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The Picard group of the moduli of G-bundles on a curve

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Abstract. Let G be a complex semi-simple group, and X a compact Riemann surface. The moduli space of principal G-bundles on X, and in particular the holomorphic line bundles on this space and their global sections, play an important role in the recent applications of Conformal Field Theory to algebraic geometry. In this paper we determine the Picard group of this moduli space when G is of classical or G_2 type (we consider both the coarse moduli space and the moduli stack).

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Introduction

This paper is concerned with the moduli space of principal G-bundles on an algebraic curve of positive genus, for G a complex semi-simple group. While the case $G = \mathbf{SL}_r$, which corresponds to vector bundles, has been extensively studied in algebraic geometry, the general case has attracted much less attention until recently, when it became clear that these spaces play an important role in Quantum Field Theory. In particular, if L is a holomorphic line bundle on the moduli space M_G, the space $H^0(M_G, L)$ is essentially independent of the curve X, and can be naturally identified with what physicists call the *space of conformal blocks* associated to the most standard Conformal Field Theory, the so-called WZW-model. This gives a strong motivation to determine the group $\mathrm{Pic}(M_G)$ of holomorphic line bundles on the moduli space.

Up to this point we have been rather vague about what we should call the moduli space of G-bundles on X. Unfortunately there are two possible choices, and both are meaningful. Because G-bundles have usually nontrivial automorphisms, the natural solution to the moduli problem is not an algebraic variety, but a slightly more complicated object, the algebraic stack $\mathcal{M}_G$. This has all the good properties one expects from a moduli space; in particular, a line bundle on $\mathcal{M}_G$ is the functorial

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assignment, for every variety S and every G-bundle on $X \times S$, of a line bundle on S. There is also a more down-to-earth object, the coarse moduli space M_G of semi-stable G-bundles; the group $\mathrm{Pic}(M_G)$ is a subgroup of $\mathrm{Pic}(\mathcal{M}_G)$, but its geometric meaning is less clear.

In this paper we determine the groups $\mathrm{Pic}(M_G)$ and $\mathrm{Pic}(\mathcal{M}_G)$ for essentially all cassical semi-simple groups, i.e. of type A, B, C, D and G_2. Since the simply-connected case was already treated in [L-S] (see also [K-N]), we are mainly concerned with non simply-connected groups. One new difficulty appears: the moduli space is no longer connected, its connected components are naturally indexed by $\pi_1(G)$. Let $\widetilde{G}$ be the universal covering of G; for each $\delta \in \pi_1(G)$, we construct a natural 'twisted' moduli stack $\mathcal{M}^{\delta}_{\widetilde{G}}$ which dominates $\mathcal{M}^{\delta}_{G}$. (For instance if $G = \mathbf{PGL}_r$, it is the moduli stack of vector bundles on X of rank r and fixed determinant of degree d, with $e^{2\pi i d/r} = \delta$.) This moduli stack carries in each case a natural line bundle $\mathcal{D}$, the determinant bundle associated to the standard representation of $\widetilde{G}$. We can now state some of our results; for simplicity we only consider the adjoint groups.

Theorem. *Put $\varepsilon_G = 1$ if the rank of G is even, 2 if it is odd. Let $\delta \in \pi_1(G)$.*

(a) *The torsion subgroup of* $\mathrm{Pic}(\mathcal{M}^{\delta}_G)$ *is isomorphic to $H^1(X, \pi_1(G))$. The torsion-free quotient is infinite cyclic, generated by $\mathcal{D}^r$ if $G = \mathbf{PGL}_r$, by $\mathcal{D}^{\varepsilon_G}$ if $G = \mathbf{PSp}_{2l}$ or $\mathbf{PSO}_{2l}$.*

(b) *The group* $\mathrm{Pic}(M^{\delta}_G)$ *is infinite cyclic, generated by $\mathcal{D}^{r\varepsilon_G}$ if $G = \mathbf{PGL}_r$, by $\mathcal{D}^{2\varepsilon_G}$ if $G = \mathbf{PSp}_{2l}$ or $\mathbf{PSO}_{2l}$.* [1]

Unfortunately, though our method has some general features, it requires a case-by-case analysis; after our preprint appeared a uniform topological determination of $\mathrm{Pic}(\mathcal{M}_G)$ has been outlined by C. Teleman [T]. As a consequence of our analysis we prove that when G is of classical or G_2 type, the moduli space M_G is locally factorial exactly when G is special in the sense of Serre (this is now also proved for exceptional groups [So]). Nevertheless it is always a Gorenstein variety.

Notation

Throughout this paper we denote by X a smooth projective connected curve over $\mathbf{C}$ of positive genus (see [La] for the genus 0 case); we fix a point p of X. We let G be a complex semi-simple group; by a G-bundle we always mean a principal bundle with structure group G. We denote by $\mathcal{M}_G$ the moduli stack parameterizing G-bundles on X, and by M_G the coarse moduli variety of semi-stable G-bundles (see Section 7).

[1] The statement '$\mathrm{Pic}(M_G)$ is generated by $\mathcal{D}^k$' must be interpreted as '$\mathcal{D}^k$ descends to M_G, and the line bundle on M_G thus obtained generates $\mathrm{Pic}(M_G)$' – and similarly for (a).

Part I: The Picard group of the moduli stack

1. The stack $\mathcal{M}_G$

(1.1) Our main tool to study Pic$(\mathcal{M}_G)$ will be the uniformization theorem of [B-L], [F2] and [L-S], which we now recall. We denote by LG the loop group $G(\mathbf{C}((z)))$, viewed as an ind-scheme over $\mathbf{C}$, by L^+G the sub-group scheme $G(\mathbf{C}[[z]])$, and by $\mathcal{Q}_G$ the infinite Grassmannian LG/L^+G; it is a direct limit of projective integral varieties (*loc. cit.*). Finally let L_XG be the sub-ind-group $G(\mathcal{O}(X-p))$ of LG. The uniformization theorem defines a canonical isomorphism of stacks

$$\mathcal{M}_G \xrightarrow{\sim} L_XG\backslash\mathcal{Q}_G.$$

Let $\widetilde{G}\to G$ be the universal cover of G; its kernel is canonically isomorphic to $\pi_1(G)$. We want to compare the stacks $\mathcal{M}_G$ and $\mathcal{M}_{\widetilde{G}}$. For each integer n, we identify the group μ_n of n-roots of 1 to $\mathbf{Z}/n\mathbf{Z}$ using the generator $e^{2\pi i/n}$.

LEMMA 1.2. (i) *The group $\pi_0(LG)$ is canonically isomorphic to $\pi_1(G)$.*

(ii) *The quotient map $LG\to\mathcal{Q}_G$ induces a bijection $\pi_0(LG)\to\pi_0(\mathcal{Q}_G)$. Each connected component of $\mathcal{Q}_G$ is isomorphic to $\mathcal{Q}_{\widetilde{G}}$.*

(iii) *The group $\pi_0(L_XG)$ is canonically isomorphic to $H^1(X,\pi_1(G))$.*

(iv) *The group L_XG is contained in the neutral component $(LG)^\circ$ of LG.*

Proof. Let us first prove (i) when G is simply connected. In that case, there exists a finite family of homomorphisms $x_\alpha\colon\mathbf{G}_a\to G$ such that for any extension K of $\mathbf{C}$, the subgroups $x_\alpha(K)$ generate $G(K)$ [S1]. Since the ind-group $\mathbf{G}_a(\mathbf{C}((z)))$ is connected, it follows that LG is connected.

In the general case, consider the exact sequence $1\to\pi_1(G)\to\widetilde{G}\to G\to 1$ as an exact sequence of étale sheaves on $D^*:=\operatorname{Spec}\mathbf{C}((z))$. Since $H^1(D^*,\widetilde{G})$ is trivial [S2], it gives rise to an exact sequence of $\mathbf{C}$-groups

$$1\to L\widetilde{G}/\pi_1(G)\longrightarrow LG\longrightarrow H^1(D^*,\pi_1(G))\to 1. \tag{1.2a}$$

The assertion (i) follows from the connectedness of $L\widetilde{G}$ and the canonical isomorphism $H^1(D^*,\pi_1(G))\xrightarrow{\sim}\pi_1(G)$ (Puiseux theorem).

To prove (ii), we first observe that the group L^+G is connected: for any $\gamma\in L^+G(\mathbf{C})$, the map $F_\gamma\colon G\times\mathbf{A}^1\to L^+G$ defined by $F_\gamma(g,t)=g^{-1}\gamma(tz)$ satisfies $F_\gamma(\gamma(0),0)=1$ and $F_\gamma(1,1)=\gamma$, hence connects γ to the origin. Therefore the canonical map $\pi_0(LG)\to\pi_0(LG/L^+G)$ is bijective. Moreover it follows from (1.2a) that $(LG)^\circ$ is isomorphic to $L\widetilde{G}/\pi_1(G)$, which gives (ii).

Consider now the cohomology exact sequence on X^* associated to the exact sequence $1\to\pi_1(G)\to\widetilde{G}\to G\to 1$. Since $H^1(X^*,\widetilde{G})$ is trivial [Ha], we get an exact sequence of $\mathbf{C}$-groups

$$1\to L_X\widetilde{G}/\pi_1(G)\to L_XG\to H^1(X^*,\pi_1(G))\to 1. \tag{1.2b}$$

Since the restriction map $H^1(X,\pi_1(G)) \to H^1(X^*,\pi_1(G))$ is bijective and $L_X\widetilde{G}$ is connected ([L-S], Proposition 5.1), we obtain (iii).

Comparing (1.2a) and (1.2b) we see that (iv) is equivalent to saying that the restriction map $H^1(X^*,\pi_1(G)) \to H^1(D^*,\pi_1(G))$ is zero. This follows at once from the commutative diagram of restriction maps

$$\begin{tikzcd} H^1(X,\pi_1(G)) \arrow[r, "\sim"] \arrow[d] & H^1(X^*,\pi_1(G)) \arrow[d] \\ H^1(D,\pi_1(G)) \arrow[r] & H^1(D^*,\pi_1(G)) \end{tikzcd}$$

and the vanishing of $H^1(D,\pi_1(G))$. □

For $\delta \in \pi_1(G)$, let us denote by $(LG)^\delta$ the component of LG corresponding to δ via Proposition 1.2 (i).

PROPOSITION 1.3. (a) *There is a canonical bijection* $\pi_0(\mathcal{M}_G) \xrightarrow{\sim} \pi_1(G)$.

(b) *For* $\delta \in \pi_1(G)$, *let* $\mathcal{M}_G^\delta$ *be the corresponding component of* $\mathcal{M}_G$; *let* ζ *be any element of* $(LG)^\delta(\mathbf{C})$. *There is a canonical isomorphism*

$$\mathcal{M}_G^\delta \xrightarrow{\sim} (\zeta^{-1}\, L_X G\, \zeta)\backslash \mathcal{Q}_{\widetilde{G}}.$$

Proof. The first assertion follows from the uniformization theorem and Lemma 1.2, (i), (ii) and (iv). Again by the uniformization theorem, $\mathcal{M}_G^\delta$ is isomorphic to $L_XG\backslash(LG)^\delta/L^+G$; left multiplication by ζ^{-1} induces an isomorphism of $(LG)^\delta/L^+G$ onto $(LG)^\circ/L^+G = \mathcal{Q}_{\widetilde{G}}$, and therefore an isomorphism of $L_XG\backslash(LG)^\delta/L^+G$ onto $(\zeta^{-1}\, L_X G\, \zeta)\backslash\mathcal{Q}_{\widetilde{G}}$. □

Proposition 1.3(a) assigns to any G-bundle P on X an element δ of $\pi_1(G)$ such that P defines a point of $\mathcal{M}_G^\delta$; we will refer to δ as the *degree* of P.

We will use Proposition 1.3 to determine the Picard group of $\mathcal{M}_G^\delta$; therefore we first need to compute Pic$(\mathcal{Q}_{\widetilde{G}})$. We denote by s the number of simple factors of Lie(G).

LEMMA 1.4. *The Picard group of* $\mathcal{Q}_{\widetilde{G}}$ *is isomorphic to* $\mathbf{Z}^s$.

Proof. Write $\widetilde{G}$ as a product $\prod\limits_{i=1}^{s} \widetilde{G}_i$ of almost simple simply connected groups. Put $\mathcal{Q} = \mathcal{Q}_{\widetilde{G}}$ and $\mathcal{Q}_i = \mathcal{Q}_{\widetilde{G}_i}$; the Grassmannian $\mathcal{Q}$ is isomorphic to $\prod \mathcal{Q}_i$. The Picard group of $\mathcal{Q}_i$ is free of rank 1 [M]; we denote by $\mathcal{O}_{\mathcal{Q}_i}(1)$ its positive generator. The projections $\mathcal{Q} \to \mathcal{Q}_i$ define a group homomorphism $\prod$ Pic$(\mathcal{Q}_i) \to$ Pic$(\mathcal{Q})$; we claim that it is bijective.

Let $\mathcal{L}$ be a line bundle on $\mathcal{Q}$; there are integers (m_i) such that the restriction of $\mathcal{L}$ to $\{q_1\} \times \cdots \times \mathcal{Q}_j \times \cdots \times \{q_s\}$, for any $(q_i) \in \prod \mathcal{Q}_i$ and any j, is isomorphic to $\mathcal{O}_{\mathcal{Q}_j}(m_j)$. Then $\mathcal{L}$ is isomorphic to $\boxtimes_i \mathcal{O}_{\mathcal{Q}_i}(m_i)$: by writing each $\mathcal{Q}_i$ as a direct limit of varieties $\mathcal{Q}_i^{(n)}$, we are reduced to prove that these two line bundles are isomorphic over $\prod\limits_i \mathcal{Q}_i^{(n)}$, which follows immediately from the theorem of the square. $\square$

PROPOSITION 1.5. *For $\delta \in \pi_1(G)$, let $q_G^\delta : \mathcal{Q}_{\widetilde{G}} \to \mathcal{M}_G^\delta$ be the canonical projection* (Proposition 1.3). *The kernel of the homomorphism*

$$(q_G^\delta)^* \colon \mathrm{Pic}(\mathcal{M}_G^\delta) \to \mathrm{Pic}(\mathcal{Q}_{\widetilde{G}}) \cong \mathbf{Z}^s$$

is canonically isomorphic to $H^1(X, \pi_1(G))$, and its image has finite index.

Proof. Since q_G^δ identifies $\mathcal{M}_G^\delta$ to the quotient of $\mathcal{Q}_{\widetilde{G}}$ by $\zeta^{-1} L_X G\, \zeta$, line bundles on $\mathcal{M}_G^\delta$ correspond in a one-to-one way to line bundles on $\mathcal{Q}_{\widetilde{G}}$ with a $(\zeta^{-1} L_X G\, \zeta)$-linearization ([V], ex. 7.21); in particular, the kernel of $(q_G^\delta)^*$ is canonically isomorphic to the character group $\mathrm{Hom}(L_X G, \mathbf{C}^*)$. From the exact sequence (1.2b) and the triviality of the character group of $L_X \widetilde{G}$ ([L-S], Corollary 5.2) we see that the group $\mathrm{Hom}(L_X G, \mathbf{C}^*)$ is isomorphic to $\mathrm{Hom}(H^1(X, \pi_1(G)), \mathbf{C}^*)$, which can be identified by duality with $H^1(X, \pi_1(G))$.

Write $\widetilde{G} \cong \prod\limits_{i=1}^{s} \widetilde{G}_i$ as in Lemma 1.4. The image of $\pi_1(G)$ under the i-th projection $p_i \colon \widetilde{G} \to \widetilde{G}_i$ is a central subgroup A_i of $\widetilde{G}_i$; we denote by G_i the quotient $\widetilde{G}_i / A_i$, so that p_i induces a homomorphism $G \to G_i$. Let δ_i be the image of δ in $\pi_1(G_i)$. Choosing a nontrivial representation $\rho \colon G_i \to \mathbf{SL}_r$ gives rise to a commutative diagram

$$\begin{tikzcd}
\mathcal{Q}_{\widetilde{G}} \arrow[r, "pr_i"] \arrow[d, "q_G^\delta"'] & \mathcal{Q}_{\widetilde{G}_i} \arrow[r] \arrow[d, "q_{G_i}^{\delta_i}"'] & \mathcal{Q}_{\mathbf{SL}_r} \arrow[d, "q_{\mathbf{SL}_r}"'] \\
\mathcal{M}_G^\delta \arrow[r] & \mathcal{M}_{G_i}^{\delta_i} \arrow[r] & \mathcal{M}_{\mathbf{SL}_r} .
\end{tikzcd}$$

The pull back of the determinant bundle $\mathcal{D}$ on $\mathcal{M}_{\mathbf{SL}_r}$ to $\mathcal{Q}_{\mathbf{SL}_r}$ is $\mathcal{O}_{\mathcal{Q}}(1)$ [B-L], and the pull back of $\mathcal{O}_{\mathcal{Q}}(1)$ to $\mathcal{Q}_i := \mathcal{Q}_{\widetilde{G}_i}$ is $\mathcal{O}_{\mathcal{Q}_i}(d_\rho)$ for some integer d_ρ (the Dynkin index of ρ, see [L-S]). Therefore $pr_i^* \, \mathcal{O}_{\mathcal{Q}_i}(d_\rho)$ belongs to the image of $(q_G^\delta)^*$. It follows that this image has finite index. $\square$

Remark 1.6. In the sequel we will be mostly interested in the case where G is almost simple; then $\pi_1(G)$ is canonically isomorphic to $\boldsymbol{\mu}_n$ or to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$. We thus get that the torsion subgroup of $\mathrm{Pic}(\mathcal{M}_G^\delta)$ is J_n in the first case and $J_2 \times J_2$ in the second, where J_n denotes the kernel of the multiplication by n in the Jacobian of X.

