety of lines contained in a cubic fourfold.

Remark

For S K3 over $\overline{\mathbb{Q}}$, Beilinson conjectures imply $CH(S) = R(S)$.

Reminder on derived categories

X smooth projective variety. $D(X) =$ (bounded) derived category of X (objects $=$ bounded complexes of vector bundles).

We are interested in equivalences $D(X) \xrightarrow{\sim} D(Y)$. Recall:

Theorem (Bondal-Orlov)

Assume K_X or $-K_X$ ample.

(i) *If $D(X) \cong D(Y)$, $X \cong Y$.*

(ii) *The group $\mathrm{Aut}(D(X))$ is generated by $\mathrm{Aut}(X)$, the shift, and $\mathrm{Pic}(X)$ acting by $E \mapsto E \otimes L$.*

Much more interesting when K_X trivial, in particular X K3:

- There are non-isomorphic K3 X, S with $D(X) \cong D(S)$;
- The group $\mathrm{Aut}(D(X))$ is large.

Huybrechts' theorem

FACT : An equivalence $F : D(X) \xrightarrow{\sim} D(Y)$ induces

$$\begin{array}{ccc} D(X) & \xrightarrow{F} & D(Y) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ CH(X) & \xrightarrow[F_{CH}]{} & CH(Y) \end{array} \qquad \begin{array}{ccc} D(X) & \xrightarrow{F} & D(Y) \\ \downarrow{\scriptstyle \mathrm{ch}_H} & & \downarrow{\scriptstyle \mathrm{ch}_H} \\ H^*(X,\mathbb{Z}) & \xrightarrow[F_H]{} & H^*(Y,\mathbb{Z}) \end{array}$$

(Experts rather use the *Mukai vector* $v_X(E) := \mathrm{Todd}_X^{1/2} \cdot \mathrm{ch}(E)$; irrelevant for our purpose.)

Theorem (Huybrechts)

X, S K3 with Picard number ≥ 2, $F : D(X) \xrightarrow{\sim} D(S) \implies$

F_{CH} maps $R(X)$ onto $R(S)$.

(The result should hold with no restriction on the Picard number.)

Strategy of the proof

1. $F_{CH}(R(X))$ is spanned by $\mathrm{ch}(E)$ for certain objects E in $D(S)$, called spherical.

2. $F_{CH}(R(X))$ is spanned by $\mathrm{ch}(E)$ for E spherical vector bundle.

3. For $\mathrm{Pic}(S) = \mathbb{Z}$, the Lazarsfeld bundles $F_{C,A}$ on S are spherical; check $\mathrm{ch}(F_{C,A}) \in R(S)$.

4. For $\mathrm{rk}\,\mathrm{Pic}(S) \geq 2$, deduce the result by specialization. ∎

Step 1

1. $F_{CH}(R(X))$ is spanned by $\mathrm{ch}(E)$ for certain objects E in $D(S)$, called spherical.

Observation : $R(X)$ spanned by $\mathrm{ch}(L)$ for $L \in \mathrm{Pic}(X)$,

hence $F_{CH}(R(X))$ is spanned by $F_{CH}(L) = \mathrm{ch}(F(L))$.

What do we know about $F(L)$? It is spherical :

$$E \in D(X) \text{ } \textit{spherical} \text{ if } \begin{cases} \mathrm{Ext}^i(E,E) = 0 \text{ for } i \neq 0,2, \\ \mathrm{Ext}^0(E,E) = \mathrm{Ext}^2(E,E) = \mathbb{C} \end{cases}$$

Spherical objects in $D(X)$ are poorly understood, but :

- A vector bundle is spherical iff it is simple and rigid.

$L \in \mathrm{Pic}(X)$ is spherical, hence also $F(L)$.

Step 2

1. X
2. $F_{CH}(R(X))$ is spanned by $\mathrm{ch}(E)$ for E spherical vector bundle.

Key ingredient:

Theorem

$$F, G : D(X) \xrightarrow{\sim} D(S)\ ,\ F_H = G_H \ \Rightarrow\ F_{CH} = G_{CH}\ .$$

Compare with Bloch's conjecture : $\Gamma, \Delta \in CH^2(X \times S)$,
$\Gamma_* = \Delta_*$ on $H^{2,0}(X)\ \Rightarrow\ \Gamma_* = \Delta_*$ on $CH^2(X)_{\deg=0}$.

The proof uses formal deformation to the general non-algebraic case.

Corollary (not immediate)

$E, E' \in D(X)$ spherical, $\mathrm{ch}_H(E) = \mathrm{ch}_H(E')\ \Rightarrow \mathrm{ch}(E) = \mathrm{ch}(E')$.

Step 2, continued

Want to prove: $E \in D(S)$ spherical $\Rightarrow$ $\mathrm{ch}(E) \in R(S)$.

Define quadratic form q on $H^*(S, \mathbb{Z})$ by

$$q(r, \alpha, s) = \alpha^2 - 2r(r+s) \qquad \textit{(Mukai pairing)}$$

For $E \in D(S)$ spherical, Riemann-Roch $\rightsquigarrow$

$$q(\mathrm{ch}_H(E)) = -\chi(\mathcal{E}nd(E)) = -2\ .$$

Theorem (Kuleshov)

For $\xi \in H^*_{\mathrm{alg}}(S, \mathbb{Z})$ with $q(\xi) = -2$, there exists F spherical vector bundle with $\mathrm{ch}_H(F) = \xi$.

