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ON THE CHOW RING OF A K3 SURFACE

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Abstract

We show that the Chow group of 0-cycles on a K3 surface contains a class of degree 1 with remarkable properties: any product of divisors is proportional to this class, and so is the second Chern class c_2.

1. Introduction

An important algebraic invariant of a projective manifold X is the Chow ring $CH(X)$ of algebraic cycles on X modulo rational equivalence. It is graded by the codimension of cycles; the ring structure comes from the intersection product. For a surface we have

$$CH(X) = \mathbf{Z} \oplus Pic(X) \oplus CH_0(X),$$

where the group $CH_0(X)$ parametrizes 0-cycles on X. While the structure of the Picard group $Pic(X)$ is well understood, this is not the case for $CH_0(X)$: if X admits a nonzero holomorphic 2-form, it is a huge group, which cannot be parametrized by an algebraic variety [M].

Among the simplest examples of such surfaces are the K3 surfaces, which carry a nowhere vanishing holomorphic 2-form. In this case $Pic(X)$ is a lattice, while $CH_0(X)$ is very large; the following result is therefore somewhat surprising:

Theorem 1. *Let X be a* K3 *surface.*

a) *All points of X which lie on some (possibly singular) rational curve have the same class c_X in $CH_0(X)$.*
b) *The image of the intersection product*

$$Pic(X) \otimes Pic(X) \to CH_0(X)$$

is contained in $\mathbf{Z}\, c_X$.
c) *The second Chern class $c_2(X) \in CH_0(X)$ is equal to $24\, c_X$.*

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The proof is elementary in the sense that it only appeals to simple geometric constructions, based on the existence of sufficiently many rational and elliptic curves on X. We prove a) and b) in section 2; the proof of c), which is more involved, is given in section 3. If we represent the class c_X by a point c of X, a key property of this class is the formula

$$(x,x)-(x,c)-(c,x)+(c,c)=0 \quad \text{in} \quad CH_0(X\times X),$$

valid for any $x\in X$. In section 4 we discuss the importance of this formula and its relation with property b) of the theorem. We prove that an analogous formula holds when X is replaced by a hyperelliptic curve, but that it cannot hold for a generic curve C of genus ≥ 3 – we show that this would imply that C is algebraically equivalent to $-C$ in its Jacobian, contradicting a result of Ceresa.

2. The image of the intersection product

We work over the complex numbers. By a *rational curve* on a surface we mean an irreducible (but possibly singular) curve of geometric genus zero. If V is an algebraic variety and $p\in \mathbf{N}$, we denote by $CH_p(V)$ the group of p-dimensional cycles on V modulo rational equivalence; we put $CH_p(V)_{\mathbf{Q}} = CH_p(V)\otimes \mathbf{Q}$.

2.1. Proof of a) and b). Let R be a rational curve on X; it is the image of a generically injective map $j:\mathbf{P}^1\to X$. Put $c_R=j_*(p)\in CH_0(X)$, where p is an arbitrary point of $\mathbf{P}^1$. For any divisor D on X, we have in $CH_0(X)$

$$R\cdot D = j_*j^*D = j_*(n\,p) = n\,c_R\ , \quad \text{with } n=\deg(R\cdot D).$$

Let S be another rational curve. If $\deg(R\cdot S)\neq 0$, the above equality applied to $R\cdot S$ gives $c_S=c_R$ (recall that $CH_0(X)$ is torsion free [R]). If $\deg(R\cdot S)=0$, choose an ample divisor H; by a theorem of Bogomolov and Mumford [M-M], H is linearly equivalent to a sum of rational curves (this is proved in [M-M] assuming that the class of H in $Pic(X)$ is primitive; but any ample class is a multiple of an ample primitive class). Since H is connected, we can find a chain $R_0,\ldots,R_k$ of distinct rational curves such that $R_0=R$, $R_k=S$ and $R_i\cap R_{i+1}\neq\emptyset$ for $i=0,\ldots,k-1$. We conclude from the preceding case that $c_R=c_{R_1}=\ldots=c_S$. Thus the class c_R does not depend on the choice of R: this is assertion a) of the Theorem. Let us denote it by c_X.