2. The twisted moduli stack $\mathcal{M}_G^\delta$

(2.1) Proposition 1.5 takes care of the torsion subgroup of $\mathrm{Pic}(\mathcal{M}_G^\delta)$; to complete the description of this group we need to determine the image of $(q_G^\delta)^*$, or more precisely to describe geometrically the generators of this image. To do this we will again compare with the simply connected case, by constructing for every $\delta\in\pi_1(G)$ a 'twisted' moduli stack $\mathcal{M}_{\tilde{G}}^\delta$ which dominates $\mathcal{M}_G^\delta$.

Let A be a central subgroup of G, together with an isomorphism $A \xrightarrow{\sim} \prod_{j=1}^{s}\boldsymbol{\mu}_{r_j}$. Using this isomorphism we identify A to a subgroup of the torus $T=(\mathbf{G}_m)^s$; let C_AG be the quotient of $G\times T$ by the diagonal subgroup A. The projection $\partial\colon C_AG\to T/A\cong T$ induces a morphism of stacks det: $\mathcal{M}_{C_AG}\to\mathcal{M}_T$. For each element $\mathbf{d}=(d_1,\dots,d_s)$ of $\mathbf{Z}^s$, let us denote by $\mathcal{O}_X(\mathbf{d}p)$ the rational point of $\mathcal{M}_T$ defined by $(\mathcal{O}_X(d_1p),\dots,\mathcal{O}_X(d_sp))$. The fiber $\mathcal{M}_{G,A}^{\mathbf{d}}$ of det at $\mathcal{O}_X(\mathbf{d}p)$ depends only, up to a canonical isomorphism, of the class of $\mathbf{d}$ modulo $\mathbf{r}=(r_1,\dots,r_s)$.

If S is a complex scheme, an object of $\mathcal{M}_{G,A}^{\mathbf{d}}(S)$ is by definition a C_AG-bundle P on $X\times S$ together with a T-bundle isomorphism of $P\times^{C_AG}T$ with the T-bundle associated to $\mathcal{O}_X(\mathbf{d}p)$. If $\mathbf{d}=0$, giving such an isomorphism amounts to reduce the structure group of P to $\operatorname{Ker}\partial=G$: in other words, the stack $\mathcal{M}_{G,A}^{\mathbf{0}}$ is canonically isomorphic to $\mathcal{M}_G$.

(2.2) The projection $p\colon C_AG\to G/A$ induces a morphism of stacks $\pi\colon\mathcal{M}_{G,A}^{\mathbf{d}}\to\mathcal{M}_{G/A}$. The exact sequence

$$1\to A\to C_AG\xrightarrow{(p,\partial)}(G/A)\times T\to 1$$

gives rise to a cohomology exact sequence

$$H^1(X,A)\to H^1(X,C_AG)\to H^1(X,G/A)\times H^1(X,T)\to H^2(X,A)$$

from which we deduce that the degree $\delta\in\pi_1(G)$ of the (G/A)-bundle $\pi(P)$, for $P\in\mathcal{M}_{G,A}^{\mathbf{d}}(\mathbf{C})$, satisfies $\rho(\delta)\,e^{2\pi i\mathbf{d}/\mathbf{r}}=1$, where ρ is the natural homomorphism of $\pi_1(G/A)$ onto $A\subset(\mathbf{G}_m)^s$ and $e^{2\pi i\mathbf{d}/\mathbf{r}}$ stands for the element $(e^{2\pi id_1/r_1},\dots,e^{2\pi id_s/r_s})$ of $(\mathbf{G}_m)^s$. We denote by $\mathcal{M}_{G,A}^\delta$ the open and closed substack $\pi^{-1}(\mathcal{M}_{G/A}^\delta)$ of $\mathcal{M}_{G,A}^{\mathbf{d}}$, where $\mathbf{d}=(d_1,\dots,d_s)$ is the unique element of $\mathbf{Z}^s$ such that $0\le d_t<r_t$ and $\rho(\delta)\,e^{2\pi i\mathbf{d}/\mathbf{r}}=1$ (if G is simply connected, ρ is bijective and $\mathcal{M}_{G,A}^\delta$ is simply $\mathcal{M}_{G,A}^{\mathbf{d}}$). The induced morphism $\pi\colon\mathcal{M}_{G,A}^\delta\to\mathcal{M}_{G/A}^\delta$ is surjective.

We will be mostly interested in the case when A is the center of G; then we will denote simply by $\mathcal{M}_{\tilde{G}}^\delta$ the stack $\mathcal{M}_{G,A}^\delta$, for any choice of the isomorphism

$A \xrightarrow{\sim} \prod_{j=1}^{s} \boldsymbol{\mu}_{r_j}$ (up to a canonical isomorphism, the stack $\mathcal{M}^{\delta}_{G,A}$ does not depend on this choice). If δ belongs to $\pi_1(G) \subset \pi_1(G_{\mathrm{ad}})$, one gets $\rho(\delta) = 1$ hence $\mathbf{d} = 0$: by the above remark, the notation $\mathcal{M}^{\delta}_{G}$ is thus coherent with the one introduced in Proposition 1.3.

Examples 2.3. (a) We take $G = \mathrm{SL}_r$, $A = \boldsymbol{\mu}_r$. The group $C_A G$ is canonically isomorphic to $\mathbf{GL}_r$; the stack $\mathcal{M}^{d}_{\mathbf{SL}_r}$ can be identified with the stack of vector bundles E on X with an isomorphism $\bigwedge^r E \xrightarrow{\sim} \mathcal{O}_X(dp)$.

(b) We take for G the group $\mathbf{O}_{2l}$ or $\mathbf{Sp}_{2l}$, for A its center, with the unique isomorphism $A \xrightarrow{\sim} \boldsymbol{\mu}_2$. The group $C_A G$ is the group $C\mathbf{O}_{2l}$ or $C\mathbf{Sp}_{2l}$ of automorphisms of $\mathbf{C}^{2l}$ respecting the bilinear form up to a (fixed) scalar. The stack $\mathcal{M}^{d}_{G}$ can therefore be viewed as parameterizing vector bundles E on X with a (symmetric or alternate) non-degenerate bilinear form with values in $\mathcal{O}_X(dp)$. Similarly, the stack $\mathcal{M}^{d}_{\mathbf{SO}_{2l}}$ parameterizes vector bundles E on X with a non-degenerate quadratic form $q\colon \mathsf{S}^2 E \to \mathcal{O}_X(dp)$ and an *orientation*, i.e. an isomorphism $\omega\colon \det E \xrightarrow{\sim} \mathcal{O}_X(dlp)$ such that $\omega^{\otimes 2}$ coincides with the quadratic form induced by q on $\det E$.

(c) We take $G = \mathbf{Spin}_r$, $A = \boldsymbol{\mu}_2$. Then $C_A G$ is the Clifford group and $\mathcal{M}^{-1}_{G,A}$ is the moduli stack $\mathcal{M}^{-}_{\mathbf{Spin}_r}$ considered in [O].

(2.4) Choose any element $\zeta \in (LG_{\mathrm{ad}})^{\delta}(\mathbf{C})$; reasoning as in Proposition 1.3, one gets a canonical isomorphism $\mathcal{M}^{\delta}_{G} \xrightarrow{\sim} (\zeta^{-1} L_X G\, \zeta)\backslash \mathcal{Q}_{\widetilde{G}}$ (see also [B-L], 3.6 for the case $G = \mathbf{SL}_r$). In particular, the stack $\mathcal{M}^{\delta}_{G}$ is connected. Moreover, we see as in the proof of Proposition 1.5 that the torsion subgroup of $\mathrm{Pic}(\mathcal{M}^{\delta}_{G})$ is canonically isomorphic to $H^1(X, \pi_1(G))$.

Let us apply the above construction to the group $\widetilde{G}$, with $A = \pi_1(G)$. Let $\delta \in \pi_1(G)$. From the exact sequence (1.2a), we see that ζ is the image of an element of $(L\widetilde{G})^{\delta}$. Comparing with Proposition 1.3, we see that the morphism $q^{\delta}_{G}\colon \mathcal{Q}_{\widetilde{G}} \to \mathcal{M}^{\delta}_{G}$ factors as

$$q^{\delta}_{G}\colon \mathcal{Q}_{\widetilde{G}} \xrightarrow{q^{\delta}_{\widetilde{G}}} \mathcal{M}^{\delta}_{\widetilde{G}} \xrightarrow{\pi} \mathcal{M}^{\delta}_{G}.$$

This shows us the way to determine the group $\mathrm{Pic}(\mathcal{M}^{\delta}_{G})$: we will first compute $\mathrm{Pic}(\mathcal{M}^{\delta}_{G})$ when G is simply connected or $G = \mathbf{SO}_{2l}$, then determine which powers of the generator(s) descend to $\mathcal{M}^{\delta}_{G}$.

3. The Picard group of $\mathcal{M}_{\mathbf{PGL}_r}$

According to (1.3), the connected components of $\mathcal{M}_{\mathbf{PGL}_r}$ are indexed by the integers d with $0 \leqslant d < r$; the component $\mathcal{M}^{d}_{\mathbf{PGL}_r}$ is dominated by the moduli stack $\mathcal{M}^{d}_{\mathbf{SL}_r}$ parameterizing vector bundles E on X with an isomorphism $\bigwedge^r E \xrightarrow{\sim} \mathcal{O}_X(dp)$ (2.3a).

Recall that the *determinant bundle* $\mathcal{D}$ on $\mathcal{M}^d_{\mathbf{SL}_r}$ is the dual of the line bundle $\det R(pr_2)_*(\mathcal{E})$, where $\mathcal{E}$ is the universal bundle on $X \times \mathcal{M}^d_{\mathbf{SL}_r}$. It follows from [B-L], Proposition 9.2, that $\mathcal{D}$ generates $\mathrm{Pic}(\mathcal{M}^d_{\mathbf{SL}_r})$ and that its inverse image on $\mathcal{Q}$ generates $\mathrm{Pic}(\mathcal{Q})$. Therefore our problem is to determine which powers of $\mathcal{D}$ descend to $\mathcal{M}^d_{\mathbf{PGL}_r}$.

PROPOSITION 3.1. *The smallest power of $\mathcal{D}$ which descends to $\mathcal{M}^d_{\mathbf{PGL}_r}$ is $\mathcal{D}^r$.*

Proof. Since it preserves the Killing form, the adjoint representation defines a homomorphism $\mathrm{Ad}\colon \mathbf{GL}_r \to \mathbf{SO}_{r^2}$. Let $f\colon \mathcal{M}^d_{\mathbf{SL}_r} \to \mathcal{M}_{\mathbf{SO}_{r^2}}$ be the induced morphism of stacks; since Ad factors through $\mathbf{PGL}_r$, f factors through $\mathcal{M}^d_{\mathbf{PGL}_r}$. By [L-S], the determinant bundle $\mathcal{D}_{\mathbf{SO}}$ on $\mathcal{M}_{\mathbf{SO}_{r^2}}$ admits a square root $\mathcal{P}$; one has $f^*\mathcal{D}_{\mathbf{SO}} \cong \mathcal{D}^{2r}$ since the Dynkin index of Ad is $2r$, hence $f^*\mathcal{P} \cong \mathcal{D}^r$, which implies that $\mathcal{D}^r$ descends.

Let J be the Jacobian of X, and $\mathcal{L}$ the Poincaré bundle on $X \times J$ whose restriction to $\{p\} \times J$ is trivial. Consider the vector bundles

$$\mathcal{F} = \mathcal{L}^{\oplus(r-1)} \oplus \mathcal{L}^{1-r}(dp) \quad \text{and} \quad \mathcal{G} = \mathcal{O}_X^{\oplus(r-1)} \oplus \mathcal{L}^{-1}(dp)$$

on $X \times J$. We denote by r_J the multiplication by r in J, and put $r_{X\times J} = \mathrm{Id}_X \times r_J$. Since $r^*_{X\times J}\,\mathcal{L} \cong \mathcal{L}^r$, one has $r^*_{X\times J}\,\mathcal{G} \cong \mathcal{F} \otimes \mathcal{L}^{-1}$, hence the projective bundles $\mathbf{P}(\mathcal{F})$ and $r^*_{X\times J}\,\mathbf{P}(\mathcal{G})$ are isomorphic. Therefore we have a commutative[1] diagram of stacks

$$\begin{tikzcd} J \arrow[r, "f"] \arrow[d, "r_J"'] & \mathcal{M}^d_{\mathbf{SL}_r} \arrow[d, "\pi"] \\ J \arrow[r, "g"] & \mathcal{M}^d_{\mathbf{PGL}_r}, \end{tikzcd}$$

where f and g are the morphisms associated to $\mathcal{F}$ and $\mathbf{P}(\mathcal{G})$ respectively.

Thus if $\mathcal{D}^k$ descends to $\mathcal{M}^d_{\mathbf{PGL}_r}$, the class of $f^*\mathcal{D}^k$ in the Néron-Severi group $NS(J)$ must be divisible by r^2. An easy computation shows that the class of $f^*\mathcal{D}$ in $NS(J)$ is $r(r-1)$ times the principal polarization; it follows that r^2 must divide $kr(r-1)$, which means that r must divide k. $\square$

Remark 3.2. One can consider more generally the group $G = \mathbf{SL}_r/\boldsymbol{\mu}_s$, for each integer s dividing r, and the corresponding stacks $\mathcal{M}^d_G$ for $d \in \frac{r}{s}\mathbf{Z}$ (mod. $r\mathbf{Z}$). It can be proved that *the line bundle $\mathcal{D}^k$ descends to $\mathcal{M}^d_G$ if and only if k is a multiple of $s/(s, \frac{r}{s})$* [La]. The 'only if' part is proved exactly as above, but the

[1] By this we always mean 2-commutative, e.g. in our case the two functors $\pi \circ f$ and $g \circ r_J$ are isomorphic.

other implication requires some descent theory on stacks which lies beyond the scope of this paper.

4. The Picard group of $\mathcal{M}_{\mathbf{PSp}_{2l}}$

According to Proposition 1.3 the moduli stack $\mathcal{M}_{\mathbf{PSp}_{2l}}$ has 2 components $\mathcal{M}^d_{\mathbf{PSp}_{2l}}$ $(d = 0, 1)$; the component $\mathcal{M}^d_{\mathbf{PSp}_{2l}}$ is dominated by the algebraic stack $\mathcal{M}^d_{\mathbf{Sp}_{2l}}$ parameterizing vector bundles of rank $2l$ on X with a symplectic form $\wedge^2 E \to \mathcal{O}_X(dp)$ (Example 2.3b). Let $\mathcal{D}$ denote the determinant bundle on $\mathcal{M}^d_{\mathbf{Sp}_{2l}}$ (i.e. the determinant of the cohomology of the universal bundle on $X \times \mathcal{M}^d_{\mathbf{Sp}_{2l}}$); it is the pull back of the determinant bundle $\mathcal{D}_0$ on $\mathcal{M}^d_{\mathbf{SL}_{2l}}$ by the morphism $f\colon \mathcal{M}^d_{\mathbf{Sp}_{2l}} \to \mathcal{M}^d_{\mathbf{SL}_{2l}}$ associated to the standard representation.

LEMMA 4.1. *The group* $\mathrm{Pic}(\mathcal{M}^d_{\mathbf{Sp}_{2l}})$ *is generated by* $\mathcal{D}$.

