BIRATIONAL INVOLUTIONS OF $\mathbf{P}^2$*

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Introduction. This paper is devoted to the classification of the elements of order 2 in the group $\operatorname{Bir}\mathbf{P}^2$ of birational automorphisms of $\mathbf{P}^2$, up to conjugacy. This is a classical problem, which seems to have been considered first by Bertini [Be]. Bertini's proof is generally considered as incomplete, as well as several other proofs which followed. We refer to the introduction of [C-E] for a more detailed story and for an acceptable proof. However the result itself, as stated by these authors, is not fully satisfactory: since they do not exclude singular fixed curves, their classification is somewhat redundant.

We propose in this paper a different approach, which provides a precise and complete classification. It is based on the simple observation that any birational involution of $\mathbf{P}^2$ is conjugate, via an appropriate birational isomorphism $S \dashrightarrow \mathbf{P}^2$, to a *biregular* involution σ of a rational surface S. We are thus reduced to the birational classification of the pairs (S, σ), a problem very similar to the birational classification of *real* surfaces. This classification has been achieved by classical geometers [C]; the case of surfaces with a finite group of automorphisms has been treated more recently along the same lines by Manin[1] [Ma].

These questions have been greatly simplified in the early 80's by the introduction of Mori theory. In our case a direct application of this theory shows that the minimal pairs (S, σ) fall into two categories, those which admit a σ-invariant base-point free pencil of rational curves, and those with $\operatorname{rk}\operatorname{Pic}(S)^\sigma = 1$. The first case leads to the so-called De Jonquières involutions; in the second case an easy lattice-theoretic argument shows that the only new possibilities are the celebrated Geiser and Bertini involutions. Any birational involution is therefore conjugate to one (and only one) of these three types.

1. Biregular involutions of rational surfaces.

We work over an algebraically closed field k of characteristic $\neq 2$. By a *surface* we mean a smooth, projective, connected surface over k.

We consider pairs (S, σ) where S is a rational surface and σ a non-trivial biregular involution of S. We will say that (S, σ) is *minimal* if any birational morphism $f : S \to S'$ such that there exists a biregular involution σ' of S' with $f \circ \sigma = \sigma' \circ f$ is an isomorphism.

Recall that an *exceptional curve* E on a surface S is a smooth rational curve with $E^2 = -1$.

LEMMA 1.1 *The pair (S, σ) is minimal if and only if each exceptional curve E on S satisfies $\sigma E \neq E$ and $E \cap \sigma E \neq \varnothing$.*

Proof. If S contains an exceptional curve E with $\sigma E = E$ (resp. $E \cap \sigma E = \varnothing$), consider the surface S' obtained by blowing down E (resp. $E \cup \sigma E$); then σ induces

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an involution σ' of S', so that (S, σ) is not minimal.

Conversely, suppose that (S, σ) is not minimal. There exists a pair (S', σ') and a birational morphism $f : S \to S'$ such that $f \circ \sigma = \sigma' \circ f$ and f contracts some exceptional curve E. Then f contracts the divisor $E + \sigma E$, which therefore has negative square; this implies $E \cdot \sigma E \leq 0$, that is $\sigma E = E$ or $E \cap \sigma E = \varnothing$. $\square$

The only piece of Mori theory we will need is concentrated in the following lemma; its proof follows closely that of [M], thm. 2.7.

LEMMA 1.2   *Let (S, σ) be a minimal pair, with* rk $\mathrm{Pic}(S)^\sigma > 1$. *Then S admits a base point free pencil stable under σ.*

Proof. Let us first recall the standard notations of Mori theory. We denote by $NE(S)$ the cone of effective divisor classes in $\mathrm{Pic}(S) \otimes \mathbf{R}$, by $\overline{NE}(S)$ its closure, and by $\overline{NE}(S)_{K \geq 0}$ the intersection of $\overline{NE}(S)$ with the half-space defined by the condition $K_S \cdot x \geq 0$. The cone theorem ([M], 1.5 and 2.1) implies

$$\overline{NE}(S) = \overline{NE}(S)_{K \geq 0} + \sum_{C \in \mathcal{E}} \mathbf{R}_+[C]$$

where $\mathcal{E}$ is a countable set and C is a smooth rational curve with $C^2 = -1, 0$ or 1; moreover if $C^2 = 1$ S is isomorphic to $\mathbf{P}^2$, and if $C^2 = 0$ $|C|$ is a base point free pencil.

