Bulletin of the London Mathematical Society Advance Access published January 20, 2016

Bull. London Math. Soc. Page 1 of 4
doi:10.1112/blms/bdv098

A very general sextic double solid is not stably rational

Arnaud Beauville

Abstract

We prove that a double covering of $\mathbb{P}^3$ branched along a very general sextic surface is not stably rational.

1. *Introduction*

A projective variety X is *stably rational* if $X \times \mathbb{P}^m$ is rational for some integer m. The paper [**10**] of Voisin introduces a new approach to show that some complex varieties are not stably rational. This is applied in [**10**] to prove that a double covering of $\mathbb{P}^3$ branched along a very general quartic hypersurface is not stably rational, and in [**4**] to prove the same result for a very general quartic threefold.

Using the same approach, we will prove the following theorem.

Theorem 1. *A double covering of $\mathbb{P}^3$ branched along a very general sextic surface is not stably rational.*

These 'sextic double solids' are the Fano threefolds with Picard number 1 of minimal anticanonical degree. They are already known to be non-rational [**8**, Theorem 2.2]; whether they are unirational or not is unknown.

We will use Voisin's method in the following form [**10**, Theorem 1.1 and Remark 1.3].

Proposition 1. *Let B be a smooth complex variety, o a point of B, $f : \mathscr{X} \to B$ a flat, projective morphism, such that the generic fiber of f is smooth, and that the only singularities of the fiber $X := \mathscr{X}_o$ are ordinary double points. Assume that for a desingularization $\tilde{X}$ of X, the torsion subgroup of $H^3(\tilde{X}, \mathbb{Z})$ is non-trivial. Then for a very general point $b \in B$, the fiber $\mathscr{X}_b$ is not stably rational.*

Thus to prove the theorem it suffices to find a nodal sextic surface $\Delta \subset \mathbb{P}^3$ such that the desingularization $\tilde{X}$ of the double cover X of $\mathbb{P}^3$ branched along Δ satisfies $\operatorname{Tors} H^3(\tilde{X}, \mathbb{Z}) \neq 0$. Such a surface is described in [**7**]. We give here another construction, perhaps simpler; it is not clear to us how the two constructions are related.

As in [**7**], we use a family of quadric surfaces over $\mathbb{P}^3$, with discriminant locus Δ of degree 6; the quadric fibration provides a natural $\mathbb{P}^1$-bundle over X_{sm}, and this gives a 2-torsion class in $H^3(X_{sm}, \mathbb{Z})$, which extends to $H^3(\tilde{X}, \mathbb{Z})$. To construct our quadric fibration we start from a cubic fivefold $V \subset \mathbb{P}^5$, and project from a 2-plane contained in V. We show that the associated $\mathbb{P}^1$-bundle has no rational section (Proposition 2), and that this provides a non-zero 2-torsion class in $H^3(\tilde{X}, \mathbb{Z})$ (Proposition 3). As the referee pointed out, this is quite close to the method used in [**5**].

Received 1 January 2015; revised 16 September 2015.

2010 *Mathematics Subject Classification* 14M20 (primary), 14E08, 14J45 (secondary).

2. *The construction*

We work over $\mathbb{C}$. Let $V \subset \mathbb{P}^6$ be a smooth cubic fivefold, and $P \subset V$ a 2-plane. We choose coordinates $(X_0, \dots, X_2;\ Y_0, \dots, Y_3)$ on $\mathbb{P}^6$ such that P is given by $Y_0 = \dots = Y_3 = 0$ and V by

$$\sum_{i,j} A_{ij} X_i X_j + \sum_i B_i X_i + C = 0, \tag{1}$$

where A_{ij}, B_i, C are homogeneous forms in $(Y_0, \dots, Y_3)$ of degree 1, 2 and 3.

Let $\hat{V}$ denote the variety obtained by blowing up V along P. The projection from P defines a rational map $V \dashrightarrow \mathbb{P}^3$, which extends to a morphism $q : \hat{V} \to \mathbb{P}^3$. The fiber of q at a point $y = (Y_0, \dots, Y_3)$ of $\mathbb{P}^3$ is the projective completion of the quadric in $\mathbb{A}^3$ defined by equation (1).

Let $\Delta \subset \mathbb{P}^3$ be the discriminant surface of the quadric fibration q, that is, the locus of points $y \in \mathbb{P}^3$ such that $q^{-1}(y)$ is singular. It is defined by the sixth degree equation

$$\det \left(\begin{array}{c|c} (A_{ij}) & (B_i) \\ \hline (B_i) & C \end{array} \right) = 0.$$

According to [**2**, Theorem 2.2], for a general choice of the forms A_{ij}, B_i and C, the surface Δ is smooth except for a finite set $\Sigma \subset \Delta$ of 31 ordinary double points. We will assume from now on that this condition holds. The quadric $q^{-1}(y)$ has rank 3 for $y \in \Delta \smallsetminus \Sigma$, and rank 2 for $y \in \Sigma$.