Proof. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{Q}_{\mathbf{Sp}_{2l}} \arrow[r, "F"] \arrow[d, "q^d_{\mathbf{Sp}_{2l}}"'] & \mathcal{Q}_{\mathbf{SL}_{2l}} \arrow[d, "q^d_{\mathbf{SL}_{2l}}"] \\ \mathcal{M}^d_{\mathbf{Sp}_{2l}} \arrow[r, "f"] & \mathcal{M}^d_{\mathbf{SL}_{2l}}, \end{tikzcd}$$

where f and F are induced by the embedding $\mathbf{Sp}_{2l} \to \mathbf{SL}_{2l}$, and $q^d_G\colon \mathcal{Q}_G \to \mathcal{M}^d_G$ is the canonical projection (2.4). One has $\mathcal{D} = f^*\mathcal{D}_0$, $(q^d_{\mathbf{SL}_{2l}})^*\mathcal{D}_0 = \mathcal{O}_{\mathcal{Q}_{\mathbf{SL}_{2l}}}(1)$ by [B-L], 5.5, and $F^*\mathcal{O}_{\mathcal{Q}_{\mathbf{SL}_{2l}}}(1) = \mathcal{O}_{\mathcal{Q}_{\mathbf{Sp}_{2l}}}(1)$ since the Dynkin index of the standard representation of $\mathbf{Sp}_{2l}$ is 1 ([L-S], Lemma 6.8). It follows that the homomorphism $(q^d_{\mathbf{Sp}_{2l}})^* : \mathrm{Pic}(\mathcal{M}^d_{\mathbf{Sp}_{2l}}) \to \mathrm{Pic}(\mathcal{Q}_{\mathbf{Sp}_{2l}}) = \mathbf{Z}\,\mathcal{O}_{\mathcal{Q}}(1)$ is surjective. On the other hand, the proof of Proposition 6.2 in [L-S] shows that it is injective; our assertion follows. □

In view of the above remarks, Proposition 1.5 and (2.4) provide us with an exact sequence

$$0 \to J_2 \to \mathrm{Pic}(\mathcal{M}^d_{\mathbf{PSp}_{2l}}) \xrightarrow{\pi^*} \mathrm{Pic}(\mathcal{M}^d_{\mathbf{Sp}_{2l}}) = \mathbf{Z}\,\mathcal{D};$$

we now determine the image of π^*:

PROPOSITION 4.2. *The smallest power of* $\mathcal{D}$ *which descends to* $\mathcal{M}^d_{\mathbf{PSp}_{2l}}$ *is* $\mathcal{D}$ *if* l *is even,* $\mathcal{D}^2$ *if* l *is odd.*

Proof. The stack $\mathcal{M}^d_{\mathbf{Sp}_{2l}}$ parameterizes vector bundles E with a symplectic form $\varphi\colon \wedge^2 E \to \mathcal{O}_X(dp)$ (2.3b). For such a pair, the form $\wedge^2\varphi$ defines a quadratic form on $\wedge^2 E$ with values in $\mathcal{O}_X(2dp)$, hence an $\mathcal{O}_X$-valued quadratic form on $\wedge^2 E(-dp)$. Put $N = l(2l-1)$; let $f_d : \mathcal{M}^d_{\mathbf{Sp}_{2l}} \to \mathcal{M}_{\mathbf{SO}_N}$ be the morphism of stacks which associates to (E, φ) the pair $(\wedge^2 E(-dp), \wedge^2\varphi)$. Since the representation $\wedge^2\colon \mathbf{Sp}_{2l} \to \mathbf{SO}_N$ factors through $\mathbf{PSp}_{2l}$, the morphism f_d factors as

$$f_d\colon \mathcal{M}^d_{\mathbf{Sp}_{2l}} \to \mathcal{M}^d_{\mathbf{PSp}_{2l}} \to \mathcal{M}_{\mathbf{SO}_N}.$$

The pull back under f_d of the determinant bundle on $\mathcal{M}_{\mathbf{SO}_N}$ is $\mathcal{D}^{2l-2}$ ($2l-2$ is the Dynkin index of the representation $\wedge^2$). But we know by [L-S] that this determinant bundle admits a square root, hence $\mathcal{D}^{l-1}$ descends to $\mathcal{M}^d_{\mathbf{PSp}_{2l}}$. On the other hand, the same argument applied to the adjoint representation shows that $\mathcal{D}^{2l}$ descends (see the proof of Proposition 3.1). We conclude that $\mathcal{D}^2$ descends, and that $\mathcal{D}$ descends when l is even.

To prove that $\mathcal{D}$ does not descend when l is odd, we use the notation of the proof of Proposition 3.1, and consider on $X \times J$ the vector bundle $\mathcal{H} = \mathcal{L}^{\oplus l} \oplus \mathcal{L}^{-1}(dp)^{\oplus l}$, endowed with the standard hyperbolic alternate form with values in $\mathcal{O}(dp)$. We see as in *loc. cit.* that the $\mathbf{PSp}_{2l}$-bundle associated to $\mathcal{H}$ descends under the isogeny 2_J (observe that $\mathcal{H} \otimes \mathcal{L}$ descends, and use the exact sequence $1 \to \mathbf{G}_m \to C\mathbf{Sp}_{2l} \to \to \mathbf{PSp}_{2l} \to 1$). Therefore the morphism $h\colon J \to \mathcal{M}^d_{\mathbf{Sp}_{2l}}$ defined by $\mathcal{H}$ fits in a commutative diagram

$$\begin{tikzcd}
J \arrow[r, "h"] \arrow[d, "2_J"'] & \mathcal{M}^d_{\mathbf{Sp}_{2l}} \arrow[d] \\
J \arrow[r] & \mathcal{M}^d_{\mathbf{PSp}_{2l}}.
\end{tikzcd}$$

Since the class of $f^*\mathcal{D}$ in $NS(J)$ is $2l$ times the principal polarization, it follows that $\mathcal{D}$ does not descend. □

5. The Picard group of $\mathcal{M}_{\mathbf{PSO}_{2l}}$

(5.1) Let us consider first the moduli stack $\mathcal{M}_{\mathbf{SO}_r}$, for $r \geqslant 3$. It has two components $\mathcal{M}^w_{\mathbf{SO}_r}$, distinguished by the second Stiefel–Whitney class $w \in \boldsymbol{\mu}_2$. The Picard group of these stacks is essentially described in [L-S]: to each theta-characteristic κ on X is associated a Pfaffian line bundle $\mathcal{P}_\kappa$ whose square is the determinant bundle $\mathcal{D}$ (determinant of the cohomology of the universal bundle on $X \times \mathcal{M}^w_{\mathbf{SO}_r}$); according to Proposition 1.5, there is a canonical exact sequence

$$0 \to J_2 \xrightarrow{\lambda} \mathrm{Pic}(\mathcal{M}^w_{\mathbf{SO}_r}) \to \mathbf{Z} \to 0,$$

where the torsion free quotient is generated by any of the $\mathcal{P}_\kappa$'s.

We can actually be more precise. Let $\theta(X)$ be the subgroup of $\mathrm{Pic}(X)$ generated by the theta-characteristics; it is an extension of $\mathbf{Z}$ by J_2.

PROPOSITION 5.2. *The map $\kappa \mapsto \mathcal{P}_\kappa$ extends by linearity to an isomorphism $\mathcal{P}\colon \theta(X) \xrightarrow{\sim} \mathrm{Pic}(\mathcal{M}^w_{\mathbf{SO}_r})$, which coincides with λ on J_2.*

In other words, we have a canonical isomorphism of extensions

$$\begin{tikzcd}
0 \arrow[r] & J_2 \arrow[r] \arrow[d, equal] & \theta(X) \arrow[r] \arrow[d, "\mathcal{P}"] & \mathbf{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J_2 \arrow[r, "\lambda"] & \mathrm{Pic}(\mathcal{M}^w_{\mathbf{SO}_r}) \arrow[r] & \mathbf{Z} \arrow[r] & 0.
\end{tikzcd}$$

Proof. It suffices to prove the formula $\mathcal{P}_{\kappa\otimes\alpha} = \mathcal{P}_\kappa \otimes \lambda(\alpha)$ for any theta-characteristic κ and element α of J_2.

Let $\mathcal{L}$ be the Poincaré bundle on $X \times J$, normalized so that its restriction to $\{p\} \times J$ is trivial. Put $d = 0$ if $w = 1$, $d = 1$ if $w = -1$. The vector bundle $\mathcal{L}(dp) \oplus \mathcal{L}^{-1}(-dp) \oplus \mathcal{O}^{r-2}$, with its natural quadratic form and orientation, defines a morphism $g\colon J \to \mathcal{M}^w_{\mathbf{SO}_r}$. Let us identify J with $\mathrm{Pic}^{\mathrm{o}}(J)$ via the principal polarization. Then the required formula is a consequence of the following two assertions:

(a) One has $g^*\mathcal{P}_{\kappa\otimes\alpha} = (g^*\mathcal{P}_\kappa) \otimes \alpha$ for every theta-characteristic κ and element α of J_2;
(b) The map $g^* : \mathrm{Pic}(\mathcal{M}^w_{\mathbf{SO}_r})_{\mathrm{tors}} \to J_2$ is the inverse isomorphism of λ.

Let us prove (a). The line bundle $g^*\mathcal{P}_\kappa$ is the pfaffian bundle associated to the quadratic bundle $\mathcal{L}(dp) \oplus \mathcal{L}^{-1}(-dp))$ and to κ. Now it follows from the construction in [L-S] that for any vector bundle E on $X \times S$, the pfaffian of the cohomology of $E \oplus (K_X \otimes E^*)$, endowed with the standard hyperbolic form with values in K_X, is the determinant of the cohomology of E. Because the choice of $\mathcal{L}$ ensures that the determinant of the cohomology is the same for $\mathcal{L}$ and $\mathcal{L}(p)$, we conclude that $g^*\mathcal{P}_\kappa$ is the determinant of the cohomology of $\mathcal{L} \otimes \kappa$, i.e. the line bundle $\mathcal{O}_J(\Theta_\kappa)$. Since $\Theta_{\kappa\otimes\alpha} = \Theta_\kappa + \alpha$, the assertion (a) follows.

Since we already know that $\mathrm{Pic}(\mathcal{M}^\pm_{\mathbf{SO}_r})_{\mathrm{tors}}$ is isomorphic to J_2 (Proposition 1.5), (a) implies that g^* is surjective, and therefore bijective. Hence $u = g^* \circ \lambda$ is an automorphism of J_2. This construction extends to any family of curves $f\colon \mathcal{X} \to \mathcal{S}$, defining an automorphism of the local system $R^1 f_*(\boldsymbol{\mu}_2)$ over $\mathcal{S}$. Since the monodromy group of this local system is the full symplectic group $\mathbf{Sp}(J_2)$ for the universal family of curves, it follows that u is the identity. □

(5.3) This settles the case of the group $\mathbf{SO}_r$; let us now consider the group $\mathbf{PSO}_r$, for $r = 2l \geqslant 4$. The moduli space $\mathcal{M}_{\mathbf{PSO}_{2l}}$ has 4 components, indexed by the center

Z of $\mathbf{Spin}_{2l}$. This group consists of the elements $\{1, -1, \varepsilon, -\varepsilon\}$ of the Clifford algebra $C(\mathbf{C}^{2l})$, with $\varepsilon^2 = (-1)^l$ ([Bo], Algèbre IX). Each component $\mathcal{M}^{\delta}_{\mathbf{PSO}_{2l}}$, for $\delta \in Z$, is dominated by the algebraic stack $\mathcal{M}^{\delta}_{\mathbf{SO}_{2l}}$ (2.1). For $\delta \in \{\pm 1\}$, this is the same stack as above; the stack $\mathcal{M}^{\varepsilon}_{\mathbf{SO}_{2l}} \cup \mathcal{M}^{-\varepsilon}_{\mathbf{SO}_{2l}}$ parameterizes vector bundles with a quadratic form with values in $\mathcal{O}_X(p)$ and an orientation (2.3b). Changing the sign of the orientation exchanges the two components $\mathcal{M}^{\varepsilon}$ and $\mathcal{M}^{-\varepsilon}$ (this corresponds to the fact that ε and $-\varepsilon$ are exchanged by the outer automorphism of $\mathbf{Spin}(2l)$ defined by conjugation by an odd degree element of the Clifford group).

LEMMA 5.4. *The torsion free quotient of* $\mathrm{Pic}(\mathcal{M}^{\pm\varepsilon}_{\mathbf{SO}_{2l}})$ *is generated by the determinant bundle* $\mathcal{D}$.

Proof. The same proof as in Lemma 4.1 shows that the pull back of $\mathcal{D}$ by the morphism $q^{\pm\varepsilon}_{\mathbf{SO}_{2l}}: \mathcal{Q}_{\mathbf{Spin}_{2l}} \to \mathcal{M}^{\pm\varepsilon}_{\mathbf{SO}_{2l}}$ is $\mathcal{O}_{\mathcal{Q}}(2)$ (the Dynkin index of the standard representation of $\mathbf{SO}_{2l}$ is 2). Therefore it suffices to prove that $\mathcal{D}$ has no square root in $\mathrm{Pic}(\mathcal{M}^{\pm\varepsilon}_{\mathbf{SO}_{2l}})$.

Let V be a l-dimensional vector space; we consider the vector bundle $T = (V \otimes_{\mathbf{C}} \mathcal{O}_X) \oplus (V^* \otimes_{\mathbf{C}} \mathcal{O}_X(p))$, with the obvious hyperbolic quadratic form $q: \mathsf{S}^2 T \to \mathcal{O}_X(p)$ and isomorphism $\omega: \det T \xrightarrow{\sim} \mathcal{O}_X(lp)$. We choose the sign of ω so that the triple $T^{\varepsilon} := (T, q, \omega)$ defines a rational point of $\mathcal{M}^{\varepsilon}_{\mathbf{SO}_{2l}}$, and put $T^{-\varepsilon} := (T, q, -\omega) \in \mathcal{M}^{-\varepsilon}_{\mathbf{SO}_{2l}}(\mathbf{C})$. The group $G = \mathbf{GL}(V)$ acts on T, and this action preserves the quadratic form and the orientation. This defines a morphism ι of the stack BG classifying G-torsors into $\mathcal{M}^{\pm\varepsilon}_{\mathbf{SO}_{2l}}$: if S is a $\mathbf{C}$-scheme and P a G-torsor on S, one puts $\iota(P) = P \times^G T^{\pm\varepsilon}_S$.

Recall [L-MB] that the $\mathbf{C}$-stack BG is the quotient of $\mathrm{Spec}\,\mathbf{C}$ by the trivial action of G; in particular, line bundles on BG correspond in a one-to-one way to G-linearizations of the trivial line bundle on $\mathrm{Spec}\,\mathbf{C}$, that is to characters of G. In our situation, the line bundle $\iota^*\mathcal{D}$ will correspond to the character of G by which G acts on $\det R\Gamma(X, T)$. As G-modules, we have

$$\det R\Gamma(X, T) \cong \det R\Gamma(X, V \otimes_{\mathbf{C}} \mathcal{O}_X) \otimes \det R\Gamma(X, V^* \otimes_{\mathbf{C}} \mathcal{O}_X(p)).$$

Now if L is a line bundle on X, the G-module $\det R\Gamma(X, V \otimes_{\mathbf{C}} L)$ is isomorphic to $\det(V \otimes H^0(L)) \otimes \det(V \otimes H^1(L))^{-1} = \det(V)^{\chi(L)}$. We conclude that $\det R\Gamma(X, T)$ is isomorphic to $\det(V^*)$, i.e. that $\iota^*\mathcal{D}$ corresponds to the character $\det^{-1}: G \to \mathbf{C}^*$. Since det generates $\mathrm{Hom}(G, \mathbf{C}^*)$, our assertion follows. [1] $\square$

PROPOSITION 5.5. *Let* $\delta \in Z$. *The line bundle* $\mathcal{D}$ (*resp.* $\mathcal{D}^2$) *descends on* $\mathcal{M}^{\delta}_{\mathbf{PSO}_{2l}}$ *if* l *is even* (*resp. odd*); *the corresponding line bundles on* $\mathcal{M}^{\delta}_{\mathbf{PSO}_{2l}}$ *generate the Picard group.*

Proof. We first prove that the Pfaffian bundles $\mathcal{P}_\kappa$ do not descend to $\mathcal{M}^{\delta}_{\mathbf{PSO}_{2l}}$. If $\delta \in \{\pm\varepsilon\}$, this follows from the above lemma. If $\delta \in \{\pm 1\}$, we consider the

[1] This argument has been shown to us by V. Drinfeld.

action of J_2 on $\mathcal{M}^{\delta}_{\mathbf{SO}_{2l}}$ deduced from the embedding $\boldsymbol{\mu}_2 \subset \mathbf{SO}_{2l}$: each element $\alpha \in J_2$ (trivialized at p) defines an automorphism – still denoted α – of the stack $\mathcal{M}^{\pm}_{\mathbf{SO}_{2l}}$, which maps a quadratic bundle (E, q, ω) onto $(E \otimes \alpha, q \otimes i_\alpha, \omega \otimes i_\alpha^{\otimes l})$, where $i_\alpha : \alpha^2 \xrightarrow{\sim} \mathcal{O}_X$ is the isomorphism which coincides at p with the square of the given trivialization.