We have $R \cdot D = \deg(R \cdot D) c_X$ for any divisor D and any rational curve R on X. Since the group $Pic(X)$ is spanned by the classes of rational curves (again by the Bogomolov-Mumford theorem), assertion b) follows. □

Remark 2.2. The result (and the proof) hold more generally for any surface X such that:

a) the Picard group of X is spanned by the classes of rational curves, and
b) there exists an ample divisor on X which is a sum of rational curves.

This is the case when X admits a non-trivial elliptic fibration over $\mathbf{P}^1$ with a section, or for some particular surfaces like Fermat surfaces in $\mathbf{P}^3$ with degree prime to 6 [S].

Remark 2.3. Let A be an abelian surface. According to [Bl], the image of the product map $Pic(A) \otimes Pic(A) \to CH_0(A)$ has finite index, so the situation looks rather different from the K3 case. There is however an analogue to the theorem. Let $Pic^+(A)$ be the subspace of $Pic(A)_{\mathbf{Q}}$ fixed by the action of the involution $a \mapsto -a$. We have a direct sum decomposition

$$Pic(A)_{\mathbf{Q}} = Pic^+(A) \oplus Pic^{\,\mathrm{o}}(A)_{\mathbf{Q}},$$

so $Pic^+(A)$ is canonically isomorphic to the image of $Pic(A)_{\mathbf{Q}}$ in $\mathrm{H}^2(A, \mathbf{Q})$. Now we claim that *the image of the map* $\mu : Pic^+(A) \otimes Pic^+(A) \to CH_0(A)_{\mathbf{Q}}$ *is* $\mathbf{Q}\,[0]$, where $[0] \in CH_0(A)$ denotes the class of the origin $0 \in A$. This is a direct consequence of the decomposition of $CH(A)_{\mathbf{Q}}$ described in [B]: let k be an integer ≥ 2, and let $\mathbf{k}$ be the multiplication by k in A. We have $\mathbf{k}^* D = k^2 D$ for any element D of $Pic^+(A)$, thus $\mathbf{k}^* c = k^4 c$ for any element c in the image of μ; but the latter property characterizes the multiples of $[0]$. □

2.4. The cycle class c_X has some remarkable properties that we will investigate in the next section. Let us observe first that for any irreducible curve C on X, there is a rational curve $R \neq C$ which intersects C; thus we can represent c_X by the class of a point $c \in C$ (namely any point of $C \cap R$).

We will need a more subtle property of c_X. Let us first prove a lemma:

Lemma 2.5. *Let E be an elliptic curve, x, y two points of E. Then*

$$(x, x) - (x, y) - (y, x) + (y, y) = 0 \qquad \textit{in} \qquad CH_0(E \times E).$$

Since the divisors $[x] - [y]$ generate the group $Pic^{\mathrm{o}}(E)$, this is equivalent to the formula $pr_1^* D \cdot pr_2^* D = 0$ in $CH_0(E \times E)$ for every D in $Pic^{\,\mathrm{o}}(E)$.

Proof. Put $\xi = (x, x) - (x, y) - (y, x) + (y, y)$. Then 2ξ is the pull-back of a 0-cycle $\eta = (x, x) + (y, y) - 2(x, y)$ on the second symmetric product $\mathsf{S}^2 E$. The addition map $a : \mathsf{S}^2 E \to E$ is a $\mathbf{P}^1$-fibration; this implies that the push-down map $a_* : CH_0(E \times E) \to CH_0(E)$ is an isomorphism. Since $a_* \eta = 0$, we have

$\eta = 0$, hence $2\xi = 0$. On the other hand ξ has degree 0 and its image in the Albanese variety of $E \times E$ is zero, so $\xi = 0$ by Rojtman's result. □

Proposition 2.6. *Let Δ be the diagonal embedding of X into $X \times X$.*

a) *For every $\alpha \in CH_1(X)$, we have*

$$\Delta_* \alpha = pr_1^* \alpha \cdot pr_2^* c_X + pr_1^* c_X \cdot pr_2^* \alpha \quad in \quad CH_1(X \times X).$$

b) *For every $\xi \in CH_0(X)$, we have*

$$\Delta_* \xi = pr_1^* \xi \cdot pr_2^* c_X + pr_1^* c_X \cdot pr_2^* \xi - (\deg \xi)\, \Delta_* c_X \quad in \quad CH_0(X \times X).$$

Proof. a) Since both sides are additive in α, it is enough to check this relation when α is the class of a rational curve; in that case it follows from the fact that the diagonal of $\mathbf{P}^1 \times \mathbf{P}^1$ is linearly equivalent to $\mathbf{P}^1 \times \{0\} + \{0\} \times \mathbf{P}^1$.