Now project the situation onto the σ-invariant subspace $\mathrm{Pic}(S)^\sigma \otimes \mathbf{R}$. We get an equality (see [M], 2.6)

$$\overline{NE}(S)^\sigma = \overline{NE}(S)^\sigma_{K \geq 0} + \sum_{C \in \mathcal{F}} \mathbf{R}_+[C + \sigma C] \ ,$$

where $\mathcal{F}$ is the subset of curves $C \in \mathcal{E}$ such that the ray $\mathbf{R}_+[C + \sigma C]$ is extremal in $\mathrm{Pic}(S)^\sigma \otimes \mathbf{R}$.

Assume rk $\mathrm{Pic}(S)^\sigma > 1$; let $R = \mathbf{R}_+[L]$ be an extremal ray in $\mathrm{Pic}(S)^\sigma \otimes \mathbf{R}$. We have $L^2 \leq 0$, because any element of $\overline{NE}(S)^\sigma$ with positive square belongs to the interior of $NE(S)^\sigma$ ([M], Lemma 2.5). This leaves the following possibilities:

α) $R = \mathbf{R}_+[F]$, where $|F|$ is a base point free pencil preserved by σ;

β) $R = \mathbf{R}_+[E + \sigma E]$, where E is an exceptional curve and $E \cdot \sigma E = 1$;

γ) $R = \mathbf{R}_+[E + \sigma E]$, where E is an exceptional curve and $E = \sigma E$ or $E \cup \sigma E = \varnothing$.

If we assume moreover that the pair (S, σ) is minimal, case γ) does not occur. In case α) the conclusion is clear. In case β), put $F = E + \sigma E$. We have $F^2 = 0$ and $h^0(F) \geq 2$ by Riemann-Roch; since E and σE do not move linearly, this implies that $|F|$ is a base point free pencil as required. $\square$

(1.3) Before stating our structure theorem for minimal pairs, let us recall two classical examples. Let S be a Del Pezzo surface of degree 2. The linear system $|-K_S|$ defines a double covering $S \to \mathbf{P}^2$, branched along a smooth quartic curve (see [D]). The involution σ which exchanges the two sheets of this covering is called the *Geiser involution*; it satisfies $\mathrm{Pic}(S)^\sigma \otimes \mathbf{Q} \cong \mathrm{Pic}(\mathbf{P}^2) \otimes \mathbf{Q} = \mathbf{Q}$.

Similarly, let S be a Del Pezzo surface of degree 1. The map $S \to \mathbf{P}^3$ defined by the linear system $|-2K_S|$ induces a degree 2 morphism of S onto a quadric cone $Q \subset \mathbf{P}^3$, branched along the vertex v of Q and a smooth genus 4 curve [D]. The corresponding involution, the *Bertini involution*, satisfies again rk $\mathrm{Pic}(S)^\sigma = 1$.

THEOREM 1.4   *Let (S, σ) be a minimal pair. One of the following holds:*

(i) *There exists a smooth $\mathbf{P}^1$-fibration $f : S \to \mathbf{P}^1$ and a non-trivial involution τ of $\mathbf{P}^1$ such that $f \circ \sigma = \tau \circ f$.*

(ii) *There exists a fibration $f : S \to \mathbf{P}^1$ such that $f \circ \sigma = f$; the smooth fibres of f are rational curves, on which σ induces a non-trivial involution; any singular fibre is the union of two rational curves exchanged by σ, meeting at one point.*

(iii) *S is isomorphic to $\mathbf{P}^2$.*

(iv) *(S, σ) is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ with the involution $(x, y) \mapsto (y, x)$.*

(v) *S is a Del Pezzo surface of degree 2 and σ the Geiser involution.*

(vi) *S is a Del Pezzo surface of degree 1 and σ the Bertini involution.*

Proof. Assume first rk $\text{Pic}(S)^\sigma > 1$. By lemma 1.2 S admits a σ-invariant pencil $|F|$ of rational curves. This defines a fibration $f : S \to \mathbf{P}^1$ with fibre F, and an involution τ of $\mathbf{P}^1$ such that $f \circ \sigma = \tau \circ f$. If f is smooth this gives either (i) or a particular case of (ii).

Let F_0 be a singular fibre of f; it contains an exceptional divisor E. Since (S, σ) is minimal, we have $\sigma E \neq E$ and $E \cdot \sigma E \geq 1$. Thus $E + \sigma E \leq F_0$ and $(E + \sigma E)^2 \geq 0$, which implies $F_0 = E + \sigma E$ and $E \cdot \sigma E = 1$.