Let $\pi : X \to \mathbb{P}^3$ be the double covering branched along Δ. Then X is smooth except for the 31 ordinary double points lying above Σ. The generatrices of the quadric $q^{-1}(y)$ are parameterized by two disjoint rational curves for $y \in \mathbb{P}^3 \smallsetminus \Delta$, one rational curve for $y \in \Delta \smallsetminus \Sigma$. This defines a $\mathbb{P}^1$-bundle $\varphi : G \to X_{sm}$ onto the smooth locus of X.

PROPOSITION 2. (a) *The fibration $q : \hat{V} \to \mathbb{P}^3$ admits no rational section.*

(b) *The $\mathbb{P}^1$-bundle φ admits no rational section.*

Proof. (a) If q admits a rational section, then the closure of its image is a subvariety Z of $\hat{V}$ whose class $[Z] \in H^4(\hat{V}, \mathbb{Z})$ satisfies $([Z] \cdot q^*y) = 1$ for $y \in \mathbb{P}^3$. Let us show that this is impossible.

Consider the blowing up

$$\begin{array}{ccc} E & \stackrel{i}{\hookrightarrow} & \hat{V} \\ \downarrow p & & \downarrow b \\ P & \hookrightarrow & V \end{array}$$

The exceptional divisor E is the hypersurface in $P \times \mathbb{P}^3$ given by $\sum A_{ij}(y) X_i X_j = 0$; the projections of E onto P and $\mathbb{P}^3$ are p and $q' := q \circ i$. The group $H^2(E, \mathbb{Z})$ is generated by the classes $p^*\ell$ and $q'^*\pi$, where ℓ is the class of a line in $H^2(P, \mathbb{Z})$ and π the class of a plane in $\mathbb{P}^3$. Let $h \in H^2(V, \mathbb{Z})$ be the class of a hyperplane section of V; the group $H^4(\hat{V}, \mathbb{Z})$ is generated by the classes b^*h^2, $i_*p^*\ell$ and $i_*q'^*\pi$ (see, for example, [**1**, Proposition 0.1.3]).

Let us compute the intersection number of these classes with the fiber of q at a point $y \in \mathbb{P}^3$. The class b^*h^2 induces on the quadric $q^{-1}(y)$ the intersection with a line, hence $(b^*h^2 \cdot q^*y) = 2$. For $d \in H^2(E, \mathbb{Z})$, we have $(i_*d \cdot q^*y) = (d \cdot q'^*y)$. This is zero for $d = q'^*\pi$. The class $p^*\ell$ is the class of a line $\sum a_i X_i = 0$, so its intersection with the conic $q'^{-1}(y)$ consists of two points, hence $(i_*p^*\ell \cdot q^*y) = 2$. It follows that $(\alpha \cdot q^*y)$ is even for any $\alpha \in H^4(\hat{V}, \mathbb{Z})$, so q does not admit a rational section.

(b) Suppose that φ admits a rational section. For a general point y in $\mathbb{P}^3$, this section maps the two points of $\pi^{-1}(y)$ to two generatrices of the quadric $q^{-1}(y)$, one in each system. These two generatrices intersect in one point $s(y)$ of the quadric. This gives a rational section s of q, thus contradicting (a). ☐

(The implication $(a) \Rightarrow (b)$ is classical; it appears, for instance, in [3].)

Let $\tilde{X} \to X$ be the resolution of singularities obtained by blowing up the double points; the exceptional divisor Q is a disjoint union of 31 smooth quadrics.

Proposition 3. *The 2-torsion subgroup of $H^3(\tilde{X}, \mathbb{Z})$ is non-zero.*

Proof. Put $U := \tilde{X} \smallsetminus Q \cong X_{sm}$. The Gysin exact sequence

$$H^1(Q,\mathbb{Z}) \longrightarrow H^3(\tilde{X},\mathbb{Z}) \longrightarrow H^3(U,\mathbb{Z}) \longrightarrow H^2(Q,\mathbb{Z})$$

shows that the restriction map induces an isomorphism on the torsion subgroups of $H^3(-,\mathbb{Z})$. Thus it suffices to prove the statement for $H^3(U,\mathbb{Z})$.