b) Again both sides are additive in ξ, so we may assume that ξ is the class of a point $x \in X$. The Bogomolov-Mumford theorem tells us that x lies on the image of a curve E of genus ≤ 1; by 2.4 we can represent c_X by a point $c \in E$. We have $(x,x) - (x,c) - (c,x) + (c,c) = 0$ in $CH_0(E \times E)$ by Lemma 2.5 (the case when E is rational is trivial); by push-down this gives the same formula in $CH_0(X \times X)$. □

3. The formula $c_2(X) = 24\, c_X$

3.1. Let c be a point of X lying on some rational curve. We will denote by (x,x,x), (x,x,c), (x,c,c), etc. the classes in $CH_2(X \times X \times X)$ of the image of X by the maps $x \mapsto (x,x,x)$, $x \mapsto (x,x,c)$, $x \mapsto (x,c,c)$, etc. With this notation we have the following key result:

Proposition 3.2. *The cycle*

$$\mathfrak{x} = (x,x,x) - (c,x,x) - (x,c,x) - (x,x,c) + (x,c,c) + (c,x,c) + (c,c,x)$$

is zero in $CH_2(X \times X \times X)_{\mathbf{Q}}$.

Corollary 3.3. *Let Δ, i_c and j_c be the maps of X into $X \times X$ defined by $\Delta(x) = (x,x)$, $i_c(x) = (x,c)$ and $j_c(x) = (c,x)$. For every ξ in $CH_2(X \times X)$, we have an equality in $CH_0(X)$*

$$\Delta^* \xi = i_c^* \xi + j_c^* \xi + n\, c, \qquad with \qquad n = \deg(\Delta^* \xi - i_c^* \xi - j_c^* \xi).$$

From this formula we recover part b) of the theorem by taking $\xi = \mathrm{pr}_1^* \alpha \cdot \mathrm{pr}_2^* \beta$, with $\alpha, \beta \in Pic(X)$, and we get part c) by taking for ξ the class of the diagonal, so that $\Delta^* \xi = c_2(X)$.

3.4. Proof of the Corollary. We will denote by p_i, for $1 \leq i \leq 3$, the projection of $X \times X \times X$ onto the i-th factor, and by p_{ij}, for $1 \leq i < j \leq 3$, the projection $(x_1, x_2, x_3) \mapsto (x_i, x_j)$.

Let us compute $p_{3*}(\mathfrak{x} \cdot p_{12}^*\xi)$. Let $\delta : X \to X \times X \times X$ be the map $x \mapsto (x, x, x)$. We have $p_3 \circ \delta = \mathrm{Id}_X$ and $p_{12} \circ \delta = \Delta$, hence

$$p_{3*}((x,x,x) \cdot p_{12}^*\xi) = p_{3*}\delta_*(\delta^* p_{12}^*\xi) = \Delta^*\xi.$$

The same argument applied to the maps $x \mapsto (c, x, x)$, $x \mapsto (x, c, x), \ldots$ gives

$$p_{3*}((c,x,x) \cdot p_{12}^*\xi) = i_c^*\xi, \qquad p_{3*}((x,c,x) \cdot p_{12}^*\xi) = j_c^*\xi,$$
$$p_{3*}((x,x,c) \cdot p_{12}^*\xi) = \deg(\Delta^*\xi) \cdot c, \qquad p_{3*}((x,c,c) \cdot p_{12}^*\xi) = \deg(i_c^*\xi) \cdot c,$$
$$p_{3*}((c,x,c) \cdot p_{12}^*\xi) = \deg(j_c^*\xi) \cdot c, \qquad p_{3*}((c,c,x) \cdot p_{12}^*\xi) = 0,$$

hence our formula. □

Remark 3.5. One also recovers Proposition 2.6 b) by restricting the class $\mathfrak{x}$ to the slices $X \times X \times \{x\} \subset X \times X \times X$ corresponding to all $x \in X$.

For the proof of the proposition we will need two results on products of elliptic curves. Let F be an elliptic curve over an arbitrary field. We denote by $Pic(F^3)^{\mathrm{inv}}$ the subgroup of elements of $Pic(F^3)_{\mathbf{Q}}$ which are invariant under permutations of the factors and under the involution (-1_{F^3}). We keep the notation of 3.1.