We claim that $\alpha^* \mathcal{P}_\kappa$ *is isomorphic to* $\mathcal{P}_{\kappa\otimes\alpha}$ for every theta-characteristic κ and element α of J_2. This is easily seen by using the following characterization of $\mathcal{P}_\kappa$ ([L-S], 7.10): let $\mathcal{E}$ be the universal bundle on $X \times \mathcal{M}^{\pm}_{\mathbf{SO}_{2l}}$; then the divisor $\Theta_\kappa :=$ div $Rpr_{2*}(\mathcal{E} \otimes \kappa)$ is divisible by 2 in Div $\mathcal{M}^{\pm}_{\mathbf{SO}_{2l}}$, and $\mathcal{P}_\kappa$ is the line bundle associated to $\frac{1}{2}\Theta_\kappa$. By construction $(1_X \times \alpha)^* \mathcal{E}$ is isomorphic to $\mathcal{E} \otimes \alpha$, hence

$$\alpha^* \Theta_\kappa = \text{div } Rpr_{2*}((1_X \times \alpha)^* \mathcal{E} \otimes \kappa = \text{div } Rpr_{2*}(\mathcal{E} \otimes \alpha \otimes \kappa) = \Theta_{\kappa\otimes\alpha},$$

which implies our claim. Since the map $\kappa \mapsto \mathcal{P}_\kappa$ is injective (Proposition 5.2), we conclude that $\mathcal{P}_\kappa$ does not descend.

The rest of the proof follows closely the symplectic case (Proposition 4.2). For $d = 0, 1$, the representation $\wedge^2$ defines a morphism of stacks $g_d : \mathcal{M}^d_{\mathbf{SO}_{2l}} \to \mathcal{M}_{\mathbf{SO}_N}$, which factors through $\mathcal{M}^d_{\mathbf{PSO}_{2l}}$. The pull back under g_d of a square root of the determinant bundle is $\mathcal{D}^{l-1}$; since $\mathcal{D}^{2l}$ descends, one concludes that $\mathcal{D}$ descends when l is even and $\mathcal{D}^2$ when l is odd.

To prove that $\mathcal{D}$ does not descend when l is odd, one considers the quadratic bundle $\mathcal{H}^\delta$ on $X \times J$ defined by

$$\begin{aligned}
\mathcal{H}^\delta &= \mathcal{L}^{\oplus l} \oplus (\mathcal{L}^{-1})^{\oplus l} && \text{if } \delta = 1\\
&= \mathcal{L}(p)^{\oplus l} \oplus \mathcal{L}^{-1}(-p)^{\oplus l} && \text{if } \delta = -1\\
&= (\mathcal{L} \oplus \mathcal{L}^{-1}(p))^{\oplus l} && \text{if } \delta = \pm\varepsilon,
\end{aligned}$$

with the standard hyperbolic quadratic form, and opposite orientations for the cases $\delta = \varepsilon$ and $\delta = -\varepsilon$.

As above, this gives rise to a commutative diagram

$$\begin{tikzcd}
J \arrow[r, "h"] \arrow[d, "2_J"'] & \mathcal{M}^{\delta}_{\mathbf{SO}_{2l}} \arrow[d] \\
J \arrow[r] & \mathcal{M}^{\delta}_{\mathbf{PSO}_{2l}},
\end{tikzcd}$$

from which one deduces that $\mathcal{D}$ does not descend, since the class of $h^*\mathcal{D}$ in $NS(J)$ is $2l$ times the principal polarization. □

Part II: The Picard group of the moduli space

6. $\mathbf{C}^*$-extension associated to group actions

This part is devoted to the Picard group of the moduli space M_G. The case of a simply connected group being known, we will consider M_G as a quotient of $M_{\widetilde{G}}$ by the finite group $H^1(X, \pi_1(G))$. Therefore we will first develop some general tools to study the Picard group of a quotient variety.

(6.1) Let H be a finite group acting on a normal projective variety M over $\mathbf{C}$ (or, more generally, any variety with $H^0(M, \mathcal{O}_M^*) = \mathbf{C}^*$), and L a line bundle on M such that $h^*L \cong L$ for all $h \in H$. We associate to this situation a central $\mathbf{C}^*$-extension

$$1 \to \mathbf{C}^* \to \mathcal{E}(H, L) \xrightarrow{p} H \to 1;$$

the group $\mathcal{E}(H, L)$ consists of pairs $(h, \tilde{h})$, where h runs over H and $\tilde{h}$ is an automorphism of L covering h, and p is the first projection.

(6.2) We will need a few elementary properties of these groups:

(a) Let $f: M' \to M$ be a H-equivariant rational map. Pulling back automorphisms induces an isomorphism $f^*: \mathcal{E}(H, L) \to \mathcal{E}(H, f^*L)$.

(b) Recall that the isomorphism classes of central $\mathbf{C}^*$-extensions of H form a commutative group, canonically isomorphic to $H^2(H, \mathbf{C}^*)$. The class in this group of $\mathcal{E}(H, L^r)$, for $r \in \mathbf{N}$, equals r times the class of $\mathcal{E}(H, L)$. More precisely, the map $\varphi_r: \mathcal{E}(H, L) \to \mathcal{E}(H, L^r)$ given by $\varphi_r(h, \tilde{h}) = (h, \tilde{h}^{\otimes r})$ is a surjective homomorphism, with kernel the group $\boldsymbol{\mu}_r$ of r-roots of unity, and therefore induces an isomorphism of the push-forward of $\mathcal{E}(H, L)$ by the endomorphism $t \mapsto t^r$ of $\mathbf{C}^*$ onto $\mathcal{E}(H, L^r)$.

(c) Let M' be another projective variety, H' a finite group acting on M', L' a line bundle on M' preserved by H'. The map $\mathcal{E}(H, L) \times \mathcal{E}(H', L') \to \mathcal{E}(H \times H', L \boxtimes L')$ given by $((h, \tilde{h}), (h', \tilde{h}')) \mapsto (h \times h', \tilde{h} \boxtimes \tilde{h}')$ is a surjective homomorphism, with kernel $\mathbf{C}^*$ embedded by $t \mapsto (t, t^{-1})$.

(d) Let K be a normal subgroup of H. The group H/K acts on M/K; let L_0 be a line bundle on M/K preserved by this action, and L the pull back of L_0 to M. Then the extension $\mathcal{E}(H, L)$ is the pull back of $\mathcal{E}(H/K, L_0)$ by the projection $H \to H/K$.

(6.3) A H-*linearization* of L is a section of the extension $\mathcal{E}(H, L)$. Such a linearization allows us to define, for each point m of M, an action of the stabilizer H_m of m in H on the fibre L_m; this action is given by a character of H_m, denoted by χ_m.

Let $\pi\colon M\to M/H$ be the quotient map; if L_0 is a line bundle on the quotient M/H, the line bundle $L=\pi^*L_0$ has a canonical H-linearization. By construction it has the property that at each point m of M, the character χ_m of H_m is *trivial*. The converse is true ('Kempf's lemma'), and is actually quite easy to prove in our set-up. Since two H-linearizations differ by a character of H, we can restate this lemma as follows: assume that L admits a H-linearization; then *L descends to M/H if and only if there exists a character χ of H such that $\chi_m=\chi_{|H_m}$ for all $m\in M$.*

It follows from the above description that the kernel of the homomorphism $\pi^*\colon \mathrm{Pic}(M/H)\to \mathrm{Pic}(M)$ consists of the H-linearizations of $\mathcal{O}_M$ such that the associated characters χ_m are trivial, i.e. of the characters of H which are trivial on the stabilizers H_m. In particular, if the subgroups H_m generate H, π^* is injective.

(6.4) Let M' be another projective variety with an action of H, and L' a line bundle admitting a H-linearization. The H-linearizations of L and L' define a H-linearization of $L\boxtimes L'$; at each point (m,m'), the corresponding character of $H_{(m,m')}=H_m\cap H_{m'}$ is the product of the characters χ_m of H_m and $\chi'_{m'}$ of $H_{m'}$ associated to the linearizations of L and L'. As a consequence, assume that L and $L\boxtimes L'$ descend to M/H and $(M\times M')/H$ respectively, and that $H=\cup_m H_m$; *then L' descends to M'/H.*

(6.5) From (6.2b) we see that *the smallest positive integer n such that L^n admits a H-linearization is the order of $\mathcal{E}(H,L)$ in $H^2(H,\mathbf{C}^*)$*. We want to know which powers of L^n descend to M/H.

Let r be a multiple of n. The class e of $\mathcal{E}(H,L)$ in $H^2(H,\mathbf{C}^*)$ comes from an element of $H^2(H,\boldsymbol{\mu}_r)$, which means that there exists a cocycle $c\in Z^2(H,\boldsymbol{\mu}_r)$ representing e, or in other words a map $\sigma\colon H\to\mathcal{E}(H,L)$ such that $p\circ\sigma=\mathrm{Id}_H$ and $\sigma(hh')\equiv\sigma(h)\,\sigma(h')$ (mod. $\boldsymbol{\mu}_r$) – let us call such a map a *section* (mod. $\boldsymbol{\mu}_r$) *of* $\mathcal{E}(H,L)$. Composing σ with the homomorphism $\varphi_r\colon\mathcal{E}(H,L)\to\mathcal{E}(H,L^r)$ (6.2b) gives a section of the extension $\mathcal{E}(H,L^r)$, that is a H-linearization of L^r.

Let m be a point of M. Using this H-linearization we get a character χ_m of H_m (6.3), which can be computed as follows: for $h\in H_m$ the element $\sigma(h)$ acts on L_m, and we have $\chi_m(h)=(\sigma(h)_m)^r$. Assume moreover that $h^r=1$ for all $h\in H$; then the element $\sigma(h)^r$ of $\mathcal{E}(H,L)$ belongs to the center $\mathbf{C}^*$. Thus *L^r endowed with the H-linearization deduced from σ descends to M/H if and only if $\sigma(h)^r=1$ for all h in $\cup H_m$*. Using 6.3 we can conclude:

PROPOSITION 6.6. *Assume that the order of $\mathcal{E}(H,L)$ in $H^2(H,\mathbf{C}^*)$ and of every element of H divides r. Let $\sigma\colon H\to\mathcal{E}(H,L)$ be a section* (mod. $\boldsymbol{\mu}_r$). *Then L^r descends to M/H if and only if there exists a character χ of H such that $\sigma(h)^r=\chi(h)$ for all $h\in H$ fixing some point of M.*

In the applications we have in mind we will always have $\cup H_m = H$. In this case we get the following condition, which depends only on the extension $\mathcal{E}(H, L)$ and not on the variety M:

COROLLARY 6.7. *Assume that every element of H fixes some point in M. Then L^r descends to M/H if and only if the map $h \mapsto \sigma^r(h)$ from H to $\mathbf{C}^*$ is a homomorphism.*

(6.8) From now on we will assume that the finite group H is *abelian*. In that case there is a canonical isomorphism of $H^2(H, \mathbf{C}^*)$ onto the group $\mathrm{Alt}(H, \mathbf{C}^*)$ of bilinear alternate forms on H with values in $\mathbf{C}^*$ (see e.g. [Br], V.6, exer. 5) : to a central $\mathbf{C}^*$-extension $\widetilde{H} \xrightarrow{p} H$ corresponds the form e such that $e(p(x), p(y)) = xyx^{-1}y^{-1} \in \mathrm{Ker}\, p = \mathbf{C}^*$. Conversely, given $e \in \mathrm{Alt}(H, \mathbf{C}^*)$, one defines an extension of H in the following way: choose any bilinear form $\varphi\colon H \times H \to \mathbf{C}^*$ such that $e(\alpha, \beta) = \varphi(\alpha, \beta)\varphi(\beta, \alpha)^{-1}$; take $\widetilde{H} = H \times \mathbf{C}^*$, with the multiplication law given by

$$(\alpha, t)\,(\beta, u) = (\alpha + \beta, tu\,\varphi(\alpha, \beta)),$$

the homomorphism $p\colon \widetilde{H} \to H$ being given by the first projection.

(6.9) Let r be an integer such that $rH = 0$. Then the group $H^2(H, \mathbf{C}^*) \cong \mathrm{Alt}(\mathrm{H}, \mathbf{C}^*)$ is also annihilated by r. Let $e \in \mathrm{Alt}(H, \mathbf{C}^*)$; we can choose the bilinear form φ with values in $\boldsymbol{\mu}_r$. Consider the extension defined as above by φ. The map $\sigma\colon H \to \widetilde{H}$ defined by $\sigma(\alpha) = (\alpha, 1)$ is a section (mod. $\boldsymbol{\mu}_r$). An easy computation gives $\sigma(\alpha)^r = \varphi(\alpha, \alpha)^{\frac{1}{2}r(r-1)} \in \{1, -1\}$. One has $\sigma^{2r}(\alpha) = 1$, and $\sigma(\alpha)^r = 1$ for all α if r is odd. If r is even, the function $\varepsilon\colon \alpha \mapsto \sigma(\alpha)^r$ is ‘quadratic’ in the sense that $\varepsilon(\alpha + \beta) = \varepsilon(\alpha)\varepsilon(\beta)\, e(\alpha, \beta)^{r/2}$. In particular, we see that σ^r is a homomorphism if and only if the alternate form $e^{r/2}$ (with values in $\boldsymbol{\mu}_2$) is trivial. In summary:

PROPOSITION 6.10. *Assume H is commutative, annihilated by r; let e be the alternate form associated to $\mathcal{E}(H, L)$. The line bundle L^{2r} descends to M/H; moreover L^r descends, except if r is even and the form $e^{r/2}$ is not trivial. In this last case, if every element of H has some fixed point on M, L^r does not descend.*

Example 6.11. Let A be an abelian variety of dimension $g \geqslant 1$, and Θ a divisor on A defining a principal polarization. Let A_r be the kernel of the multiplication by r in A. The group $\mathcal{E}(A_r, \mathcal{O}(r\Theta))$ is the Heisenberg group which plays a fundamental role in Mumford’s theory of theta functions; the corresponding alternate form $e_r\colon A_r \times A_r \to \boldsymbol{\mu}_r$ is the Weil pairing. The group A_r acts on the linear system $|r\Theta|$, and the morphism $A \to |r\Theta|^*$ associated to this linear system is A_r-equivariant;

therefore by (6.2a), the extension $\mathcal{E}(A_r, \mathcal{O}_{|r\Theta|^*}(1))$ is isomorphic to $\mathcal{E}(A_r, \mathcal{O}(r\Theta))$. It follows easily that $\mathcal{E}(A_r, \mathcal{O}_{|r\Theta|}(1))$ corresponds to the nondegenerate form e_r^{-1}. Let s be a positive integer dividing r; an easy computation shows that

(6.12) *the restriction of* e_r *to* A_s *is equal to* $e_s^{r/s}$.

We conclude from the proposition that:

– the line bundle $\mathcal{O}(2s)$ descends to $|r\Theta|/A_s$;
– the line bundle $\mathcal{O}(s)$ descends to $|r\Theta|/A_s$ if s is odd or r/s is even, but does *not* descend if s is even and r/s odd.

7. The moduli space M_G

For the rest of this paper we assume that the genus of X is $\geqslant 2$.

(7.1) Recall [R1, R2] that a G-bundle P on X is *semi-stable* (resp. *stable*) if for every parabolic subgroup Π, every dominant character χ of Π and every Π-bundle P' whose associated G-bundle is isomorphic to P, the line bundle P'_χ has nonpositive (resp. negative) degree.

Let $\rho: G \to G'$ be a homomorphism of semi-simple groups, and P a G-bundle. If P is semi-stable the G'-bundle $P_\rho = P \times^G G'$ is semi-stable; the converse is true if ρ has finite kernel. In particular P is semi-stable if and only if its adjoint bundle $\mathrm{Ad}(P)$ is semi-stable.

We denote by M_G the coarse moduli space of semi-stable principal G-bundles on X (*loc. cit.*). It is a projective normal variety. Let $\mathcal{M}_G^{ss}$ be the open substack of $\mathcal{M}_G$ corresponding to semi-stable G-bundles; there is a canonical surjective morphism $f: \mathcal{M}_G^{ss} \to M_G$. For $\delta \in \pi_1(G)$, f maps the component $(\mathcal{M}_G^{ss})^\delta$ onto the connected component M_G^δ of M_G parameterizing G-bundles of degree δ.

The definition of (semi-) stability extends to any reductive group H: a H-bundle P is semi-stable (resp. stable) if and only if the (H/Z^{o})-bundle P/Z^{o} has the same property, where Z^{o} is the neutral component of the center of H. The construction of the moduli space M_H makes sense in this set-up; each component of M_H is normal and projective.

Let Z be the center of G; we choose an isomorphism $Z \xrightarrow{\sim} \prod \boldsymbol{\mu}_{r_j}$. Let $\delta \in \pi_1(G_{\mathrm{ad}})$. The construction of the 'twisted' moduli stack $\mathcal{M}_G^\delta$ (Section 2) obviously makes sense in the framework of coarse moduli spaces. We get a coarse moduli space M_G^δ, which parameterizes semi-stable $C_Z G$-bundles with determinant $\mathcal{O}_X(\mathbf{d}p)$, such that the associated G_{ad}-bundle has degree δ, with $\rho(\delta)\, e^{2\pi i \mathbf{d}/\mathbf{r}} = 1$ (2.2). The open substack $\mathcal{M}_G^{\delta,ss}$ of $\mathcal{M}_G^\delta$ parameterizing semi-stable bundles maps surjectively onto M_G^δ. If A is a central subgroup of G, there is a canonical morphism $\pi: M_G^\delta \to M_{G/A}^\delta$ which is a (ramified) Galois covering with Galois group $H^1(X, A)$. The next lemma will allow us to compare the Picard groups of these moduli spaces by applying the results of Section 6, in particular Proposition 6.10, to the action of $H^1(X, A)$ on M_G^δ.

