On the Second Lower Quotient of the Fundamental Group

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Dedicated to Klaus Hulek on his 60th birthday

Abstract Let X be a topological space, $G = \pi_1(X)$ and $D = (G,G)$. We express the second quotient $D/(D,G)$ of the lower central series of G in terms of the homology and cohomology of X. As an example, we recover the isomorphism $D/(D,G) \cong \mathbb{Z}/2$ (due to Collino) when X is the Fano surface parametrizing lines in a cubic threefold.

1 Introduction

Let X be a connected topological space. The group $G := \pi_1(X)$ admits a lower central series

$$G \supseteq D := (G,G) \supseteq (D,G) \supseteq \dots$$

The first quotient G/D is the homology group $H_1(X,\mathbb{Z})$. We consider in this note the second quotient $D/(D,G)$. In particular when $H_1(X,\mathbb{Z})$ is torsion free, we obtain a description of $D/(D,G)$ in terms of the homology and cohomology of X (see Corollary 2 below).

As an example, we recover in the last section the isomorphism $D/(D,G) \cong \mathbb{Z}/2$ (due to Collino) for the Fano surface parametrizing the lines contained in a cubic threefold.

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2 The Main Result

Proposition 1. *Let X be a connected space homotopic to a CW-complex, with $H_1(X,\mathbb{Z})$ finitely generated. Let $G = \pi_1(X)$, $D = (G,G)$ its derived subgroup, $\tilde{D}$ the subgroup of elements of G which are torsion in G/D. The group $D/(\tilde{D},G)$ is canonically isomorphic to the cokernel of the map*

$$\mu : H_2(X,\mathbb{Z}) \to \mathrm{Alt}^2(H^1(X,\mathbb{Z})) \quad \textit{given by} \;\; \mu(\sigma)(\alpha,\beta) = \sigma \frown (\alpha \wedge \beta) \ ,$$

where $\mathrm{Alt}^2(H^1(X,\mathbb{Z}))$ is the group of skew-symmetric integral bilinear forms on $H^1(X,\mathbb{Z})$.

Proof. Let H be the quotient of $H_1(X,\mathbb{Z})$ by its torsion subgroup; we put $V := H \otimes_{\mathbb{Z}} \mathbb{R}$ and $T := V/H$. The quotient map $\pi : V \to T$ is the universal covering of the real torus T.

Consider the surjective homomorphism $\alpha : \pi_1(X) \to H$. Since T is a $K(H,1)$, there is a continuous map $a : X \to T$, well defined up to homotopy, inducing α on the fundamental groups. Let $\rho : X' \to X$ be the pull back by a of the étale covering $\pi : V \to T$, so that $X' := X \times_T V$ and ρ is the covering associated to the homomorphism α.

Our key ingredient will be the map $f : X \times V \to T$ defined by $f(x,v) = a(x) - \pi(v)$. It is a locally trivial fibration, with fibers isomorphic to X'. Indeed the diagram

$$\begin{array}{ccc} X' \times V & \xrightarrow{\;g\;} & X \times V \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{\;\pi\;} & T \end{array}$$

where $g((x,v),w) = (x, v-w)$, is cartesian.

It follows from this diagram that the monodromy action of $\pi_1(T) = H$ on $H_1(X',\mathbb{Z})$ is induced by the action of H on X'; it is deduced from the action of $\pi_1(X)$ on $\pi_1(X')$ by conjugation in the exact sequence

$$1 \to \pi_1(X') \xrightarrow{\rho_*} \pi_1(X) \to H \to 1 \ . \tag{1}$$

The homology spectral sequence of the fibration f (see for instance [5]) gives rise in low degree to a five terms exact sequence

$$H_2(X,\mathbb{Z}) \xrightarrow{a_*} H_2(T,\mathbb{Z}) \longrightarrow H_1(X',\mathbb{Z})_H \xrightarrow{\rho_*} H_1(X,\mathbb{Z}) \longrightarrow H_1(T,\mathbb{Z}) \longrightarrow 0 \ , \tag{2}$$

where $H_1(X',\mathbb{Z})_H$ denote the coinvariants of $H_1(X',\mathbb{Z})$ under the action of H.

The exact sequence (1) identifies $\pi_1(X')$ with $\tilde{D}$, hence $H_1(X', \mathbb{Z})$ with $\tilde{D}/(\tilde{D}, \tilde{D})$, the action of H being deduced from the action of G by conjugation. The group of coinvariants is the largest quotient of this group on which G acts trivially, that is, the quotient $\tilde{D}/(\tilde{D}, G)$.