Apply to $\xi = \mathrm{ch}_H(E)$ $\rightsquigarrow$ F spherical vector bundle with $\mathrm{ch}_H(E) = \mathrm{ch}_H(F)$ $\Rightarrow$ $\mathrm{ch}(E) = \mathrm{ch}(F)$ by Corollary.

So it suffices to prove $\mathrm{ch}(F) \in R(S)$ for each spherical v.b. F. ■

Step 3: the Lazarsfeld bundles

3. For $\mathrm{Pic}(X) =$, the Lazarsfeld bundles $F_{C,A}$ on S are spherical; check $\mathrm{ch}(F_{C,A}) \in R(S)$.

S K3, $i : C \hookrightarrow S$ smooth curve, A line bundle on C generated by $H^0(C, A)$. Define vector bundle $F_{C,A}$ on S by

$$0 \to F^*_{C,A} \longrightarrow H^0(C, A) \otimes \mathcal{O}_S \longrightarrow A \to 0 \ .$$

Then: $\mathrm{rk}\, F_{C,A} = h^0(A)$, $c_1(F_{C,A}) = [C]$, $c_2(F_{C,A}) = i_*[A]$, $\chi(\mathcal{E}nd(F)) = 2 - 2\rho(A)$ with $\rho(A) := g(C) - h^0(A)h^1(A)$ (*Brill-Noether number*)

Proposition (Lazarsfeld)

Assume $\mathrm{Pic}(S) = \mathbb{Z}\,[C]$ *and* $\rho(A) = 0$. *Then* $F_{C,A}$ *is spherical.*

Step 3, continued

Proposition

$\mathrm{ch}(F_{C,A}) \in R(S)$ *(equivalently,* $c_2(F_{C,A}) \in R(S)$*).*

Proof :

Dual sequence: $0 \to H^0(C,A)^* \otimes \mathcal{O}_S \to F_{C,A} \to \omega_C \otimes A^{-1} \to 0.$

$\mathcal{S} := \{\text{subspaces } V \subset H^0(F_{C,A}), \dim V = h^0(A), V \otimes \mathcal{O}_S \xhookrightarrow{j} F_{C,A}\}$

Map $\mathcal{S} \to |C|$, $V \mapsto C_V := \mathrm{Supp}\,\mathrm{Coker}(j)$;

$c_2(F_{C,A})$ is supported on C_V.

Brill-Noether $\Rightarrow$ $\mathcal{S} \to |C|$ dominant, so C_V specializes to $R \in |C|$ rational, with $c_2(F_{C,A})$ supported on R, hence multiple of c_S. ∎

End of Step 3

Corollary

$\mathrm{Pic}(S) = \mathbb{Z}[C]$, *E spherical bundle with* $c_1(E) = [C] \Rightarrow$
$\mathrm{ch}(E) \in R(S)$.

Proof :

Put $\mathrm{ch}_H(E) = (r, C, s)$; $q(r, C, s) = -2 \Leftrightarrow r(r+s) = g(C)$.

By B-N theory there exists A on C with $h^0(A) = r$, $h^1(A) = r + s$

$\Rightarrow \mathrm{ch}_H(F_{C,A}) = (r, C, s) = \mathrm{ch}_H(E)$, hence $\mathrm{ch}(E) = \mathrm{ch}(F_{C,A})$

(Step 2) and $\mathrm{ch}(E) \in R(S)$. ∎

4. For $\operatorname{rk}\operatorname{Pic}(S) \geq 2$, deduce the result by specialization.

Want to prove : E spherical bundle $\Rightarrow \operatorname{ch}(E) \in R(S)$.

1) Assume $H := \det E$ primitive and ample. Deform (S, H) to (S_η, H_η) with $\operatorname{Pic}(S_\eta) = \mathbb{Z}[H_\eta]$.

Deformation theory $\Rightarrow$ E extends to E_η spherical bundle on S_η

$$\left(\begin{array}{l} \text{Obstruction lies in } H^2(\mathcal{E}nd(E)) \xrightarrow[\operatorname{Tr}]{\sim} H^2(\mathcal{O}_S) \\ \qquad\qquad\qquad\qquad = \text{obstruction to deform } \det(E) \\ \qquad\qquad\qquad\qquad = 0 \,. \end{array} \right)$$

$\operatorname{ch}(E_\eta) \in R(S_\eta)$ by Step 3 $\Rightarrow$ $\operatorname{ch}(E) \in R(S)$.

Step 4, end

2) General case (with $\operatorname{rk}\operatorname{Pic}(S) \geq 2$). Write $\operatorname{ch}_H(E) = (r, kL, s)$ with L primitive in $\operatorname{Pic}(S)$.

$$q(r, kL, s) = k^2L^2 - 2r(r+s) = -2 \;\Rightarrow\; (k, r) = 1 \;.$$

Pick $M \in \operatorname{Pic}(S)$ primitive, $\neq L^j$, sufficiently ample. Then $kL + rM$ primitive and ample, so $\operatorname{ch}(E \otimes M) \in R(S)$ by 1). ■

THE END