Let p be the intersection point of E and σE. The involution induced by σ on the tangent space to S at p exchanges the directions of E and σE, hence has eigenvalues $+1$ and -1. It follows that there is a fixed curve of σ passing through p; this curve must be horizontal, which forces the involution τ to be trivial. Moreover the involution induced by σ on a smooth fibre cannot be trivial, since the fixed curve of σ must be smooth. This gives all the properties stated in (ii).

Assume now rk $\text{Pic}(S)^\sigma = 1$. Since $\text{Pic}(S)^\sigma$ contains an ample class, it follows that $-K_S$ is ample, that is, S is a Del Pezzo surface. If rk $\text{Pic}(S) = 1$ we get case (iii). If rk $\text{Pic}(S) > 1$, σ acts as -1 on the orthogonal of K_S in $\text{Pic}(S)$, or in other words $-\sigma$ is the orthogonal reflection with respect to $K_S^\perp$. Such a reflection is of the form $x \mapsto x - 2\dfrac{(\alpha \cdot x)}{(\alpha \cdot \alpha)}\alpha$, with $(\alpha \cdot \alpha) \in \{1, 2\}$ and K_S proportional to α. If K_S is divisible, S is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, and we get case (iv) because σ must act non-trivially on $\text{Pic}(S)$. The remaining possibilities are $K_S^2 = 1$ or 2. In these cases we have seen that the Geiser and Bertini involutions have the required properties (1.3); they are the only ones, since an automorphism γ of S which acts trivially on $\text{Pic}(S)$ is the identity (consider S as $\mathbf{P}^2$ blown up at $9 - d$ points in general position: γ induces an automorphism of $\mathbf{P}^2$ which must fix these points). $\square$

COMPLEMENT 1.5   Let us consider case (ii) more closely. Let $F_1, \ldots, F_s$ be the singular fibres of f, and p_i ($1 \leq i \leq s$) the singular point of F_i. The fixed locus of σ is a smooth curve C, which passes through $p_1, \ldots, p_s$; the covering $C \to \mathbf{P}^1$ induced by f is of degree 2, ramified at $p_1, \ldots, p_s$. This leads us to distinguish the following cases:

(ii)$_{sm}$: if f is smooth, we have $s = 0$ and C is the union of two sections of f which do not intersect.

(ii)$_g$: if f is not smooth, C is a hyperelliptic curve of genus $g \geq 0$, and $s = 2g + 2$. $\square$

Theorem 1.4 is sufficient for our purpose, but it does not tell us which pairs in the list 1.4 are indeed minimal. Before answering that question we need to work out two more examples:

EXAMPLES 1.6   a) Let $\mathbf{F}_1$ be the surface obtained by blowing up a point $p \in \mathbf{P}^2$; projecting from p defines a $\mathbf{P}^1$-bundle $f : \mathbf{F}_1 \to \mathbf{P}^1$. Any biregular involution σ of

$\mathbf{F}_1$ preserves this fibration, hence defines a pair $(\mathbf{F}_1, \sigma)$ of type (i) or (ii)$_{sm}$. On the other hand σ preserves the unique exceptional curve E_1 of $\mathbf{F}_1$, so the pair $(\mathbf{F}_1, \sigma)$ is not minimal: σ induces a biregular involution of $\mathbf{P}^2$. In case (i) p lies on the fixed line of σ, in case (ii)$_{sm}$ it is the isolated fixed point.

b) Let Q be a smooth conic in $\mathbf{P}^2$, and p a point of $\mathbf{P}^2 - Q$. We define a birational involution of $\mathbf{P}^2$ by mapping a point x to its harmonic conjugate on the line $<p, x>$ with respect to the two points of $<p, x> \cap Q$. It is not defined at p and at the two points q, r where the tangent line to Q passes through p. Let S be the surface obtained by blowing up p, q, r in $\mathbf{P}^2$; the above involution extends to a biregular involution σ of S, the De Jonquières involution of degree 2. The projection from p defines a fibration $S \to \mathbf{P}^1$ of type (ii) above, with 2 singular fibres. The pair (S, σ) is not minimal: if E denotes the exceptional curve above p, σE is the proper transform of the line $<q, r>$, so it does not meet E. Blowing down E and σE we get a pair (T, τ) with rk Pic$(T) = 2$ and rk Pic$(T)^\tau = 1$; inspection of the list (1.4) shows that it is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ with the involution exchanging the two factors.