Lemma 3.6. a) *The cycle class*

$$\mathfrak{v} = (u,u,u) - (0,u,u) - (u,0,u) - (u,u,0) + (u,0,0) + (0,u,0) + (0,0,u)$$

in $CH_1(F^3)_{\mathbf{Q}}$ *is zero.*

b) *The divisors* $\alpha_F = \sum_i p_i^* 0$ *and* $\beta_F = \sum_{i<j} p_{ij}^*\Delta$ *form a basis of* $Pic(F^3)^{\mathrm{inv}}$.

Proof. a) The class $\mathfrak{v}$ is symmetric, hence comes from a cycle class $\bar{\mathfrak{v}}$ in the third symmetric product S^3F. This variety is a $\mathbf{P}^2$-bundle over F, through the addition map $a : \mathsf{S}^3F \to F$. Thus we have $CH_1(\mathsf{S}^3F) = a^*\, Pic(F) \cdot h \oplus \mathbf{Z}h^2$, where h is any divisor class on S^3F which induces on a fibre $a^{-1}(u) \cong \mathbf{P}^2$ the class of a line.

Write $\bar{\mathfrak{v}} = (a^*d) \cdot h + nh^2$. We have $n = \deg(\bar{\mathfrak{v}} \cdot a^*0) = 3^2 - 3 \cdot 2^2 + 3 \cdot 1 = 0$, hence $d = a_*(\bar{\mathfrak{v}} \cdot h)$. We can represent h by the image of the divisor p_1^*0 in $F \times F \times F$; since $\mathfrak{v} \cdot p_1^*0 = 0$, we get $d = 0$ and finally $\mathfrak{v} = 0$.

b) As above we have $Pic(\mathsf{S}^3F) = a^*\, Pic(F) \oplus \mathbf{Z}h$. Taking the invariants under (-1_{F^3}), we see that $Pic(F^3)^{\mathrm{inv}}$ has rank 2. Thus it suffices to prove that the divisors α_F and β_F are not proportional in $Pic(F^3)$; but their restriction to F^2 (embedded in F^3 by $(u,v)\mapsto(u,v,0)$) are $p_1^*0+p_2^*0$ and $\Delta+p_1^*0+p_2^*0$, which are clearly non-proportional. $\square$

3.7. Proof of Proposition 3.2. It will make our life easier to assume that $Pic(X)$ is generated by an ample divisor class H; the general case will follow by specialization (see [SGA6], X.7.14). By the Bogomolov-Mumford theorem, we can find in the linear system $|H|$ a one-dimensional family $(E'_b)_{b\in B}$ of (singular) elliptic curves; that is, we can find a surface E with a fibration $p:E\rightarrow B$ onto a smooth curve, with general fibre a smooth curve E_b of genus 1, and a generically finite map $\pi:E\rightarrow X$ which maps each fibre E_b of p birationally onto the singular curve E'_b. Passing to a covering of B if necessary, we may assume that:

a) p has a section $0:B\rightarrow E$, and

b) the curve $\pi(0_B)$ is *rational*.

(To see b), replace B by a component of $\pi^{-1}(R)$, where R is a rational curve on X not contained in any E'_b.)

Note that because of the assumption on $Pic(X)$ every fibre E_b is irreducible.

3.8. Using again the notation of 3.1, we consider on the fibre product $E^3_B=E\times_B E\times_B E$ the cycle class

$$\mathfrak{u}=(u,u,u)-(0_{pu},u,u)-(u,0_{pu},u)-(u,u,0_{pu})$$
$$+(u,0_{pu},0_{pu})+(0_{pu},u,0_{pu})+(0_{pu},0_{pu},u).$$

For $b\in B$, the class in $CH_2(X\times X\times X)$ of the cycle

$$\{c\}\times E'_b\times E'_b\ +\ E'_b\times\{c\}\times E'_b\ +\ \{c\}\times E'_b\times E'_b$$

does not depend on b, since the curves E'_b all belong to the same linear system $|H|$; let us denote it by $\mathfrak{z}$. Let $\pi^3:E^3_B\rightarrow X^3$ be the morphism deduced from π.