LEMMA 7.2. *Let* $\delta \in \pi_1(G_{\mathrm{ad}})$.

(a) *The moduli space* M_G^δ *is unirational.*

(b) *Any finite order automorphism of* M_G^δ *has fixed points.*

Proof. (a) The proof in [K-N-R], Corollary 6.3, for the untwisted case extends in a straightforward way: by (2.4) we have a surjective morphism $\mathcal{Q}_{\widetilde{G}} \to \mathcal{M}_G^\delta$; so the open subset of $\mathcal{Q}_{\widetilde{G}}$ parameterizing semi-stable bundles maps surjectively onto M_G^δ. Since $\mathcal{Q}_{\widetilde{G}}$ is a direct limit of an increasing sequence of generalized Schubert varieties, which are rational, the lemma follows.

(b) This is actually true for any finite order automorphism g of a projective unirational variety M. One (rather sophisticated) proof goes as follows: there exists a desingularization $\widetilde{M}$ of M to which g lifts to an automorphism $\tilde{g}$ [H], necessarily of finite order. Since $H^i(\widetilde{M}, \mathcal{O}_{\widetilde{M}})$ is zero for $i > 0$, we deduce from the holomorphic Lefschetz formula that $\tilde{g}$ has fixed points, hence also g. □

Recall that the moduli space M_G is constructed as a *good quotient* [Se] of a smooth scheme R by a reductive group Γ [R1] – this implies in particular that the closed points of M_G correspond to the closed orbits of Γ in R. In order to compare the Picard groups of Pic(M_G) and Pic$(\mathcal{M}_G)$, we will need a more precise result:

LEMMA 7.3. *There exists a presentation of* $\mathcal{M}_G^{\delta,ss}$ *as a quotient of a smooth scheme* R *by a reductive group* Γ*, such that the moduli space* M_G^δ *is a good quotient of* R *by* Γ.

Proof. We will explain the proof in some detail for the untwisted case, then indicate how to adapt the argument to the general situation.

We fix a faithful representation $\rho: G \to \mathbf{SL}_r$ and an integer N such that for every semi-stable G-bundle P, the vector bundle $P_\rho(Np)$ is generated by its global sections and satisfies $H^1(X, P_\rho(Np)) = 0$. Let $M = r(N+1-g)$. For any complex scheme S, we denote by $\underline{R}_G(S)$ the set of isomorphism classes of pairs (P, α), where P is a G-bundle on $X \times S$ whose restriction to $X \times \{s\}$, for each closed point $s \in S$, is semi-stable, and $\alpha: \mathcal{O}_S^M \xrightarrow{\sim} pr_{2*}P_\rho(Np)$ an isomorphism. We define in this way a functor $\underline{R}_G$ from the category of $\mathbf{C}$-schemes to the category of sets; we claim that it is representable by a scheme R_G. If $G = \mathbf{SL}_r$, this follows from Grothendieck theory of the Hilbert scheme [G1]. In the general case, we observe that reductions to G of the structure group of a $\mathbf{SL}_r$-bundle P correspond canonically to global sections of the bundle P/G; it follows that $\underline{R}_G$ is isomorphic to the functor of global sections of $\mathcal{P}/G$, where $\mathcal{P}$ is the universal $\mathbf{SL}_r$-bundle on $X \times R_{\mathbf{SL}_r}$. Again by [G1], this functor is representable by a scheme R_G, which is affine over $R_{\mathbf{SL}_r}$.

Put $\Gamma = \mathbf{GL}_M$. The group Γ acts on $\underline{R}_G$, and therefore on R_G, by the rule $g \cdot (P, \alpha) = (P, \alpha g^{-1})$. This action lifts to the universal G-bundle $\mathcal{P}$ over $X \times R_G$ as follows: by construction the universal pair $(\mathcal{P}, \alpha)$ is isomorphic to $((\mathrm{Id} \times g)^*\mathcal{P}, \alpha \circ g)$, hence there is an isomorphism of G-bundles $\sigma_g: (\mathrm{Id} \times g)^*\mathcal{P} \to \mathcal{P}$ such that $\alpha \circ g^{-1} = pr_{2*}(\sigma_{g,\rho}) \circ \alpha$. Since ρ is faithful this isomorphism is uniquely determined by $pr_{2*}(\sigma_{g,\rho})$, hence depends only on g and defines the required lifting.

The Γ-equivariant morphism $\varphi_{\mathcal{P}}: R_G \to \mathcal{M}_G$ induces a morphism of stacks $\overline{\varphi}_{\mathcal{P}}: [R_G/\Gamma] \to \mathcal{M}_G^{ss}$ which is easily seen to be an isomorphism. We also have a Γ-equivariant morphism $\psi_{\mathcal{P}}: R_G \to M_G$; if there exists a good quotient $R_G//\Gamma$, the universal property of M_G implies that $\psi_{\mathcal{P}}$ must induce an isomorphism of this quotient onto M_G. The existence of such a good quotient is classical in the case $G = \mathbf{SL}_r$ (possibly after increasing N); for general G, since the canonical map $R_G \to R_{\mathbf{SL}_r}$ is Γ-equivariant and affine, the existence of a good quotient of $R_{\mathbf{SL}_r}$ by Γ implies the same property for R_G thanks to a lemma of Ramanathan ([R1], lemma 4.1).

Let us finally consider the twisted case. We choose an embedding of the center Z of G in a torus $T = \mathbf{G}_m^s$, and an embedding $\rho: G \to \prod_{i=1}^{s} \mathbf{GL}_{r_i}$ such that $\rho(Z)$ is central; we put $S = (\prod \mathbf{GL}_{r_i}) \times (T/Z)$. The map $(g,t) \mapsto (t^{-1}\rho(g), t \text{ mod. } Z)$ of $G \times T$ into S defines an embedding of $C_Z G$ into S, which maps the center of $C_Z G$ into the center of S, so that a $C_Z G$-bundle P on X is semi-stable if and only if the associated S-bundle is semi-stable. We then argue as before, replacing $\mathbf{SL}_r$ by S. □

PROPOSITION 7.4. *Assume that the group G is almost simple. The group* $\mathrm{Pic}(M_G^\delta)$ *is infinite cyclic, and the homomorphism* $\pi^*: \mathrm{Pic}(M_G^\delta) \to \mathrm{Pic}(M_{\tilde{G}}^\delta)$ *is injective.*

The second assertion follows from Lemma 7.2(b) and (6.3); it is therefore enough to prove the first one when G is simply connected. The proof then is the same as in the untwisted case ([L-S] or [K-N]): since the stack $\mathcal{M}_G^\delta$ is smooth, the restriction map $\mathrm{Pic}(\mathcal{M}_G^\delta) \to \mathrm{Pic}(\mathcal{M}_G^{\delta,ss})$ is surjective, hence by Proposition 1.5 the group $\mathrm{Pic}(\mathcal{M}_G^{\delta,ss})$ is cyclic; it remains to prove that the pull back homomorphism $\mathrm{Pic}(M_G^\delta) \to \mathrm{Pic}(\mathcal{M}_G^{\delta,ss})$ is injective.

We choose a presentation of $\mathcal{M}_G^{\delta,ss}$ as a quotient of a smooth scheme R by a reductive group Γ, such that the moduli space M_G^δ is a good quotient of R by Γ (Lemma 7.3); then line bundles on $\mathcal{M}_G^{\delta,ss}$ (resp. on M_G^δ) correspond to line bundles on R with a Γ-linearization (resp. a Γ-linearization σ such that $\sigma(\gamma)_r = 1$ for each $(\gamma, r) \in \Gamma \times R$ such that $\gamma r = r$), hence our assertion. □

In what follows we will identify the group $\mathrm{Pic}(M_G^\delta)$ with its image in $\mathrm{Pic}(M_{\tilde{G}}^\delta)$; our aim will be to find its generator.

8. The Picard groups of $M_{\mathbf{Spin}_r}$ and M_{G_2}

(8.1) In this section we complete the results of [L-S] in the simply connected case. The cases $G = \mathbf{SL}_r$ or $\mathbf{Sp}_{2l}$ are dealt with in *loc. cit.*. We now consider the case

$G = \mathbf{Spin}_r$; we denote by $\mathcal{D}$ the determinant bundle on $M_{\mathbf{Spin}_r}$ associated to the standard representation σ of $\mathbf{Spin}_r$ in $\mathbf{C}^r$.

PROPOSITION 8.2. *Let r be an integer $\geqslant 7$. The group* $\mathrm{Pic}(M_{\mathbf{Spin}_r})$ *is generated by $\mathcal{D}$.*

Proof. Choose a presentation of $\mathcal{M}^{ss}_{\mathbf{Spin}_r}$ as a quotient of a smooth scheme R by a reductive group Γ, such that $M_{\mathbf{Spin}_r}$ is a good quotient of R by Γ (Lemma 7.3). Let $\mathcal{S}$ be the universal $\mathbf{Spin}_r$-bundle on $X \times R$. We fix a theta-characteristic κ on X; this allows us to define the pfaffian line bundle $\mathcal{P}_\kappa$ on R, which is a square root of $\det Rpr_{2*}(\mathcal{S}_\sigma \otimes \kappa)$ [L-S]. The action of Γ on $\mathcal{S}$ defines a Γ-linearization of $\mathcal{P}_\kappa$.

By [L-S] we know that the group of Γ-linearized line bundles on R (isomorphic to $\mathrm{Pic}(\mathcal{M}^{ss}_{\mathbf{Spin}_r})$) is generated by $\mathcal{P}_\kappa$, so all we have to prove is that $\mathcal{P}_\kappa$ itself does not descend to $R//\Gamma$, i.e. to exhibit a closed point $s \in R$ whose stabilizer in Γ acts nontrivially on the fibre of $\mathcal{P}_\kappa$ at s. If s corresponds to a semi-stable $\mathbf{Spin}_r$-bundle P, its stabilizer is the group $\mathrm{Aut}(P)$; since the formation of pfaffians commutes with base change, its action on $(\mathcal{P}_\kappa)_s$ is the natural action of $\mathrm{Aut}(P)$ on $(\bigwedge^{\max} H^0(X, P_\sigma \otimes \kappa))^{-1}$ [L-S].

To construct P we follow [L-S], Proposition 9.5: we choose a stable $\mathbf{SO}_4$-bundle Q and a stable $\mathbf{SO}_{r-4}$-bundle Q' with $w_2(Q) = w_2(Q') = 1$. Let H be the subgroup $\mathbf{SO}_4 \times \mathbf{SO}_{r-4}$ of $\mathbf{SO}_r$, and $\widetilde{H}$ its inverse image in $\mathbf{Spin}_r$. The H-bundle $Q \times Q'$ has $w_2 = 0$ by construction, hence admits a $\widetilde{H}$-structure; we choose one, and take for P the associated $\mathbf{Spin}_r$-bundle. Let γ be a central element of $\widetilde{H}$ lifting the element $(-1, 1)$ of H. Then γ defines an automorphism of P, which acts on the associated vector bundle $P_\sigma = Q_\sigma \oplus Q'_\sigma$ as $(-\mathrm{Id}, \mathrm{Id})$ (we use the same letter σ to denote the standard representation of all orthogonal groups in sight). Therefore γ acts on $(\mathcal{P}_\kappa)_s$ by multiplication by $(-1)^{h^0(Q_\sigma \otimes \kappa)}$. But $h^0(Q_\sigma \otimes \kappa)$ is congruent to $w_2(Q)$ (mod. 2) [L-S, 7.10.1], hence our assertion. $\square$

Remark 8.3. For $r \leqslant 6$ the group $\mathbf{Spin}_r$ is of type A or C, so we already have a complete description of $\mathrm{Pic}(M_{\mathbf{Spin}_r})$. It is worth noticing that the above result does not hold for $r \leqslant 6$: using the exceptional isomorphisms one checks easily that $\mathrm{Pic}(M_{\mathbf{Spin}_r})$ is generated by a square root of $\mathcal{D}$ for $r = 5$ or 6 and a fourth root for $r = 3$ – while it is isomorphic to $\mathbf{Z}^2$ for $r = 4$.

We now consider the case when G is of type G_2. The group G is the automorphism group of the octonion algebra $\mathbf{O}$ over $\mathbf{C}$ ([Bo], Algèbre III, App.); in particular it has a natural orthogonal representation σ in the 7-dimensional space $\mathbf{O}/\mathbf{C}$. We denote by $\mathcal{D}$ the determinant bundle on M_G associated to this representation.

PROPOSITION 8.4. *The group* $\mathrm{Pic}(M_G)$ *is generated by $\mathcal{D}$.*

Proof. As before we choose a presentation of $\mathcal{M}^{ss}_G$ as a quotient of a smooth scheme R by a reductive group Γ, such that $M_G = R//\Gamma$; choosing a theta-characteristic κ on X allows to define a pfaffian line bundle $\mathcal{P}_\kappa$ on R, with a natural

Γ-linearization. By [L-S], Theorem 1.1, $\mathcal{P}_\kappa$ generates the group of Γ-linearized line bundles on R; we must again prove that it does not descend to $R//\Gamma$, i.e. exhibit a G-bundle P such that $\mathrm{Aut}(P)$ acts nontrivially on $\bigwedge^{\max} H^0(P_\sigma \otimes \kappa)$.

Let V be a 3-dimensional vector space over $\mathbf{F}_2$. The algebra $\mathbf{O}$ has a basis $(e_\alpha)_{\alpha\in V}$, with multiplication rule

$$e_\alpha\, e_\beta = \varepsilon(\alpha,\beta)\, e_{\alpha+\beta},$$

for a certain function $\varepsilon\colon V\times V \to \{\pm 1\}$. Suppose given a homomorphism $\alpha \mapsto L_\alpha$ of V into J. We view J as the moduli space for degree 0 line bundles with a trivialization at p; for each pair (α,β) in V we have a unique isomorphism $u_{\alpha\beta}\colon L_\alpha \otimes L_\beta \to L_{\alpha+\beta}$ compatible with these trivializations. We endow the $\mathcal{O}_X$-module $\mathcal{A} = \bigoplus_{\alpha\in V} L_\alpha$ with the algebra structure defined by the map $\mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ which coincides with $\varepsilon(\alpha,\beta)\, u_{\alpha\beta}$ on $L_\alpha \otimes L_\beta$. It is a sheaf of $\mathcal{O}_X$-algebras, locally isomorphic to $\mathcal{O}_X \otimes_{\mathbf{C}} \mathbf{O}$. Let P be the associated G-bundle (the sections of P over an open subset U of X are algebra isomorphisms of $\mathcal{O}_U \otimes_{\mathbf{C}} \mathbf{O}$ onto $\mathcal{A}_{|U}$). Since the pull back of P to any finite covering of X on which the L_α's are trivial is trivial, P is semi-stable. The vector bundle P_σ is simply $\bigoplus_{\alpha\neq 0} L_\alpha$. Let $\chi\colon V \to \{\pm 1\}$ be a nontrivial character; the diagonal endomorphism $(\chi(\alpha))_{\alpha\in V}$ of $\mathcal{A}$ is an algebra automorphism, and therefore defines an automorphism ι of P, which acts on P_σ with eigenvalues $(\chi(\alpha))_{\alpha\neq 0}$. Hence ι acts on $\bigwedge^{\max} H^0(P_\sigma \otimes \kappa)$ by multiplication by $(-1)^h$, with $h = \sum_{\chi(\alpha)=-1} h^0(L_\alpha \otimes \kappa)$. Since the function $\alpha \mapsto h^0(L_\alpha\otimes\kappa)$ (mod. 2) is quadratic, an easy computation gives that h is even if and only if the image of $\mathrm{Ker}\,\chi$ in J_2 is totally isotropic with respect to the Weil pairing. Clearly we can choose our map $V \to J_2$ and the character χ so that this does not hold; this provides the required example. $\square$

9. The Picard group of M_G^0

In the study of $\mathrm{Pic}(M_G^\delta)$, contrary to what we found for the moduli stacks, the degree δ plays an important role. We treat first the degree 0 case, which is easier. Let us start with the case A_l. We recall that the determinant bundle $\mathcal{D}$ exists on the moduli space $M_{\mathbf{SL}_r}$, and generates its Picard group.