PROPOSITION 1.7 *The pairs* (S, σ) *in the list* (1.4) *are minimal, with the following exceptions:*
Case (i): $S \cong \mathbf{F}_1$
Case (ii): $S \cong \mathbf{F}_1$ *or* S *is* $\mathbf{P}^2$ *with* 3 *non-collinear points blown up and* σ *is a De Jonquières involution of degree* 2.

Proof. The pairs (S, σ) in (iii) to (vi) have rk Pic$(S)^\sigma = 1$ and therefore are minimal. The pairs (S, σ) of type (i) or (ii) have rk Pic$(S)^\sigma = 2$; thus we have to eliminate the pairs of these types which can be obtained by blowing up either a fixed point or two conjugate points in a pair (T, τ) of type (iii) to (vi). Let E be the corresponding exceptional divisor in S (which may be reducible), H the pull back to S of the positive generator of Pic$(T)^\tau$. The group Pic$(S)^\sigma$ is spanned by the classes of H and E, with $H \cdot E = 0$, $H^2 = 1$ or 2, $E^2 = -1$ or -2. The σ-invariant pencil F is linearly equivalent to $pH - qE$ for some integers p, q which are non-negative (because $|F|$ is base point free) and coprime (because F is not divisible). The condition $F^2 = 0$ implies $p = q = 1$, and $E^2 = -H^2$. Using $F \cdot K_S = -2$ the only possibilities are $S = \mathbf{P}^2$ with one fixed point blown up, or $S = \mathbf{P}^1 \times \mathbf{P}^1$ with the involution exchanging the factors and two conjugate points blown up. $\square$

2. Birational involutions of $\mathbf{P}^2$.
The following simple observation provides the link between biregular involutions of rational surfaces and birational involutions of the plane:

LEMMA 2.1 *Let* ι *be a birational involution of a surface* S_1. *There exists a birational morphism* $f : S \to S_1$ *and a biregular involution* σ *of* S *such that* $f \circ \sigma = \iota \circ f$.

Proof. There exists a birational morphism $f : S \to S_1$ such that the rational map $g = \iota \circ f$ is everywhere defined (elimination of indeterminacies, see for instance [B], II.7); moreover, f is a composition

$$f : S = S_n \xrightarrow{\varepsilon_{n-1}} S_{n-1} \longrightarrow \cdots \longrightarrow S_2 \xrightarrow{\varepsilon_1} S_1 ,$$

where $\varepsilon_i : S_{i+1} \to S_i$ $(1 \le i \le n-1)$ is obtained by blowing up a point $p_i \in S_i$. Since ι is not defined at p_1, so is $g^{-1} = f^{-1} \circ \iota$; by the universal property of blowing up [B, II.8], this implies that g factors as $S \xrightarrow{g_1} S_2 \xrightarrow{\varepsilon_1} S_1$. Proceeding by induction we see that g factors as $f \circ \sigma$, where σ is a birational morphism; since $f \circ \sigma^2 = f$, σ is an

involution. $\square$

(2.2) We now consider birational involutions $\iota : S \dashrightarrow S$, where S is a rational surface. We will say that two such involutions $\iota : S \dashrightarrow S$ and $\iota' : S' \dashrightarrow S'$ are birationally equivalent if there exists a birational map $\varphi : S \dashrightarrow S'$ such that $\varphi \circ \iota = \iota' \circ \varphi$. In particular, two birational involutions of $\mathbf{P}^2$ are birationally equivalent if and only if they are conjugate in the group $\mathrm{Bir}\,\mathbf{P}^2$.

Suppose that ι fixes a curve C; then $\iota' = \varphi \circ \iota \circ \varphi^{-1}$ fixes the proper transform of C under φ, which is a curve birational to C except possibly if C is rational – in which case it may be contracted to a point. Let us define the *normalized fixed curve* of ι to be the union of the normalizations of the non-rational curves fixed by ι; it follows from the above discussion that this is an invariant of the birational equivalence class of ι.