Lemma 3.9. *The class $\pi^3_*(\mathfrak{u})$ is proportional to $\mathfrak{z}$.*

Proof. By Lemma 3.6.a), the restriction of $\mathfrak{u}$ to the generic fibre of the fibration $E^3_B\rightarrow B$ is zero. It follows that $\mathfrak{u}$ is a sum of cycles of the form $i_{b*}D_b$, where i_b is the inclusion of E^3_b into E^3_B and D_b a (Weil) divisor on E^3_b [Bl-S].

The involution σ of E which coincides with $u\mapsto -u$ on each smooth fibre gives rise to an involution σ^3 of E^3_B which commutes with the action of $\mathfrak{S}_3$ by permutations of the factors. The cycle $\mathfrak{u}$ is invariant by this action of $\mathfrak{S}_3\times\mathbf{Z}/2$. By averaging on this group we may choose the above divisor classes D_b in the invariant subgroup of $CH_2(E^3_b)_{\mathbf{Q}}$. We want to prove that each cycle class $i_{b*}D_b$ is pushed down to a multiple of $\mathfrak{z}$ by π^3.

Assume first that the curve E_b is smooth. By Lemma 3.6.b) the class D_b is a $\mathbf{Q}$-linear combination of α_{E_b} and β_{E_b}. By 3.7.b), $\pi(0_b)$ is linearly equivalent to c; thus we have $\pi^3_*(i_{b*}\,\alpha_{E_b}) = \mathfrak{z}$. The cycle $\pi^3_*(i_{b*}\,\beta_{E_b})$ is the sum of $(\Delta_* E'_b)\times E'_b$ and the two cycles obtained by permutation of the factors. Now, using Lemma 2.4, this class is equivalent to $2\mathfrak{z}$; hence the result in this case.

If E_b is singular, its normalization $\widetilde{E}_b$ is a smooth rational curve, and we have a surjective homomorphism $CH_2(\widetilde{E}_b^3)_{\mathbf{Q}} \to CH_2(E_b^3)_{\mathbf{Q}}$. The $\mathfrak{S}_3$-invariant part of $CH_2(\widetilde{E}_b^3)_{\mathbf{Q}}$ is spanned by the divisor $\alpha_{\widetilde{E}_b} = \sum p_i^* 0$, which again maps to a cycle linearly equivalent to $\mathfrak{z}$ under π^3. □

Lemma 3.10. *Let $d = \deg \pi$, and let $\mathfrak{x}$ be the cycle class defined in* 3.2. *Then*

$$\pi^3_*(\mathfrak{u}) = d\,\mathfrak{x}.$$

Proof. We compute the images under π^3_* of the cycles which appear in the definition of $\mathfrak{x}$.

a) We have $\pi^3_*(u,u,u) = d\,(x,x,x)$ in $CH_2(X\times X\times X)$.

b) Let $\Gamma \subset X\times X$ be the image of the surface $(u, 0_{pu})$ (that is, the graph of $0\circ p$) in $E\times E$. We have $\pi^3_*(u,u,0_{pu}) = p_{12}^*\Delta\cdot p_{23}^*\Gamma$. The normalization $\widetilde{R}$ of $R = \pi(0_B)$ is a smooth rational curve (3.7.b). Since our cycle Γ is supported by $X\times R$, it comes from a divisor Γ_0 in $X\times\widetilde{R}$. Such a divisor is of the form $D\times\widetilde{R} + mX\times\{r\}$ for some divisor D on X, some point $r\in\widetilde{R}$ and some integer m; this integer is equal to the degree of Γ_0 over X, that is, d. Therefore Γ is linearly equivalent to $d\,(X\times c) + D\times R$; since we assume $Pic(X) = \mathbf{Z}$, we have $D\times R = a\,E'_b\times E'_b$ in $CH_2(X\times X)$ for some integer a and any $b\in B$. Intersecting with $p_{12}^*\Delta$, we get

$$\pi^3_*(u,u,0_{pu}) = d\,(x,x,c) + a\,(\Delta_* E'_b)\times E'_b.$$

c) We have $\pi^3_*(u,0_{pu},0_{pu}) = p_{12}^*\Gamma\cdot p_{23}^*\Delta$; reasoning as in b), we find that

$$\pi^3_*(u,0_{pu},0_{pu}) = d\,(x,c,c) + a\,E'_b\times(\Delta_* E'_b).$$

d) The lemma follows by permuting and summing. □

3.11. Therefore $\mathfrak{x} = d^{-1}\,\pi^3_*(\mathfrak{u})$ is proportional to the effective cycle $\mathfrak{z}$ (Lemma 3.9). On the other hand, $\mathfrak{x}$ is homologically trivial: this follows from the Künneth formula and the fact that the cycles $p_{ij*}\mathfrak{x}$ are identically zero. Thus we obtain $\mathfrak{x} = 0$, which concludes the proof of the proposition. □