PROPOSITION 9.1. *Let $G = \mathbf{SL}_r/\boldsymbol{\mu}_s$, with s dividing r.*

(a) *If s is odd or r/s is even,* $\mathrm{Pic}(M_G^0)$ *is generated by $\mathcal{D}^s$.*
(b) *If s is even and r/s is odd,* $\mathrm{Pic}(M_G^0)$ *is generated by $\mathcal{D}^{2s}$.*

In particular, $\mathrm{Pic}(M^0_{\mathbf{PGL}_r})$ *is generated by $\mathcal{D}^r$ if r is odd and by $\mathcal{D}^{2r}$ if r is even.*

Proof. We identify $M_{\mathbf{SL}_r}$ with the moduli space of semi-stable vector bundles of rank r and trivial determinant on X. Let J^{g-1} be the component of the Picard variety of X parameterizing line bundles of degree $g-1$, and $\Theta\subset J^{g-1}$ the canonical theta divisor. It is shown in [B-N-R] that for E general in $M_{\mathbf{SL}_r}$, the condition $H^0(X,E\otimes L)\neq 0$ defines a divisor $D(E)$ in J^{g-1} which belongs to the linear system $|r\Theta|$, and that the rational map $D\colon M_{\mathbf{SL}_r}\dashrightarrow|r\Theta|$ thus defined satisfies $D^*\mathcal{O}(1)=\mathcal{D}$. Using (6.2a) we deduce that the alternate form associated to $\mathcal{E}(J_r,\mathcal{D})$ is the inverse of the Weil pairing e_r; its restriction to J_s is $e_s^{-r/s}$ (6.12). From Proposition 6.10, we conclude that the line bundles $\mathcal{D}^s$ in case (a) and $\mathcal{D}^{2s}$ in case b) descend to M_G^0.

It remains to prove that these line bundles are indeed in each case generators of $\mathrm{Pic}(M_G^0)$. Consider first the case $s=r$. Since the extension $\mathcal{E}(J_r,\mathcal{D})$ is of order r in $H^2(J_r,\mathbf{C}^*)$, the smallest power of $\mathcal{D}$ which admits a J_r-linearization is $\mathcal{D}^r$, so the conclusion follows from Proposition 6.10. In the general case, put $M:=M_{\mathbf{SL}_r}$, and assume that some power $\mathcal{D}^k$ of $\mathcal{D}$ descends to M/J_s. Observe that M/J_r can be viewed as the quotient of M/J_s by $J_{r/s}$.

Assume that r/s is even. We know by Proposition 6.10 that $\mathcal{D}^{2kr/s}$ descends to M/J_r; since r is even, this implies by what we have seen that $2r$ divides $2kr/s$, hence that k is a multiple of s. If r/s is odd, then $\mathcal{D}^{kr/s}$ descends by Proposition 6.10, and therefore k is a multiple of s or $2s$ according to the parity of r. □

(9.2) We now consider the case of the orthogonal and symplectic group. If $G=\mathbf{SO}_r$ or $\mathbf{Sp}_r$ (r even), we will denote by $\mathcal{D}$ the determinant bundle on M_G, i.e. the pull back of the determinant bundle on $M_{\mathbf{SL}_r}$ by the morphism associated to the standard representation. We know that the group $\mathrm{Pic}(M_{\mathbf{Sp}_r})$ is generated by $\mathcal{D}$ ([L-S], 1.6), and that $\mathrm{Pic}(M_{\mathbf{Spin}_r})$ is generated by the pull back of $\mathcal{D}$ (Proposition 8.2); it follows that the Picard group of each component of $M_{\mathbf{SO}_r}$ is generated by $\mathcal{D}$. It remains to consider the groups $\mathbf{PSp}_{2l}$ and $\mathbf{PSO}_{2l}$.

PROPOSITION 9.3. *Let* $G=\mathbf{PSp}_{2l}$ *or* $\mathbf{PSO}_{2l}$ $\quad(l\geqslant 2)$.

(a) *If l is even,* $\mathrm{Pic}(M_G^0)$ *is generated by* $\mathcal{D}^2$.
(b) *If l is odd,* $\mathrm{Pic}(M_G^0)$ *is generated by* $\mathcal{D}^4$.

Proof. The extension $\mathcal{E}(J_2,\mathcal{D})$ is the pull back to J_2 of the Heisenberg extension of J_{2l}, and the corresponding alternate form is e_2^l (6.12). We deduce from Proposition 6.10 that $\mathcal{D}^2$ descends to M_G^0 if l is even, and that $\mathcal{D}^4$ descends but $\mathcal{D}^2$ does not if l is odd.

It remains to prove that $\mathcal{D}$ does not descend when l is even. Let us consider for instance the case of the symplectic group; for every integer n, we put $M_n=M_{\mathbf{Sp}_{2n}}$ and denote by $\mathcal{D}_n$ the determinant line bundle on M_n. Write $l=p+q$, where p and q are odd (e.g. $p=1$, $q=l-1$), and consider the morphism $u\colon M_p\times M_q\to M_l$ given by $u((E,\varphi),(E',\varphi'))=(E\oplus E',\varphi\oplus\varphi')$. It is J_2-equivariant and satisfies $u^*\mathcal{D}_l=\mathcal{D}_p\boxtimes\mathcal{D}_q$. The group $J_2\times J_2$ acts on $M_p\times M_q$; from (6.2c) one deduces that the alternate form e corresponding to the extension $\mathcal{E}(J_2\times J_2,\mathcal{D}_p\boxtimes\mathcal{D}_q)$ is given by $e((\alpha,\alpha'),(\beta,\beta'))=e_2(\alpha,\beta)\,e_2(\alpha',\beta')$. If $\mathcal{D}_l$ descends to M_l/J_2, then $\mathcal{D}_p\boxtimes\mathcal{D}_q$ descends to $(M_p\times M_q)/J_2$, and we can apply (6.2d) to the variety $M_p\times M_q$ and the diagonal embedding $J_2 \subset J_2 \times J_2$. We conclude that the form e is the pull back of an alternate form on J_2 by the sum map $J_2 \times J_2 \to J_2$. This is clearly impossible, which proves that $\mathcal{D}_l$ does not descend to M_l/J_2.

The same argument applies to the orthogonal groups, except that one has to be careful about the definition of M_1: we take it to be the Jacobian of X, by associating to a line bundle α on X the vector bundle $\alpha \oplus \alpha^{-1}$ with the standard isotropic form. Then $\mathcal{D}_1$ is the line bundle $\mathcal{O}(2\Theta)$. The alternate form associated to $\mathcal{E}(J_2, \mathcal{D}_1)$ is e_2, and the rest of the argument applies without any change. $\square$

Remark 9.4. There remains one case to deal with. When l is even, the center Z of $\mathbf{Spin}_{2l}$ is isomorphic to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$, so it contains two subgroups of order 2 (besides the kernel of the homomorphism $\mathbf{Spin}_{2l} \to \mathbf{SO}_{2l}$). These subgroups are exchanged by the outer automorphisms of $\mathbf{Spin}_{2l}$, so the corresponding quotient groups are canonically isomorphic; let us denote them by G. Since M_G^0 dominates $M^0_{\mathbf{PSO}_{2l}}$, it follows from Proposition 9.3 that $\mathcal{D}^2$ descends to M_G^0. If l is not divisible by 4, one can show that $\mathcal{D}$ does not descend to M_G^0, so $\operatorname{Pic}(M_G^0)$ *is generated by* $\mathcal{D}^2$. If $l = 4$, one sees using the triality automorphism that $\mathcal{D}$ descends; we do not know what happens for $l = 4m$, $m \geqslant 2$.

10. The Picard group of $M^d_{\mathbf{PGL}_r}$

In this section we consider the component $M^d_{\mathbf{PGL}_r}$ of the moduli space $M_{\mathbf{PGL}_r}$, for $0 < d < r$. It is the quotient by J_r of the moduli space $M^d_{\mathbf{SL}_r}$ of semi-stable vector bundles of rank r and determinant $\mathcal{O}_X(dp)$. We denote by δ the g.c.d. of r and d. If A is a vector bundle on X of rank r/δ and degree $(r(g-1)-d)/\delta$ which is general enough, the condition $H^0(X, E \otimes A) \neq 0$ defines a Cartier divisor on $M^d_{\mathbf{SL}_r}$; the associated line bundle $\mathcal{L}$ (sometimes called the *theta line bundle*) is independent of the choice of A, and generates $\operatorname{Pic}(M^d_{\mathbf{SL}_r})$ [D-N].

PROPOSITION 10.1. *The group* $\operatorname{Pic}(M^d_{\mathbf{PGL}_r})$ *is generated by* $\mathcal{L}^\delta$ *if* r *is odd and by* $\mathcal{L}^{2\delta}$ *if* r *is even.*

Choose a stable vector bundle A of rank r/δ and determinant $\mathcal{O}_X(-\frac{d}{\delta}p)$, and consider the morphism $a\colon E \mapsto E \otimes A$ of $M^d_{\mathbf{SL}_r}$ into $M^0_{\mathbf{SL}_{r^2/\delta}}$. By definition $\mathcal{L}$ is the pull back of the determinant bundle $\mathcal{D}$ on the target. The map a is J_r-equivariant, hence induces an isomorphism $\mathcal{E}(J_r, \mathcal{D}) \cong \mathcal{E}(J_r, \mathcal{L})$. We have seen in the proof of Proposition 9.1 that the alternate form associated to $\mathcal{E}(J_r, \mathcal{D})$ is $e_r^{-r/\delta}$; hence the smallest power of $\mathcal{L}$ which descends to $M^d_{\mathbf{PGL}_r}$ is $\mathcal{L}^\delta$. Therefore it is enough to prove that $\mathcal{L}^\delta$ descends to $M^d_{\mathbf{PGL}_r}$ when r is odd and that $\mathcal{L}^{2\delta}$ but not $\mathcal{L}^\delta$ descends when r is even.

We will prove this by reducing to the degree 0 case with the help of the Hecke correspondence. Let us denote simply by M the moduli space $M^1_{\mathbf{SL}_r}$ of stable vector bundles of rank r and determinant $\mathcal{O}_X(p)$ on X. There exists a Poincaré bundle $\mathcal{E}$ on $X \times M$; we denote by $\mathcal{E}_p$ its restriction to $\{p\} \times M$, viewed as a vector bundle on M. We fix an integer h with $0 < h < r$ and let $\mathcal{P} = \mathbf{G}_M(h, \mathcal{E}_p)$ the Grassmann bundle parameterizing rank h locally free quotients of $\mathcal{E}_p$. A point of $\mathcal{P}$ can be viewed as a pair (E, F) of vector bundles with $E \in M$, $E(-p) \subset F \subset E$ and $\dim(E_p/F_p) = h$.

LEMMA 10.2. *If E is general enough in M, for any pair (E, F) in $\mathcal{P}$ the vector bundle F is semi-stable, and stable if $g \geqslant 3$.*

Proof. We will actually prove a more precise result. If G is a vector bundle on X, define the *stability degree* $s(G)$ of G as the minimum of the rational numbers $\mu(G'') - \mu(G')$ over all exact sequences $0 \to G' \to G \to G'' \to 0$. One has $s(G) \geqslant 0$ if and only if G is semi-stable, $s(G) > 0$ if and only if G is stable, and $s(G) = g - 1$ when G is a general stable vector bundle [L, Hi].

Let E, F be two vector bundles on X, with $E(-p) \subset F \subset E$. The lemma will follow from the inequality

$$s(F) \geqslant s(E) - 1$$

(note that since E and F play a symmetric role, this implies $|s(E) - s(F)| \leqslant 1$). Let Q_p be the sheaf E/F (with support $\{p\}$), and h the dimension of its fibre at p. Let F' be a subbundle of F, of rank r'. From the exact sequence $0 \to F/F' \to E/F' \to Q_p \to 0$ we get

$$\mu(F/F') - \mu(F') = \mu(E/F') - \frac{h}{r - r'} - \mu(F') \geqslant s(E) - \frac{h}{r - r'}.$$

Let $K_p := \mathrm{Ker}(E_p \to Q_p)$. The exact sequence $0 \to E(-p) \to F \to K_p \to 0$ induces an exact sequence $0 \to E' \to F' \to K_p$, with $E' := F' \cap E(-p)$. Therefore

$$\mu(F/F') - \mu(F') \geqslant \mu(E(-p)/E') - \mu(E') - \frac{r - h}{r'} \geqslant s(E) - \frac{r - h}{r'}.$$

Since one of the two numbers $\dfrac{h}{r - r'}$ and $\dfrac{r - h}{r'}$ is $\leqslant 1$, we get the required inequality. $\square$

Let us denote by M' the moduli space $M^{1-h}_{\mathbf{SL}_r}$. Using the lemma we get a diagram

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, "q"'] \arrow[dr, dotted, "q'"] & \\ M & & M' \end{tikzcd}$$

('Hecke diagram'), where q (resp. q') associates to a pair (E, F) the vector bundle E (resp. F, provided F is semi-stable).

Let $\mathcal{L}$ and $\mathcal{L}'$ be the theta line bundles on M and M'. Let δ be the g.c.d. of r and $1-h$.

LEMMA 10.3. *One has* $K_{\mathcal{P}} = q^*\mathcal{L}^{-1} \otimes q'^*\mathcal{L}'^{-\delta}$.

Proof. Let E be a general vector bundle in M; let us compute the restriction of $q'^*\mathcal{L}'$ to the fibre $q^{-1}(E)$. On $X \times \mathcal{P}$ we have a canonical exact sequence

$$0 \to \mathcal{F} \to (1_X \times q)^*\mathcal{E} \to (i_p)_*\mathcal{Q}_p \to 0,$$

where $\mathcal{Q}_p$ is the universal quotient bundle of $q^*\mathcal{E}_p$ on $\mathcal{P}$ and i_p the embedding of $\mathcal{P} = \{p\} \times \mathcal{P}$ in $X \times \mathcal{P}$. For each point $P = (E, F)$ of $\mathcal{P}$ this exact sequence gives by restriction to $X \times \{P\}$ the exact sequence $0 \to F \to E \to Q_p \to 0$ defining P; in particular, one has $\mathcal{F}_{X \times \{P\}} = F$, and the map $q' \colon \mathcal{P} -- \to M'$ is the classifying map associated to $\mathcal{F}$. It follows that $q'^*\mathcal{L}'$ is the determinant bundle associated to $\mathcal{F} \otimes A$, where A is a vector bundle of rank r/δ and appropriate degree.

Now let $E \in M$; put $\mathbf{G} = q^{-1}(E) = \mathbf{G}(h, E_p)$, and denote by $\pi \colon X \times \mathbf{G} \to X$ and $\rho \colon X \times \mathbf{G} \to \mathbf{G}$ the two projections. The restriction of the above exact sequence to $X \times \mathbf{G}$ gives, after tensor product with π^*A, an exact sequence

$$0 \to \mathcal{F} \otimes \pi^*A \to \pi^*(\mathcal{E} \otimes A) \to (i_p)_*\mathcal{Q}_p^{r/\delta} \to 0;$$

applying $R\rho_*$ and taking determinants, we obtain

$$\det R\rho_*(\mathcal{F} \otimes \pi^*A) \cong (\det \mathcal{Q}_p)^{r/\delta} = \mathcal{O}_{\mathbf{G}}(r/\delta).$$

The restriction of $K_{\mathcal{P}}$ to $\mathbf{G}$ is $K_{\mathbf{G}} = \mathcal{O}_{\mathbf{G}}(-r)$; since $\mathrm{Pic}(\mathcal{P})$ is generated by $\mathcal{O}_{\mathcal{P}}(1)$ and $q^*\,\mathrm{Pic}(M)$, one can write $K_{\mathcal{P}} = q'^*\mathcal{L}'^{-\delta} \otimes q^*\mathcal{L}^a$ for some integer a. To compute a we consider the restriction of $q^*\mathcal{L}$ to a general fibre $q'^{-1}(F)$: by lemma (10.2) this fibre can be identified with the Grassmann variety $\mathbf{G}(r-h, F_p)$, and the same argument as above shows that the restriction of $q^*\mathcal{L}$ is equal to $\mathcal{O}_{\mathbf{G}}(r)$, that is to the restriction of $K_{\mathcal{P}}^{-1}$. This gives $a = -1$, hence the lemma. □

Observe that the group J_r acts in a natural way on $\mathcal{P}$, by the rule $\alpha \cdot (E, F) = (E \otimes \alpha, F \otimes \alpha)$; the Hecke diagram is J_r-equivariant.

LEMMA 10.4. *Let s be an integer dividing r. The canonical bundle $K_{\mathcal{P}}$ descends to $\mathcal{P}/J_s$, except if s is even and h and r/s are odd; in this last case $K_{\mathcal{P}}^2$ descends.*

Proof. (a) We first prove that K_M descends to M/J_r. Let π and ρ denote the projections from $X \times M$ onto X and M respectively. By deformation theory, the tangent bundle T_M is canonically isomorphic to $R^1\rho_*(\mathcal{E}nd_0(\mathcal{E}))$, where $\mathcal{E}nd_0$ denotes the sheaf of traceless endomorphisms; it follows that K_M is the inverse of the determinant bundle $\det R\rho_*(\mathcal{E}nd_0(\mathcal{E}))$. Since $\mathcal{E}nd_0(\mathcal{E})$ has trivial determinant, this is also equal to $\det R\rho_*(\mathcal{E}nd_0(\mathcal{E}) \otimes \pi^* L)$ for any line bundle L on X (see e.g. [B-L], 3.8); therefore K_M^{-1} is the pull back of the generator $\mathcal{L}$ of $\mathrm{Pic}(M_{\mathbf{SL}_{r^2-1}})$ by the morphism $M \to M_{\mathbf{SL}_{r^2-1}}$ which maps E to $\mathcal{E}nd_0(E)$. This morphism factors through the quotient M/J_r, hence our assertion.