The exact sequence (2) gives an isomorphism $\operatorname{Ker}\rho_* \xrightarrow{\sim} \operatorname{Coker} a_*$. The map $\rho_* : H_1(X', \mathbb{Z})_H \to H_1(X, \mathbb{Z})$ is identified with the natural map $\tilde{D}/(\tilde{D}, G) \to G/D$ deduced from the inclusions $\tilde{D} \subset G$ and $(\tilde{D}, G) \subset D$. Therefore its kernel is $D/(\tilde{D}, G)$. On the other hand since T is a torus we have canonical isomorphisms

$$H_2(T, \mathbb{Z}) \xrightarrow{\sim} \operatorname{Hom}(H^2(T, \mathbb{Z}), \mathbb{Z}) \xrightarrow{\sim} \operatorname{Alt}^2(H^1(T, \mathbb{Z})) \xrightarrow{\sim} \operatorname{Alt}^2(H^1(X, \mathbb{Z})) ,$$

through which a_* corresponds to μ, hence the Proposition. □

Corollary 1. 1. *There is a canonical surjective map* $D/(D, G) \to \operatorname{Coker}\mu$ *with finite kernel.*
2. *There are canonical exact sequences*

$$H_2(X, \mathbb{Q}) \xrightarrow{\mu_{\mathbb{Q}}} \operatorname{Alt}^2(H^1(X, \mathbb{Q})) \longrightarrow D/(D, G) \otimes \mathbb{Q} \to 0$$

$$0 \to \operatorname{Hom}(D/(D, G), \mathbb{Q}) \longrightarrow \wedge^2 H^1(X, \mathbb{Q}) \xrightarrow{c_{\mathbb{Q}}} H^2(X, \mathbb{Q}) ,$$

where $c_{\mathbb{Q}}$ *is the cup-product map.*

Proof. (2) follows from (1), and from the fact that the transpose of $\mu_{\mathbb{Q}}$ is $c_{\mathbb{Q}}$. Therefore in view of the Proposition, it suffices to prove that the kernel of the natural map $D/(D, G) \to D/(\tilde{D}, G)$, that is, $(\tilde{D}, G)/(D, G)$, is finite. Consider the surjective homomorphism

$$G/D \otimes G/D \to D/(D, G)$$

deduced from $(x, y) \mapsto xyx^{-1}y^{-1}$. It maps $\tilde{D}/D \otimes G/D$ onto $(\tilde{D}, G)/(D, G)$; since $\tilde{D}/D$ is finite and G/D finitely generated, the result follows. □

Corollary 2. *Assume that* $H_1(X, \mathbb{Z})$ *is torsion free.*

1. *The second quotient* $D/(D, G)$ *of the lower central series of* G *is canonically isomorphic to* $\operatorname{Coker}\mu$.
2. *For every ring* R *the group* $\operatorname{Hom}(D/(D, G), R)$ *is canonically isomorphic to the kernel of the cup-product map* $c_R : \wedge^2 H^1(X, R) \to H^2(X, R)$.

Proof. We have $\tilde{D} = D$ in that case, so (1) follows immediately from the Proposition. Since $H_1(X, \mathbb{Z})$ is torsion free, the universal coefficient theorem provides an isomorphism $H^2(X, R) \xrightarrow{\sim} \operatorname{Hom}(H_2(X, \mathbb{Z}), R)$, hence applying $\operatorname{Hom}(-, R)$ to the exact sequence

$$H_2(X, \mathbb{Z}) \to \operatorname{Alt}^2(H^1(X, \mathbb{Z})) \to D/(D, G) \to 0$$

gives (2). □

Remark 1. The Proposition and its Corollaries hold (with the same proofs) under weaker assumptions on X, for instance for a connected space X which is paracompact, admits a universal cover and is such that $H_1(X, \mathbb{Z})$ is finitely generated. We leave the details to the reader.

Remark 2. For compact Kähler manifolds, the isomorphism $\operatorname{Hom}(D/(D, G), \mathbb{Q}) \cong \operatorname{Ker} c_{\mathbb{Q}}$ (Corollary 1) is usually deduced from Sullivan's theory of minimal models (see [1], ch. 3); it can be used to prove that certain manifolds, for instance Lagrangian submanifolds of an abelian variety, have a non-abelian fundamental group.

3 Example: The Fano Surface

Let $V \subset \mathbb{P}^4$ be a smooth cubic threefold. The Fano surface F of V parametrizes the lines contained in V. It is a smooth connected surface, which has been thoroughly studied in [2]. Its Albanese variety A is canonically isomorphic to the intermediate Jacobian JV of V, and the Albanese map $a : F \to A$ is an embedding. Recall that $A = JV$ carries a principal polarization $\theta \in H^2(A, \mathbb{Z})$; for each integer k the class $\dfrac{\theta^k}{k!}$ belongs to $H^{2k}(A, \mathbb{Z})$. The class of F in $H^6(A, \mathbb{Z})$ is $\dfrac{\theta^3}{3!}$ ([2], Proposition 13.1).

Proposition 2. *The maps* $a^* : H^2(A, \mathbb{Z}) \to H^2(F, \mathbb{Z})$ *and* $a_* : H_2(F, \mathbb{Z}) \to H_2(A, \mathbb{Z})$ *are injective and their images have index 2.*

Proof. We first recall that if $u : M \to N$ is a homomorphism between two free $\mathbb{Z}$-modules of the same rank, the integer $|\det u|$ is well-defined: it is equal to the absolute value of the determinant of the matrix of u for any choice of bases for M and N. If it is nonzero, it is equal to the index of $\operatorname{Im} u$ in N.