The $\mathbb{P}^1$-bundle φ gives a class $[\varphi]$ in the 2-torsion subgroup $\mathrm{Br}_2(U)$ of the Brauer group of U; the assertion (b) of Proposition 2 means that this class is non-zero. Let us recall how such a class gives a 2-torsion class in $H^3(U,\mathbb{Z})$, the topological Brauer class (see [6, § 1], or [9, 1.1]). The exact sequences $0 \to \{\pm 1\} \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ (for the étale topology) and $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ (for the classical topology) give rise to a commutative diagram of exact sequences

$$\begin{array}{ccccccc} \mathrm{Pic}(U) & \longrightarrow & H^2(U,\mathbb{Z}/2) & \longrightarrow & \mathrm{Br}_2(U) & \longrightarrow & 0 \\ \downarrow{\scriptstyle c_1} & & \| & & & & \\ H^2(U,\mathbb{Z}) & \longrightarrow & H^2(U,\mathbb{Z}/2) & \xrightarrow{\partial} & H^3(U,\mathbb{Z}) & & \end{array}$$

Therefore, ∂ induces a homomorphism $\bar{\partial} : \mathrm{Br}_2(U) \to H^3(U,\mathbb{Z})$, which is injective if $c_1 : \mathrm{Pic}(U) \to H^2(U,\mathbb{Z})$ is surjective. This is indeed the case: in the commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(\tilde{X}) & \xrightarrow{c_1} & H^2(\tilde{X},\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{Pic}(U) & \xrightarrow{c_1} & H^2(U,\mathbb{Z}) \end{array}$$

The top horizontal arrow is surjective because $H^2(\tilde{X}, \mathscr{O}_{\tilde{X}}) = 0$; the restriction map $H^2(\tilde{X},\mathbb{Z}) \to H^2(U,\mathbb{Z})$ is surjective because of the Gysin exact sequence $H^2(\tilde{X},\mathbb{Z}) \to H^2(U,\mathbb{Z}) \to H^1(Q,\mathbb{Z}) = 0$. Thus $\bar{\partial}([\varphi])$ is a non-zero 2-torsion class in $H^3(U,\mathbb{Z})$, hence the proposition. ☐

Theorem 1 follows by taking for B the space of sextic surfaces in $\mathbb{P}^3$, for $\mathscr{X}$ the family of double coverings of $\mathbb{P}^n$ branched along those surfaces, and for $o \in B$ the point corresponding to the discriminant surface Δ.

Acknowledgements. I am indebted to C. Shramov for pointing out the paper [7], to A. Collino for spotting an inaccuracy in the first version of this note and to J.-L. Colliot-Thélène for pointing out reference [3].

References

1. A. Beauville, 'Variétés de Prym et jacobiennes intermédiaires', *Ann. Sci. École Norm. Sup.* (4) 10 (1977) 309–391.

2. F. Catanese, 'Babbage's conjecture, contact of surfaces, symmetric determinantal varieties and applications', *Invent. Math.* 63 (1981) 433–465.

3. F. Châtelet, 'Relations entre l'arithmétique et la géométrie sur une quadrique', *Bull. Soc. Math. France* 76 (1948) 108–113.

4. J.-L. Colliot-Thélène and A. Pirutka, 'Hypersurfaces quartiques de dimension 3 : non-rationalité stable', *Ann. Sci. Éc. Norm. Sup.* 49 (2016).

5. M. Gross, 'Some calculations of Brauer groups', *Commun. Math. Phys.* 178 (1996) 129–134. Appendix to *Stable singularities in string theory* by P. Aspinwall and D. Morrison.

6. A. Grothendieck, 'Le groupe de Brauer', *I. Dix Exposés sur la Cohomologie des Schémas* (North-Holland, Amsterdam, Masson, Paris.1968) 46–66.

7. A. Iliev, L. Katzarkov and V. Przyjalkowski, 'Double solids, categories and non-rationality', *Proc. Edinb. Math. Soc.* (2) 57 (2014) 145–173.

8. V. A. Iskovskikh, 'Birational automorphisms of three-dimensional algebraic varieties', *J. Soviet Math.* 13 (1980) 815–868.

9. N. Nitsure, 'Topology of conic bundles', *J. London Math. Soc.* (2) 35 (1987) 18–28.

10. C. Voisin, 'Unirational threefolds with no universal codimension 2 cycle', *Invent. Math.* 201 (2015) 207–237.

Arnaud Beauville
Laboratoire J.-A. Dieudonné
UMR 7351 du CNRS
Université de Nice
Parc Valrose
F-06108 Nice cedex 2
France

arnaud.beauville@unice.fr