(b) Therefore we need only to consider the relative canonical bundle $K_{\mathcal{P}/M}$, with its canonical J_r-linearization. Let $\alpha \in J_r$, and let $P = (E, F)$ be a fixed point of α in $\mathcal{P}$; we want to compute the tangent map $T_P(\alpha)$ to α at P. The vector bundle $E \in M$ is fixed by α, and the action of α on the fibre $q^{-1}(E) = \mathbf{G}(E_p)$ is induced by the automorphism $\tilde{\alpha}$ of E_p obtained from the isomorphism $\varphi_\alpha: E \to E \otimes \alpha$ (note that φ_α, hence also $\tilde{\alpha}$, are uniquely determined up to a scalar, since E is stable).

Let $0 \to K_p \to E_p \to Q_p \to 0$ be the exact sequence corresponding to P. The tangent space to $\mathbf{G}(E_p)$ at P is canonically isomorphic to $\mathrm{Hom}(K_p, Q_p)$, hence its determinant is canonically isomorphic to $(\det E_p)^{-h} \otimes (\det Q_p)^r$; we conclude that $\det T_P(\alpha)$ is equal to $(\det \tilde{\alpha})^h$, where $\tilde{\alpha}$ is normalized so that $\tilde{\alpha}^r = 1$.

(c) It remains to compute $\det \tilde{\alpha}$. Now the fixed points of α on M are easy to describe [N-R]: let s be the order of α, and $\pi: \widetilde{X} \to X$ the associated étale s-sheeted covering; a vector bundle E on X satisfies $E \otimes \alpha \cong E$ if and only if it is of the form $\pi_* \widetilde{E}$ for some vector bundle $\widetilde{E}$ on $\widetilde{X}$, of rank r/s. To evaluate φ_α at p, we can trivialize $\widetilde{E}$ in a neighborhood of $\pi^{-1}(p)$: write $\widetilde{E} = \pi^* T$, where $T = \mathcal{O}_X^{r/s}$. Then one has $\pi_* \widetilde{E} = \bigoplus_{i \in \mathbf{Z}/s} T \otimes \alpha^i$, and the isomorphism φ_α maps identically $T \otimes \alpha^i$ onto $(T \otimes \alpha^{i-1}) \otimes \alpha$. It follows that the eigenvalues of $\tilde{\alpha}$ are the s-th roots of 1, each counted with multiplicity r/s. This implies in particular $\det \tilde{\alpha} = \zeta^{r(s-1)/2}$, where ζ is a primitive s-th root of 1, and therefore $\det T_P(\alpha) = (-1)^{h(s-1)\frac{r}{s}}$. The lemma follows. $\square$

(10.5) *Proof of Proposition* (10.1). We first observe that a line bundle L on M descends to M/J_s if and only if its pull back to $\mathcal{P}$ descends to $\mathcal{P}/J_s$. In fact, we know by (6.2a) that G-linearizations of L correspond bijectively by pull back to G-linearizations of $q^* L$; for $\alpha \in J_s$, any fixed point E of α in M is the image of a point $P \in \mathcal{P}$ fixed by α, so with the notation of (6.3) one has $\chi_E(\alpha) = \chi_P(\alpha)$, which implies our assertion.

Similarly, a line bundle on M' descends to M'/J_s if and only if its pull back to $\mathcal{P}$ descends to $\mathcal{P}/J_s$: what we have to check in order to apply the same argument is that every component of the fixed locus $\mathrm{Fix}_{M'}(\alpha)$ is dominated by a component of $\mathrm{Fix}_{\mathcal{P}}(\alpha)$, and conversely that every component of $\mathrm{Fix}_{\mathcal{P}}(\alpha)$ dominates a component of $\mathrm{Fix}_{M'}(\alpha)$. But this follows easily from the description of the fixed points of α given above (10.4c).

We first consider the case $h = 1$. If r is odd, we know from Proposition 9.1 and Lemma 10.4 that $\mathcal{L}'^r$ and $K_{\mathcal{P}} = q^* \mathcal{L}^{-1} \otimes q'^* \mathcal{L}'^{-r}$ descend to $\mathcal{P}/J_r$; it follows that $\mathcal{L}$ descends to M/J_r. Assume that r is even. Endow $K_{\mathcal{P}}$ with its canonical J_r-linearization, $\mathcal{L}^r$ with the J_r-linearization defined in (6.10), and $q^* \mathcal{L}$ with the

J_r-linearization deduced from the isomorphism $K_{\mathcal{P}} \cong q^*\mathcal{L}^{-1} \otimes q'^*\mathcal{L}'^{-r}$. Let α be an element of order r in J_r, and P a fixed point of α in $\mathcal{P}$; we know that α acts on $(K_{\mathcal{P}})_P$ by multiplication by -1 (10.4c) and on $(q'^*\mathcal{L}'^r)_P$ by multiplication by $\varepsilon(\alpha)$ (6.10), hence it acts on $(q^*\mathcal{L})_P$ by multiplication by $-\varepsilon(\alpha)$. Since $-\varepsilon(\alpha+\beta) \neq (-\varepsilon(\alpha))\,(-\varepsilon(\beta)))$ when α and β are two elements of order r orthogonal for the Weil pairing, we conclude that $\mathcal{L}$ does not descend, while of course $\mathcal{L}^2$ descends.

We now apply the same argument with h arbitrary. If r is odd, $K_{\mathcal{P}}$ and $q^*\mathcal{L}$ descend, hence $\mathcal{L}'^{\delta}$ descends. If r is even, we get a J_r-linearization on $q'^*\mathcal{L}'^{\delta}$ such that an element α of order r in J_r acts by multiplication by $(-1)^{h+1}\varepsilon(\alpha)$; again this implies that $\mathcal{L}'^{\delta}$ does not descend, while $\mathcal{L}'^{2\delta}$ descends. □

Remark 10.6. The methods of this section allow to treat more generally in most cases the group $\mathbf{SL}_r/\boldsymbol{\mu}_s$, for s dividing r. We will contend ourselves with an example, which we will need below: the case $G = \mathbf{SL}_{2l}/\boldsymbol{\mu}_2$ ($l \geqslant 1$). The moduli space M_G has two components, namely M_G^0 (treated in Proposition 9.1) and the quotient M_G^l of $M_{\mathbf{SL}_{2l}}^l$ by J_2. The theta line bundle $\mathcal{L}$ on $M_{\mathbf{SL}_{2l}}^l$ is the pull back of the determinant bundle on $M_{\mathbf{SL}_{4l}}^0$ under the map $E \mapsto E \otimes A$, where A is a stable vector bundle of rank 2 and degree -1. It follows from Proposition 9.1 that $\mathcal{L}^2$ descends to M_G^l; on the other hand, by Proposition 10.1, $\mathcal{L}^l$ and therefore $\mathcal{L}$ do not descend if l is odd. We shall now prove that *$\mathcal{L}$ descends to M_G^l when l is even*.

Let $\lambda\colon M_{\mathbf{SL}_{2l}}^l \to M_{\mathbf{SL}_{l(2l-1)}}$ be the morphism $E \mapsto \mathsf{\Lambda}^2E(-p)$. One checks easily that the pull back of the determinant bundle $\mathcal{D}$ on $M_{\mathbf{SL}_{l(2l-1)}}$ is $\mathcal{L}^{l-1}$ (e.g. by pulling back to the moduli stack, and using the fact that the Dynkin index of the representation $\mathsf{\Lambda}^2$ is $2l-2$). Now λ factors through $M_{\mathbf{SL}_{2l}}^l/J_2$, therefore $\mathcal{L}^{l-1}$ descends to this quotient. When l is even, this implies that $\mathcal{L}$ itself descends.

11. The Picard groups of $M_{\mathbf{PSp}_{2l}}$ and $M_{\mathbf{PSO}_{2l}}$

(11.1) In the case C_l, it remains only to consider the component $M_{\mathbf{PSp}_{2l}}^1$, which is the quotient by J_2 of the moduli space $M_{\mathbf{PSp}_{2l}}^1$ of semi-stable pairs (E,φ), where E is a vector bundle of rank $2l$ on X and $\varphi\colon \mathsf{\Lambda}^2E \to \mathcal{O}_X(p)$ a non-degenerate alternate form. Let $\mathcal{L}$ denote the pull back of the theta line bundle by the natural map $M_{\mathbf{Sp}_{2l}}^1 \to M_{\mathbf{SL}_{2l}}^1$.

PROPOSITION 11.2. (a) *The group* $\operatorname{Pic}(M_{\mathbf{Sp}_{2l}}^1)$ *is generated by* $\mathcal{L}$.

(b) *The group* $\operatorname{Pic}(M_{\mathbf{PSp}_{2l}}^1)$ *is generated by $\mathcal{L}$ if l is even, and by $\mathcal{L}^2$ if l is odd.*

Proof. By Proposition 7.4 to prove (a) it suffices to prove that $\mathcal{L}$ is not divisible. Choose an element (A,ψ) of $M_{\mathbf{Sp}_{2l-2}}^1$, and consider the map $u\colon M_{\mathbf{SL}_2}^1 \to M_{\mathbf{Sp}_{2l}}^1$ given by $u(E) = (E,\det) \oplus (A,\psi)$. The pull back of $\mathcal{L}$ is the theta line bundle Θ on $M_{\mathbf{SL}_2}^1$, hence the assertion (a).

Let us prove (b). By Remark 10.6 we already know that $\mathcal{L}^2$ descends to $M^1_{\mathbf{PSp}_{2l}}$, and that $\mathcal{L}$ descends if l is even. Consider the morphism $\mu\colon M^1_{\mathbf{SL}_2} \to M^1_{\mathbf{Sp}_{2l}}$ given by $\mu(E) = (E, \det)^{\oplus l}$. One has $\mu^*\mathcal{L} = \Theta^l$, so if $\mathcal{L}$ descends Θ^l descends to $M^1_{\mathbf{PGL}_2}$; by Proposition 10.1 this implies that l is even. □

(11.3) Let us consider the group $G = \mathbf{PSO}_{2l}$. The moduli space M_G has 4 components, indexed by the center $\{1, -1, \varepsilon, -\varepsilon\}$ of $\mathbf{Spin}_{2l}$ (5.3).

The component $M^1_{\mathbf{PSO}_{2l}}$ has already been dealt with in Proposition 9.3. The component $M^{-1}_{\mathbf{PSO}_{2l}}$ is the quotient by the action of J_2 of the moduli space $M^{-1}_{\mathbf{SO}_{2l}}$ of semi-stable quadratic bundles with $w_2 = 1$. Let $\mathcal{D}$ denote the determinant bundle on this moduli space.

PROPOSITION 11.4. *The group* $\mathrm{Pic}(M^{-1}_{\mathbf{PSO}_{2l}})$ *is generated by* $\mathcal{D}^2$ *if l is even, by* $\mathcal{D}^4$ *if l is odd.*

Proof. The same proof as in 9.3 shows that $\mathcal{D}^2$ descends to $M^{-1}_{\mathbf{PSO}_{2l}}$ if l is even, and that $\mathcal{D}^4$ descends but $\mathcal{D}^2$ does not if l is odd. To prove that $\mathcal{D}$ does not descend when l is even $\geqslant 3$, we apply the argument of *loc. cit.* to the morphism $u\colon JX \times M^{-1}_{\mathbf{SO}_{2l-2}} \to M^{-1}_{\mathbf{SO}_{2l}}$ deduced from the natural embedding $\mathbf{SO}_2 \times \mathbf{SO}_{2l-2} \hookrightarrow \mathbf{SO}_{2l}$ (note that w_2 is additive and $w_2(\alpha \oplus \alpha^{-1}) = 0$ for $\alpha \in JX$).

When $l = 2$ we consider the morphism $v\colon M^1_{\mathbf{SL}_2} \times M^1_{\mathbf{SL}_2} \to M^{-1}_{\mathbf{SO}_4}$ which associates to a pair (E, F) the vector bundle $\mathcal{H}om(E, F)$ with the quadratic form defined by the determinant and the orientation deduced from the canonical isomorphism $\det(E^* \otimes F) \xrightarrow{\sim} (\det E)^{-2} \otimes (\det F)^2$. One has $v^*\mathcal{D} = \mathcal{L} \boxtimes \mathcal{L}$, where $\mathcal{L}$ is the theta line bundle on $M^1_{\mathbf{SL}_2}$. Since $\mathcal{L}$ does not descend to $M^1_{\mathbf{PGL}_2}$ (Proposition 10.1), it follows from the commutative diagram

$$\begin{array}{ccc} M^1_{\mathbf{SL}_2} \times M^1_{\mathbf{SL}_2} & \xrightarrow{v} & M^{-1}_{\mathbf{SO}_4} \\ \downarrow & & \downarrow \\ M^1_{\mathbf{PGL}_2} \times M^1_{\mathbf{PGL}_2} & \longrightarrow & M^{-1}_{\mathbf{PSO}_4} \end{array}$$

that $\mathcal{D}$ does not descend to $M^{-1}_{\mathbf{PSO}_4}$. □

We now consider the components $M^{\pm\varepsilon}_{\mathbf{PSO}_{2l}}$ corresponding to the elements $+\varepsilon$ and $-\varepsilon$ of the center of $\mathbf{Spin}_{2l}$. Each of these is the quotient by J_2 of the moduli space $M^{\pm\varepsilon}_{\mathbf{SO}_{2l}}$ of semi-stable quadratic bundles (E, q, ω), where E is a vector bundle of rank $2l$, $q\colon \mathsf{S}^2E \to \mathcal{O}_X(p)$ a quadratic form and $\omega\colon \det E \to \mathcal{O}_X(lp)$ an isomorphism compatible with q; changing the sign of ω exchanges M^{ε} and $M^{-\varepsilon}$ (5.3). We denote by $\mathcal{L}_l$ the pull back of the theta line bundle on $M_{\mathbf{SL}_{2l}^l}$ under the natural map $M^{\pm\varepsilon}_{\mathbf{SO}_{2l}} \to M_{\mathbf{SL}_{2l}^l}$.

PROPOSITION 11.5. *The group* $\mathrm{Pic}(M^{\pm\varepsilon}_{\mathbf{PSO}_{2l}})$ *is generated by* $\mathcal{L}_l$ *when* l *is even, and by* $\mathcal{L}_l^2$ *when* l *is odd.*

Proof. We already know that the theta line bundle descends to $M^l_{\mathbf{SL}_{2l}}/J_2$ when l is even and that its square descends when l is odd (10.6), so we have only to prove that $\mathcal{L}_l$ does not descend when l is odd.

Let us first consider the case $l = 3$. If E is a vector bundle of rank 4 and determinant $\mathcal{O}_X(p)$ on X, the bundle $\wedge^2 E$ carries a quadratic form with values in $\mathcal{O}_X(p)$ (defined by the exterior product) and an orientation. We thus get a morphism $\lambda\colon M^1_{\mathbf{SL}_4} \to M^{\pm\varepsilon}_{\mathbf{SO}_6}$ such that $\lambda(E \otimes \alpha) = \lambda(E) \otimes \alpha^2$ for $\alpha \in J_4$. An easy computation shows that $\lambda^*\mathcal{L}_3$ is the theta line bundle on $M^1_{\mathbf{SL}_4}$, which does not descend to $M^1_{\mathbf{PGL}_4}$ (10.1); our assertion follows.