Poincaré duality identifies a_* with the Gysin map $a_* : H^2(F, \mathbb{Z}) \to H^8(A, \mathbb{Z})$, and also to the transpose of a^*. The composition

$$f : H^2(A, \mathbb{Z}) \xrightarrow{a^*} H^2(F, \mathbb{Z}) \xrightarrow{a_*} H^8(A, \mathbb{Z})$$

is the cup-product with the class $[F] = \dfrac{\theta^3}{3!} \cdot$ We have $|\det a^*| = |\det a_*| \neq 0$ ([2], 10.14), so it suffices to show that $|\det f| = 4$.

The principal polarization defines a unimodular skew-symmetric form on $H^1(A, \mathbb{Z})$; we choose a symplectic basis $(\varepsilon_i, \delta_j)$ of $H^1(A, \mathbb{Z})$. Then

$$\theta = \sum_i \varepsilon_i \wedge \delta_i \qquad \text{and} \qquad \frac{\theta^3}{3!} = \sum_{i<j<k} (\varepsilon_i \wedge \delta_i) \wedge (\varepsilon_j \wedge \delta_j) \wedge (\varepsilon_k \wedge \delta_k)\,.$$

If we identify by Poincaré duality $H^8(A, \mathbb{Z})$ with the dual of $H^2(A, \mathbb{Z})$, and $H^{10}(A, \mathbb{Z})$ with $\mathbb{Z}$, f is the homomorphism associated to the bilinear symmetric form $b : (\alpha, \beta) \mapsto \alpha \wedge \beta \wedge \frac{\theta^3}{3!}$, hence $|\det f|$ is the absolute value of the discriminant of b. Let us write $H^2(A, \mathbb{Z}) = M \oplus N$, where M is spanned by the vectors $\varepsilon_i \wedge \varepsilon_j$, $\delta_i \wedge \delta_j$ and $\varepsilon_i \wedge \delta_j$ for $i \neq j$, and N by the vectors $\varepsilon_i \wedge \delta_i$. The decomposition is orthogonal with respect to b; the restriction of b to M is unimodular, because the dual basis of $(\varepsilon_i \wedge \varepsilon_j, \delta_i \wedge \delta_j, \varepsilon_i \wedge \delta_j)$ is $(-\delta_i \wedge \delta_j, -\varepsilon_i \wedge \varepsilon_j, -\varepsilon_j \wedge \delta_i)$. On N the matrix of b with respect to the basis $(\varepsilon_i \wedge \delta_i)$ is $E - I$, where E is the 5-by-5 matrix with all entries equal to 1. Since E has rank 1 we have $\wedge^k E = 0$ for $k \geq 2$, hence

$$\det(E - I) = -\det(I - E) = -I + \operatorname{Tr} E = 4\,;$$

hence $|\det f| = 4$. □

Corollary 3. *Set $G = \pi_1(F)$ and $D = (G, G)$. The group $D/(D, G)$ is cyclic of order* 2.

Indeed $H_1(F, \mathbb{Z})$ is torsion free [3], hence the result follows from Corollary 2. □

Remark 3. The deeper topological study of [3] gives actually the stronger result that D is generated as a normal subgroup by an element σ of order 2 (see [3], and the correction in [4], Remark 4.1). Since every conjugate of σ is equivalent to σ modulo (D, G), this implies Corollary 3.

Remark 4. Choose a line $\ell \in F$, and let $C \subset F$ be the curve of lines incident to ℓ. Let $d : H^2(F, \mathbb{Z}) \to \mathbb{Z}/2$ be the homomorphism given by $d(\alpha) = (\alpha \cdot [C])$ (mod. 2). We claim that the image of $a^* : H^2(A, \mathbb{Z}) \to H^2(F, \mathbb{Z})$ is $\operatorname{Ker} d$. Indeed we have $(C^2) = 5$ (the number of lines incident to two given skew lines on a cubic surface), hence $d([C]) = 1$, so that $\operatorname{Ker} d$ has index 2; thus it suffices to prove $d \circ a^* = 0$. For $\alpha \in H^2(A, \mathbb{Z})$, we have $d(a^*\alpha) = (a^*\alpha \cdot [C]) = (\alpha \cdot a_*[C])$ mod. 2; this is 0 because the class $a_*[C] \in H^8(A, \mathbb{Z})$ is equal to $2\,\frac{\theta^4}{4!}$ ([2], Lemma 11.5), hence is divisible by 2.

We can identify a^* with the cup-product map c; thus we have an exact sequence

$$0 \to \wedge^2 H^1(F, \mathbb{Z}) \xrightarrow{c} H^2(F, \mathbb{Z}) \xrightarrow{d} \mathbb{Z}/2 \to 0 \quad \text{with } d(\alpha) = (\alpha \cdot [C]) \text{ (mod. 2)}\,.$$

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