4. 0-cycles on a product

4.1. The cycle c_X has two remarkable properties, namely the intersection property b) of the theorem, and the diagonal property $(x,x)-(x,c)-(c,x)+(c,c) = 0$ in $CH_0(X\times X)$ for any $x\in X$. These two properties may seem unrelated. However, we can rephrase them in the following way, which shows that they are in some sense dual to each other: since the Picard group of X is isomorphic to its Néron-Severi group, the degree 1 zero-cycle c_X provides a splitting of $CH(X)$ as

$$CH(X) = CH(X)_{\mathrm{hom}} \oplus \mathrm{H},$$

where $CH(X)_{\mathrm{hom}}$ is the subgroup of 0-cycles homologous to 0, and H the image of $CH(X)$ into $\mathrm{H}^*(X, \mathbf{Z})$ via the cycle map. This splitting induces the splitting

$$\begin{aligned} &CH(X) \otimes CH(X) \\ &\quad = (CH(X)_{\mathrm{hom}} \otimes CH(X)_{\mathrm{hom}}) \oplus (CH(X)_{\mathrm{hom}} \otimes \mathrm{H}) \\ &\quad\oplus (\mathrm{H} \otimes CH(X)_{\mathrm{hom}}) \oplus (\mathrm{H} \otimes \mathrm{H}) \end{aligned}$$

of $CH(X) \otimes CH(X)$. It is immediate to see that it induces one on the image $CH(X \times X)_{\mathrm{dec}}$ of $CH(X) \otimes CH(X)$ in $CH(X \times X)$. We can see these decompositions as giving gradings on $CH(X)$ and $CH(X \times X)_{\mathrm{dec}}$. (Here it is natural from the point of view of the Bloch-Beilinson conjectures to assign the degree 0 to H and the degree 2 to $CH(X)_{\mathrm{hom}}$, since our surface is regular.) Then the intersection property b) says that if $\Delta : X \to X \times X$ is the diagonal embedding, then the homomorphism

$$\Delta^* : CH(X \times X)_{\mathrm{dec}} \to CH(X)$$

is compatible with the gradings, while the diagonal relations a) and b) of Proposition 2.6 say that for $p > 0$ the homomorphism

$$\Delta_* : CH_p(X) \to CH_p(X \times X)$$

takes values in $CH_p(X \times X)_{\mathrm{dec}}$ and is also compatible with the gradings.

We are now going to investigate the corresponding diagonal property for a curve.

Proposition 4.2. *Let C be a hyperelliptic curve, and w a Weierstrass point of C. For any $x \in C$, we have*

$$(x, x) - (x, w) - (w, x) + (w, w) = 0 \quad \textit{in} \quad CH_0(C \times C).$$

(Note that the class of w is well-defined in $CH_0(C)_{\mathbf{Q}}$.)

Proof. Let J be the Jacobian variety of C; choose an Abel-Jacobi embedding $C \hookrightarrow J$. The induced map $C \times C \to J \times J$ is an Albanese map for $C \times C$.

The subgroup of degree 0 cycles in $CH_0(C \times C)$ maps onto the Albanese variety $J \times J$; let $T(C \times C)$ be the kernel of this map. The surjective map

$$CH_0(C) \otimes CH_0(C) \longrightarrow CH_0(C \times C)$$

induces a surjective map

$$J \otimes J \longrightarrow T(C \times C).$$

Let ι be the hyperelliptic involution of C; since ι acts as (-1) on J, we see that *the involution* (ι, ι) *of* $C \times C$ *acts trivially on* $T(C \times C)$.