For l odd ≥ 5, we consider the morphism $\mu\colon M^{\pm\varepsilon}_{\mathbf{SO}_{2l-6}} \times M^{\pm\varepsilon}_{\mathbf{SO}_6} \to M^{\pm\varepsilon}_{\mathbf{SO}_{2l}}$ deduced from the embedding $\mathbf{SO}_{2l-6} \times \mathbf{SO}_6 \hookrightarrow \mathbf{SO}_{2l}$. It is J_2-equivariant (with respect to the canonical action of J_2 on the spaces $M^{\pm\varepsilon}_{\mathbf{SO}_{2n}}$, and the diagonal action on the product), and the pull back $\mu^*\mathcal{L}_l$ is isomorphic to $\mathcal{L}_{l-3} \boxtimes \mathcal{L}_3$. Assume that $\mathcal{L}_l$ descends to $M^{\pm\varepsilon}_{\mathbf{PSO}_{2l}}$; since $\mathcal{L}_{l-3}$ descends, we deduce from 6.4 that $\mathcal{L}_3$ descends, contradicting what we just proved. $\square$

12. Determinantal line bundles

(12.1) We can express the above results in a more suggestive way. Assume G is of type A, B, C or D; let $\delta \in \pi_1(G)$. We identify $\mathrm{Pic}(M_G^\delta)$ to a subgroup of $\mathrm{Pic}(\mathcal{M}^{\delta,ss}_{\widetilde{G}})$. Let σ be the standard representation of $\widetilde{G}$ in $\mathbf{C}^r$ (for $\widetilde{G} = \mathbf{SL}_r$, $\mathbf{Spin}_r$ or $\mathbf{Sp}_r$), and $\mathcal{D}_\sigma$ the corresponding determinant bundle on $\mathcal{M}^{\delta,ss}_{\widetilde{G}}$. The results of sections 8 to 11 express the generator of $\mathrm{Pic}(M_G^\delta)$ as a certain power of $\mathcal{D}_\sigma$. Using the fact that the pull back to $\mathcal{M}^{d,ss}_{\mathbf{SL}_r}$ of the theta line bundle on $M^d_{\mathbf{SL}_r}$ is $(\mathcal{D}_\sigma)^{\frac{r}{(d,r)}}$, one finds:

PROPOSITION 12.2. *Assume that* G *is one of the groups* $\mathbf{PGL}_r$, $\mathbf{PSp}_{2l}$ *or* $\mathbf{PSO}_{2l}$. *Put* $\varepsilon_G = 1$ *if the rank of* G *is even, 2 if it is odd. Let* $\delta \in \pi_1(G)$. *The group* $\mathrm{Pic}(M_G^\delta)$ *is generated by* $(\mathcal{D}_\sigma)^{r\varepsilon_G}$ *for* $G = \mathbf{PGL}_r$, *and by* $(\mathcal{D}_\sigma)^{2\varepsilon_G}$ *for the other groups.*

(12.3) To produce line bundles on M_G^δ, we have already used the following recipe: to any representation $\rho\colon G \to \mathbf{SL}_N$ we associate the pull back $\mathcal{D}_\rho$ of the determinant bundle under the morphism $M_G^\delta \to M_{\mathbf{SL}_N}$ deduced from ρ. These line bundles generate a subgroup $\mathrm{Pic}_{\mathrm{det}}(M_G^\delta)$ of $\mathrm{Pic}(M_G^\delta)$. We suspect that this subgroup is actually equal to $\mathrm{Pic}(M_G^\delta)$, i.e. that all line bundles on M_G^δ can be constructed from representations of G. We have checked this in some cases:

PROPOSITION 12.4. *Assume G is of classical type or of type G_2, and either simply connected or adjoint or isomorphic to* $\mathbf{SO}_r$. *Then, for every $\delta \in \pi_1(G)$, one has* $\mathrm{Pic}_{\det}(M_G^\delta) = \mathrm{Pic}(M_G^\delta)$.

Proof. The simply connected case, and also the case $G = \mathbf{SO}_r$, follow from [L-S], Proposition 8.2 and 8.4.

The other groups are those which appear in the above Proposition; let us denote by e_G the positive integer such that $(\mathcal{D}_\sigma)^{e_G}$ generates $\mathrm{Pic}(M_G^\delta)$. If ρ is a representation of G, with Dynkin index d_ρ, the line bundle $\mathcal{D}_\rho$ on M_G^d is isomorphic to $(\mathcal{D}_\sigma)^{d_\rho/d_\sigma}$ (d_σ is 1 for the types A, C and 2 for B, D). It follows that e_G divides d_ρ/d_σ, and that our assertion is equivalent to saying that e_G is the g.c.d. of the numbers d_ρ/d_σ when ρ runs over the representations of G.

Let us consider the case $G = \mathbf{PGL}_r$. We have $d_{\mathrm{Ad}} = 2r$, which settles the case r even. If r is odd, consider the representation $\mathsf{S}^2 \otimes \mathsf{\Lambda}^{r-2}$ of $\mathbf{SL}_r$; since $\boldsymbol{\mu}_r$ acts trivially, it defines a representation ρ of $\mathbf{PGL}_r$, whose Dynkin index is

$$\begin{aligned} d_\rho &= d_{\mathsf{S}^2} \dim \mathsf{\Lambda}^{r-2} + d_{\mathsf{\Lambda}^2} \dim \mathsf{S}^2 \\ &= (r+2)\binom{r}{2} + (r-2)\binom{r+1}{2} = r^3 - 2r. \end{aligned}$$

Then $(d_{\mathrm{Ad}}, d_\rho) = r = e_G$, which proves the result in this case.

For $G = \mathbf{PSp}_{2l}$, easy computations give $d_{\mathrm{Ad}} = 2l + 2$ and $d_{\mathsf{\Lambda}^2} = 2l - 2$, hence $e_G = (d_{\mathrm{Ad}}, d_{\mathsf{\Lambda}^2})$. For $G = \mathbf{PSO}_{2l}$, one has $d_{\mathrm{Ad}} = 2(2l-2)$ and $d_{\mathsf{S}^2} = 2(2l+2)$, hence $e_G = (d_{\mathrm{Ad}}, d_{\mathsf{\Lambda}^2})/d_\sigma$. □

Remark 12.5. We can also prove the equality $\mathrm{Pic}_{\det}(M_G^0) = \mathrm{Pic}(M_G^0)$ for $G = \mathbf{SL}_r/\boldsymbol{\mu}_s$ when s and r/s are coprime. Reasoning as above and using Proposition 9.1, we need to prove that the g.c.d. of the d_ρ's is $2s$ if s is even, and s if it is odd. We consider the representation $\rho_p = \mathsf{S}^p \otimes \mathsf{\Lambda}^{s-p}$ for $1 \leqslant p \leqslant s$. Using some nontrivial combinatorics we can prove the relation $\sum_{p=1}^{s} p\, d_{\rho_p} = (-1)^s s^2$. Since $d_{\mathrm{Ad}} = 2r$ we find that the g.c.d. of the d_ρ's divides $(2r, s^2) = s(2\frac{r}{s}, s)$, hence our assertion. □

13. Local properties of the moduli spaces M_G

(13.1) A G-bundle P is called *regularly stable* if it is stable and its automorphism group is equal to the center $Z(G)$ of G. The open subset M_G^{reg} of M_G corresponding to regularly stable G-bundles is smooth, and its complement in M_G is of codimension $\geqslant 2$, except when X is of genus 2 and G maps onto $\mathbf{PGL}_2$: this is seen exactly as the analogous statement for Higgs bundles, which is proved in [F1], theorem II.6. In what follows we will assume that we are not in this exceptional case, leaving to the reader to check that our assertions extend by using the explicit description of $M_{\mathbf{SL}_2}$ in genus 2.

Let i be the natural injection of M_G^{reg} into M_G. Then the map i_* identifies $\mathrm{Pic}(M_G^{\mathrm{reg}})$ with the Weil divisor class group $\mathrm{Cl}(M_G)$, that is the group of isomorphism classes of rank 1 reflexive sheaves on M_G (see [Re], App. to Section 1); the restriction map $i^*\colon \mathrm{Pic}(M_G) \to \mathrm{Pic}(M_G^{\mathrm{reg}})$ corresponds to the inclusion $\mathrm{Pic}(M_G) \subset \mathrm{Cl}(M_G)$. Local factoriality of M_G is equivalent to the equality $\mathrm{Pic}(M_G) = \mathrm{Cl}(M_G)$.

We already know from [D-N] and [L-S] that M_G is locally factorial when G is $\mathbf{SL}_r$ or $\mathbf{Sp}_{2l}$. We want to show that these are essentially the only cases where this occurs.

PROPOSITION 13.2. *Let G be a simply connected group, containing a factor of type B_l $(l \geqslant 3)$, D_l $(l \geqslant 4)$, F_4 or G_2. Then M_G is not locally factorial.*

The same result holds if G contains a factor of type E_l [So]. This has the amusing consequence that *the semi-simple groups G for which M_G is locally factorial are exactly those which are* special *in the sense of Serre, i.e. such that all G-bundles are locally trivial for the Zariski topology* (see [G2]).

Proof of the Proposition. We can assume that G is almost simple. Choose a presentation of $\mathcal{M}_G$ as a quotient of a smooth scheme R by a reductive group Γ, such that M_G is a good quotient of R by Γ (Lemma 7.3). We denote by σ the standard representation in $\mathbf{C}^r$ in case $G = \mathbf{Spin}_r$, in $\mathbf{C}^7$ if G is of type G_2, and the orthogonal representation in $\mathbf{C}^{26}$ with highest weight ϖ_4 if G is of type F_4 (we use the standard notation of [Bo], Lie VII). Let $\mathcal{D}$ be the determinant bundle on R associated to σ. As in the proof of Proposition 8.2, the choice of a theta-characteristic κ on X allows us to define a square root $\mathcal{P}_\kappa$ of $\mathcal{D}$ on R, with a canonical Γ-linearization. We will show that $\mathcal{P}_\kappa$ descends to the open subset M_G^{reg}, but not to M_G, thus showing that the restriction map is not surjective.

The first assertion is clear if G is of type F_4 or G_2, because then $Z(G)$ is trivial, so Γ acts freely on the open subset of R corresponding to regularly stable G-bundles. Suppose $G = \mathbf{Spin}_r$; let Q be a G-bundle, and z an element of $Z(G)$. The image of z in $\mathbf{SO}_r$ is either 1 or possibly -1 if r is even; since $h^0(Q_\sigma \otimes \kappa) \equiv r h^0(\kappa)$ (mod. 2) by [L-S], 7.10.1, we conclude that z acts trivially on $\Lambda^{\max} H^0(Q_\sigma \otimes \kappa)$, i.e. on the fibre of $\mathcal{P}_\kappa$ at Q (8.2).

We already know that $\mathcal{P}_\kappa$ does not descend to M_G when $G = \mathbf{Spin}_r$ (Proposition 8.2) or G is of type G_2 (Proposition 8.4); it remains to show that the class of $\mathcal{D}$ is not divisible by 2 in $\mathrm{Pic}(M_G)$ when G is of type F_4. There is a natural inclusion $\mathbf{Spin}_8 \subset G$, which induces a morphism $f\colon M_{\mathbf{Spin}_8} \to M_G$. An easy computation gives that the Dynkin index of the restriction to $\mathbf{Spin}_8$ of the standard representation of G is 6. Since the Dynkin index of the standard representation of $\mathbf{Spin}_8$ is 2, it follows that $f^*\mathcal{D}$ is isomorphic to $\mathcal{D}_0^{\otimes 3}$, where $\mathcal{D}_0$ is the generator $\mathrm{Pic}(M_{\mathbf{Spin}_8})$; this implies that $\mathcal{D}$ is not divisible by 2 in $\mathrm{Pic}(M_G)$. □

We now treat the case of a non simply connected group. We start with two lemmas which are certainly well known, but for which we could find no reference:

LEMMA 13.3. *Let* $\pi\colon \widetilde{Y} \to Y$ *be a ramified Galois covering, with abelian Galois group* A*. If* π *is étale in codimension* 1*, the variety* Y *is not locally factorial.*

Proof. Let $\widehat{A} = \mathrm{Hom}(A, \mathbf{C}^*)$. Let Y^{o} be an open subset of Y such that $Y - Y^{\mathrm{o}}$ has codimension $\geqslant 2$ and the induced covering $\pi^{\mathrm{o}}\colon \widetilde{Y}^{\mathrm{o}} \to Y^{\mathrm{o}}$ is étale. This covering corresponds to a homomorphism $L\colon \widehat{A} \to \mathrm{Pic}(Y^{\mathrm{o}})$ such that $\pi_*\mathcal{O}_{\widetilde{Y}^{\mathrm{o}}} = \bigoplus_{\chi\in\widehat{A}} L(\chi)$. If Y is locally factorial, the restriction map $\mathrm{Pic}(Y) \to \mathrm{Pic}(Y^{\mathrm{o}})$ is bijective, so L extends to a homomorphism $\widehat{A} \to \mathrm{Pic}(Y)$ which defines an étale covering of Y extending π^{o}, and therefore equal to π. Then π is étale, contrary to our hypothesis. □

LEMMA 13.4. *Let* S *be a scheme,* H *an algebraic group,* A *a closed central subgroup of* H*,* P *a principal* H*-bundle on* S*. The cokernel of the natural homomorphism* $\mathrm{Aut}(P) \to \mathrm{Aut}(P/A)$ *is canonically isomorphic to the stabilizer of* P *in* $H^1(X, A)$ *(for the natural action of* $H^1(X, A)$ *on* $H^1(X, H)$ *).*

Proof. Denote by $\mathcal{A}ut\,(P)$ the automorphism bundle of the H-bundle P. We have an exact sequence of groups over S

$$1 \to A_S \to \mathcal{A}ut\,(P) \to \mathcal{A}ut\,(P/A) \to 1$$

(to check exactness one may replace P by the trivial H-bundle, for which this is clear). The associated cohomology exact sequence reads

$$1 \to A \to \mathrm{Aut}(P) \to \mathrm{Aut}(P/A) \to H^1(S, A) \xrightarrow{h} H^1(S, \mathcal{A}ut\,(P)).$$

The map h associates to an A-bundle α the class of the $\mathcal{A}ut\,(P)$-bundle $\alpha \times^A \mathcal{A}ut\,(P)$, which is canonically isomorphic to $\mathcal{I}som\,(P, \alpha\times^A P)$; the element $h(\alpha)$ is trivial if and only if this last bundle admits a global section, which means exactly that $\alpha \times^A P$ is isomorphic to P, hence the lemma. □

PROPOSITION 13.5. *Suppose* G *is not simply connected; let* $\delta \in \pi_1(G)$*. The moduli space* M_G^δ *is not locally factorial.*

Proof. We first prove that the Galois covering $\pi\colon M_{\widetilde{G}}^\delta \to M_G^\delta$ is étale above $(M_G^\delta)^{\mathrm{reg}}$. We put $A = \pi_1(G)$, and choose an isomorphism $A \xrightarrow{\sim} \prod_{j=1}^{s} \boldsymbol{\mu}_{r_j}$; we use freely the notation of (2.1). We denote by H the group $C_A\widetilde{G} = (\widetilde{G}\times T)/A$. Let $Q \in (M_G^\delta)^{\mathrm{reg}}$ and P a point of $M_{\widetilde{G}}^\delta$ above Q; we will use the same letters to denote the corresponding bundles. Using the isomorphism $H/A \cong G \times (T/A)$, the condition $\pi(P) = Q$ means that the (H/A)-bundle P/A is isomorphic to $Q \times \mathcal{O}_X(\mathbf{d}p)$. Since $\mathrm{Aut}(Q)$ is reduced to the center of G, the map $\mathrm{Aut}(P) \to \mathrm{Aut}(P/A)$ is surjective; we deduce from Lemma 13.4 that the stabilizer of P in $H^1(X, A)$ is trivial, i.e. π is étale at P.

It follows that the abelian cover $\pi\colon M_{\widetilde{G}}^\delta \to M_G^\delta$ is étale in codimension one. Since it is ramified by Lemma 7.2, we conclude from Lemma 13.3 that M_G^δ is not locally factorial.

Finally we observe that, though the moduli space is not locally factorial in most cases, it is always Gorenstein (this is proved in [K-N], theorem 2.8, for a simply connected G):

PROPOSITION 13.6. *The moduli space M_G is Gorenstein.*

Proof. We choose again a presentation of $\mathcal{M}_G$ as a quotient of a smooth scheme R by a reductive group Γ, such that M_G is a good quotient of R by Γ (Lemma 7.3); we denote by $\mathcal{P}$ the universal bundle on $X \times R$, and by R^{reg} the open subset of R corresponding to regularly stable bundles. Since the center of G is killed by the adjoint representation, the vector bundle $\mathrm{Ad}(\mathcal{P})$ descends to a vector bundle on $X \times M_G^{\mathrm{reg}}$, that we will still denote $\mathrm{Ad}(\mathcal{P})$. Deformation theory provides an isomorphism $T_{M_G^{\mathrm{reg}}} \xrightarrow{\sim} R^1pr_{2*}(\mathrm{Ad}\,\mathcal{P})$; since $H^0(X, \mathrm{Ad}(P)) = 0$ for $P \in M_G^{\mathrm{reg}}$, the line bundle $\det T_{M_G^{\mathrm{reg}}}$ is isomorphic to $\det Rpr_{2*}(\mathrm{Ad}\,\mathcal{P})$, that is to the restriction to M_G^{reg} of the determinant bundle $\mathcal{D}_{\mathrm{Ad}}$ associated to the adjoint representation.

Since M_G is Cohen–Macaulay, it admits a dualizing sheaf ω, which is torsion-free and reflexive ([Re], App. of Section 1). The reflexive sheaves ω and $\mathcal{D}_{\mathrm{Ad}}^{-1}$, which are isomorphic above M_G^{reg}, are isomorphic (*loc. cit.*), hence ω is invertible. □

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