(2.3) Lemma 2.1 tells us that any birational involution is birationally equivalent to a biregular involution $\sigma : S \to S$; moreover we can assume that the pair (S, σ) is minimal. Therefore the classification of conjugacy classes of involutions in $\mathrm{Bir}\,\mathbf{P}^2$ is equivalent to the classification of minimal pairs (S, σ) up to birational equivalence. We first recall the classical examples of such involutions:

EXAMPLES 2.4 a) Let S be a Del Pezzo surface of degree 2 and σ the Geiser involution (1.3). We consider S as the blow up of $\mathbf{P}^2$ along a set $\mathcal{F}$ of 7 points in general position [D], and denote by $\varepsilon : S \to \mathbf{P}^2$ the blowing up map. The birational involution $\varepsilon \circ \sigma \circ \varepsilon^{-1}$ is the classical Geiser involution of $\mathbf{P}^2$. It associates to a general point $x \in \mathbf{P}^2$ the ninth intersection point of the pencil of cubics passing through $\mathcal{F}$ and x. The normalized fixed curve is a non-hyperelliptic curve of genus 3.

b) We define similarly the Bertini involution of $\mathbf{P}^2$ from the corresponding involution on a Del Pezzo surface of degree 1 (1.3), obtained by blowing up a set $\mathcal{G}$ of 8 points in general position in $\mathbf{P}^2$. It associates to a general point $x \in \mathbf{P}^2$ the fixed point of the net of sextics in $\mathbf{P}^2$ passing through $\mathcal{G}$ and x and singular along $\mathcal{G}$. Its normalized fixed curve is a non-hyperelliptic curve of genus 4, whose canonical model lies on a singular quadric.

c) Let $C \subset \mathbf{P}^2$ be a curve of degree $d \geq 2$, and p a point of $\mathbf{P}^2$; we assume that C has an ordinary multiple point of multiplicity $d - 2$ at p and no other singularity. We associate to (C, p) the unique birational involution which preserves the lines through p and fixes the curve C: it maps a general point $x \in \mathbf{P}^2$ to its harmonic conjugate on the line $<p, x>$ with respect to the two residual points of intersection of C with $<p, x>$. This is a De Jonquières involution of degree d, with center p and fixed curve C (the case $d = 2$ was already considered in (1.6 b)). Its normalized fixed curve is a hyperelliptic curve[1] of genus $d - 2$ for $d \geq 3$; it is empty for $d = 2$.

(2.5) Finally let us recall that the $\mathbf{P}^1$-bundles over $\mathbf{P}^1$ are of the form $\mathbf{F}_n := \mathbf{P}_{\mathbf{P}^1}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n))$ for some integer $n \geq 0$. We have $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$, and $\mathbf{F}_1$ is obtained by blowing up one point in $\mathbf{P}^2$. For $n \geq 1$ the fibration $f : \mathbf{F}_n \to \mathbf{P}^1$ has a unique section E_n with negative square, and we have $E_n^2 = -n$.

Let F be a fibre of f and p a point of F. The *elementary transformation* centered at p consists in blowing up p and blowing down the proper transform of F; the surface obtained in this way is isomorphic to $\mathbf{F}_{n-1}$ if $p \notin E_n$, to $\mathbf{F}_{n+1}$ if $p \in E_n$ or $n = 0$.

Suppose moreover that we have a birational involution ι of $\mathbf{F}_n$ which is regular in a neighborhood of F and fixes p. Then after performing the elementary transformation

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[1] We consider by convention an elliptic curve as hyperelliptic.

at p we still get a birational involution of $\mathbf{F}_{n\pm1}$ which is regular in a neighborhood of the new fibre.

THEOREM 2.6   *Every non-trivial birational involution of* $\mathbf{P}^2$ *is conjugate to one and only one of the following:*
   – A De Jonquières involution of a given degree $d \geq 2$;
   – A Geiser involution;
   – A Bertini involution.

Proof. The unicity assertion follows from (2.2). By (2.3) we must prove that the involutions of the list 1.4 are birationally equivalent to one of the above types. Cases (v) and (vi) give by definition the Geiser and Bertini involutions; we have seen in 1.6 b) that an involution of type (iv) is birationally equivalent to a De Jonquières involution of degree 2.

In case (iii), we choose a point $p \in \mathbf{P}^2$ with $\sigma p \neq p$ and blow up p and σp; then we blow down the proper transform of the line $<p, \sigma p>$, which is a σ-invariant exceptional curve. We obtain a pair (T, τ) with $T \cong \mathbf{P}^1 \times \mathbf{P}^1$ (by stereographic projection) and rk $\mathrm{Pic}(T)^\tau = 1$, hence of type (iv).