Let $\mathfrak{c} : J \to CH_0(C \times C)$ be the homomorphism defined by

$$\mathfrak{c}(\alpha) = \Delta_* \alpha - pr_1^* \alpha \cdot pr_2^* w - pr_1^* w \cdot pr_2^* \alpha.$$

The cycle $\mathfrak{c}(\alpha)$ is of degree zero, and its image in $J \times J$ is $(\alpha, \alpha) - (\alpha, 0) - (0, \alpha) = 0$; hence it is invariant under (ι, ι). On the other hand, we have

$$(\iota, \iota)^* \mathfrak{c}(\alpha) = \mathfrak{c}(\iota^* \alpha) = \mathfrak{c}(-\alpha) = -\mathfrak{c}(\alpha).$$

Therefore $2\mathfrak{c}(\alpha) = 0$, and actually $\mathfrak{c}(\alpha) = 0$ by Rojtman's result. Applying this to $\alpha = [x] - [w]$ gives the result. □

In contrast, we now have:

Proposition 4.3. *Let C be a general curve of genus ≥ 3. There exists no divisor c on C such that the 0-cycle*[1] $(x, x) - (x, c) - (c, x)$ *in* $CH_0(C \times C)$ *is independent of* $x \in C$.

Proof. As above, the hypothesis on c is equivalent to the relation

$$\Delta_* \alpha = pr_1^* \alpha \cdot pr_2^* c + pr_1^* c \cdot pr_2^* \alpha$$

for all α in J. Applying $pr_{1\,*}$, we observe that this formula implies $\deg c = 1$.

Put $c' = (x, c) + (c, x) - (x, x)$, and assume that this class in $CH_0(C \times C)$ is independent of x. With the notation of 3.1, we consider in $CH_1(C \times C \times C)_{\mathbf{Q}}$ the cycle

$$\mathfrak{z} = (x, x, x) - (c, x, x) - (x, c, x) - (x, x, c) + (c, c, x) + (c, x, c) + (x, c').$$

Our hypothesis ensures that the restriction of $\mathfrak{z}$ to the generic fibre of p_1 is zero. As in [Bl-S] we conclude that $\mathfrak{z}$ is a sum of 1-cycles of the form $i_{b*} D_b$, where $i_b : C \times C \to C \times C \times C$ is the embedding $(x, y) \mapsto (b, x, y)$ and D_b is a divisor on $C \times C$.

Let us now work in the group $A_1(C \times C \times C)_{\mathbf{Q}}$ of cycles modulo algebraic equivalence. In this group the class of $i_{b*} D$, for $D \in CH_1(C \times C)_{\mathbf{Q}}$, is independent of $b \in C$; thus we can write $\mathfrak{z} = i_{b*} D$ for some fixed $b \in C$ and some divisor D in $C \times C$. Since $p_{12} \circ i_b = \mathrm{Id}_{C \times C}$, we have $D = p_{12*} \mathfrak{z}$.

Now the cycle $\mathfrak{z}$ is homologically trivial: as in 3.11 it suffices to check this for the projections $p_{ij*} \mathfrak{z}$ on $C \times C$, and this is straightforward. Thus, the divisor D is homologically, and therefore algebraically, trivial in $C \times C$; we conclude that $\mathfrak{z}$ is zero in $A_1(C \times C \times C)_{\mathbf{Q}}$.

[1] Here (x, c) stands for the 0-cycle $pr_1^* x \cdot pr_2^* c$

Now let J be the Jacobian variety of C, and $\alpha : C \to J$ the Abel-Jacobi map which maps a point x of C to the divisor class $[x] - c$; we will identify C with its image under α. Let $\alpha^3 : C^3 \to J$ be the map deduced from α. We have

$$(\alpha^3)_*(\mathfrak{z}) = \mathbf{3}_*C \,-\, 3(\mathbf{2}_*C) \,+\, 3C = 0 \quad \text{in} \quad A_1(J)_{\mathbf{Q}},$$

where $\mathbf{k}$ denotes the multiplication in J by the integer k.

According to [B] we have a decomposition

$$A_1(J)_{\mathbf{Q}} = A_1(J)_0 \oplus \cdots \oplus A_1(J)_{g-1} \ ,$$

where $\mathbf{k}_*$ acts by multiplication by k^{2+s} on $A_1(J)_s$. Since $3^\ell - 3 \cdot 2^\ell + 3 > 0$ for $\ell \geq 3$, the above equality implies that the components of the 1-cycle C in $A_1(J)_i$ are zero for $i \geq 1$, that is, $[C] \in A_1(J)_0$. Taking $k = -1$ we see that C is algebraically equivalent to $-C$; this contradicts the result of Ceresa [C]. □

Remark 4.4. The cycle class $\mathfrak{z}$ is studied in [G-S].

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