In case (i), the surface S is isomorphic to $\mathbf{F}_n$ for some $n \geq 0$; the involution σ has 2 invariant fibres, each of them containing at least 2 fixed points. One of these points does not lie on E_n, so performing successive elementary transformations we arrive at $n = 1$. As explained in 1.6 a) we conclude that σ is birationally equivalent to a biregular involution of $\mathbf{P}^2$ (case (iv)).

Let us treat case (ii)$_{sm}$ (1.5). Again by performing elementary transformations we can suppose that S is the surface $\mathbf{F}_1$. The fixed locus of σ is the union of E_1 and a section which do not meet E_1. Blowing down E_1 gives again case (iv).

It remains to treat case (ii)$_g$ for $g \geq 0$ (1.5). Blowing down one of the components in each singular fibre we get a surface $\mathbf{F}_n$ with a birational involution; the fixed curve C is embedded into $\mathbf{F}_n$. Performing successive elementary transformations at general points of C leads to the same situation on $\mathbf{F}_1$. The genus formula gives $E_1 \cdot C = g$.

Assume that C is tangent to E_1 at some point q of $\mathbf{F}_1$. Performing an elementary transformation at q, then at some general point of C, we lower by 1 the order of contact of C and E_1 at q. Proceeding in this way we arrive at a situation where E_1 and C meet transversally at g distinct points. We blow down E_1 to a point p of $\mathbf{P}^2$; the curve C maps to a plane curve $\overline{C}$ of degree $g + 2$, with an ordinary multiple point of multiplicity g at p and no other singularity. We get a birational involution of $\mathbf{P}^2$ which preserves the lines through p and admits $\overline{C}$ as fixed curve: this is the De Jonquières involution with center p and fixed curve $\overline{C}$. □

We can be more precise about the parameterization of each conjugacy class:

PROPOSITION 2.7   *The map which associates to a birational involution of* $\mathbf{P}^2$ *its normalized fixed curve (2.2) establishes a one-to-one correspondence between:*
   – conjugacy classes of De Jonquières involutions of degree d *and isomorphism classes of hyperelliptic curves of genus* $d - 2$ ($d \geq 3$);
   – conjugacy classes of Geiser involutions and isomorphism classes of non-hyperelliptic curves of genus 3;
   – conjugacy classes of Bertini involutions and isomorphism classes of non-hyperelliptic curves of genus 4 whose canonical model lies on a singular quadric.
   The De Jonquières involutions of degree 2 form one conjugacy class.

Proof. The result is clear for the Bertini involution: the canonical model of the genus 3 curve is a plane quartic; the double cover of the plane branched along that

quartic is a Del Pezzo surface of degree 2, which carries a canonical involution as explained in (1.3). Similarly the canonical model of a genus 4 curve lies on a unique quadric, so again we recover the Geiser involution by taking the double cover of this quadric branched along the curve and the singular point of the quadric.

Let $g \geq 1$. A De Jonquières involution of degree $g + 2$ is determined by a plane curve $\overline{C}$ of degree $g + 2$, with an ordinary multiple point p of multiplicity g and no other singularity[1]. The normalization C of $\overline{C}$ is a hyperelliptic curve of genus g, with g distinct points $p_1, \ldots, p_g$ mapped to p; the map $C \to \overline{C} \hookrightarrow \mathbf{P}^2$ is given by the linear system $|p_1 + \ldots + p_g + g_2^1|$, where g_2^1 denotes the degree 2 linear pencil on C and $p_1, \ldots, p_g$ the points which are mapped to p. Blowing up p we can view C as embedded in $\mathbf{F}_1$; we have $E_{1|C} = p_1 + \ldots + p_g$. This implies in particular $\sigma p_i \neq p_j$ for all pairs i, j, where σ stands for the hyperelliptic involution on C. Let p_{g+1} be any point of C such the points $p_1, \ldots p_{g+1}, \sigma p_1, \ldots, \sigma p_{g+1}$ are all distinct; performing an elementary transformation at p_1, then at σp_{g+1}, we get a birationally equivalent embedding $C \hookrightarrow \mathbf{F}_1$ such that $E_{1|C} = p_2 + \ldots + p_{g+1}$. Continuing in this way we see that all maps of C onto a plane curve of degree $g + 2$ with one ordinary g-uple point give rise to birationally equivalent involutions, so there is only one conjugacy class of De Jonquières involutions with normalized fixed curve C.

Finally any two degree 2 De Jonquières involutions are conjugate by a linear isomorphism. □

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[1] We leave to the reader the obvious modifications needed in the case $g = 1$.