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Prym Varieties and the Schottky Problem

Arnaud Beauville
Départment de Mathématiques, Université d'Angers, F-49000 Angers, France

0. Introduction

The Schottky problem is the problem of characterizing Jacobian varieties among all abelian varieties.

More precisely, let:

$$\mathscr{A}_g = H_g/Sp(\mathbf{Z}, 2g)$$

be the moduli space of principally polarized abelian varieties of dimension g, $J_g \subset \mathscr{A}_g$ the locus of Jacobians. The problem is to find explicit equations for J_g (or rather its closure $\bar{J}_g$) in $\mathscr{A}_g$.

In their beautiful paper [A-M], Andreotti and Mayer prove that $\bar{J}_g$ is an irreducible component of the locus N_{g-4} of principally polarized abelian varieties (A, Θ) with $\dim \operatorname{Sing} \Theta \geqq g-4$. Then they give a procedure to write "explicit" equations for N_{g-4}.

There is no hope that $\bar{J}_g$ be equal to N_{g-4}: already in genus 4, there is at least one other component, namely the divisor θ_{null} of principally polarized abelian varieties with one vanishing theta-null (i.e. such that $\operatorname{Sing} \Theta$ contains a point of order 2). Our aim is to prove the following:

Theorem. a) $N_0 = \bar{J}_4 \cup \theta_{\text{null}}$.

b) *The divisor* θ_{null} *is irreducible.*

In genus 5, the locus $N_1 \subset \mathscr{A}_5$ has already many components. However, we prove that $\bar{J}_5$ is the only component of N_1 not contained in θ_{null}.

The proofs use the fact that a generic principally polarized abelian variety of dimension 4 or 5 is a Prym variety. In [M 2], Mumford gives a complete list of all Prym varieties with $\dim \operatorname{Sing} \Theta \geqq g-4$. If every principally polarized abelian variety (of dimension 4 or 5) were a Prym variety, the results would follow at once; however we see immediately from Mumford's list that the product of an elliptic curve and a non-hyperelliptic Jacobian – for instance – is not a Prym variety in the classical sense.

Thus the main ingredient of the proofs is the construction of the generalized Prym varieties which appear in the closure of the locus of ordinary Prym varieties in $\mathscr{A}_g$. It turns out that the whole theory of Prym varieties, as developped in [M 1] and [M 2], extends to certain coverings of curves with ordinary double points. These generalized Prym varieties were known in the case of one ordinary double point (see [M 2] and (3.6) below); in general, they naturally appear as intermediate Jacobians of certain non-singular varieties, for instance the intersection of three quadrics in $\mathbf{P}^{2n}$ $(n \geqq 2)$.

After some preliminary results (§1 and 2), we define the generalized Prym varieties in §3. In §4, we give a list of all the generalized Prym varieties with dim Sing $\Theta \geqq g-4$. The method is that of [M 2], but there are some technical difficulties due to reducible curves. It should be noted that the proof of the result for Prym varieties of dimension 4 and 5 (the only one to be used in this paper) is considerably simpler; in particular, the hardest part of Lemma 4.9. (from (4.9.4) on) is not needed. However, we have insisted on giving a general proof because of the application to intermediate Jacobians: thus Theorem 4.10 implies for instance that every smooth intersection of three quadrics in $\mathbf{P}^6$ is non-rational.

In §5 and 6, we prove that the locus of generalized Prym varieties is closed in $\mathscr{A}_g$; this follows from the work of Deligne and Mumford on compactification of the moduli space of curves. In §7 and 8 we apply these results to principally polarized abelian varieties of dimension 4 and 5.

Most of our results are actually valid over an algebraically closed field of any characteristic different from two. In particular, we obtain as a consequence the irreducibility of the moduli space of principally polarized abelian varieties of dimension 4 or 5, over a field of characteristic $p \neq 2$[1].

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Terminology and Notation

Throughout this paper we fix an algebraically closed field k of characteristic $\neq 2$; all varieties considered are defined over k. By a point of a variety we mean a point rational over k.

A curve is a one-dimensional variety over k (that is, a one-dimensional reduced scheme of finite type over k). The genus $p_a(C)$ of a curve C is defined by:

$$p_a(C) = 1 - \chi(\mathcal{O}_C).$$

Let $C_1, \ldots, C_c$ be the irreducible components of C. For any vector bundle E on C, the multidegree $\underline{\deg}(E) = (r_1 \ldots r_c)$ is defined by $r_i = \deg E_{|C_i}$ and the degree of E by $\deg(E) = \sum r_i$. For any coherent sheaf F on C, we write:

$$\dim H^0(C, F) = h^0(C, F)$$

(or $h^0(F)$ if there is no ambiguity on C).

[1] I am grateful to F. Oort for pointing out to me that this fact was not known, and that it should be a consequence of the results contained in this paper

1. Theta-Characteristics on a Singular Curve

In this section we indicate how to modify the proofs in [M 1] to get the corresponding results for curves with arbitrary singularities.

(1.1) **Theorem.** *Let*

(i) $\pi: \mathscr{X} \to S$ *be a proper, flat family of curves.*

(ii) $\mathscr{E}$ *a coherent* $\mathcal{O}_{\mathscr{X}}$*-Module, flat over* S*, such that for all* $s \in S$ *the induced sheaf* $\mathscr{E}_s$ *is torsion-free of rank* r*.*

(iii) $Q: \mathscr{E} \to \omega_{\mathscr{X}/S}$ *a non-degenerate quadratic form.*

Then the function $s \mapsto \dim H^0(\mathscr{X}_s, \mathscr{E}_s) \bmod 2$ *is constant on connected components of* S*.*

Here $\omega_{\mathscr{X}/S}$ is the relative dualizing sheaf $f^! \mathcal{O}_S$ ([H]), that is, a sheaf whose restriction to each fibre $\mathscr{X}_s$ is the dualizing sheaf $\omega_{\mathscr{X}_s}$. By (iii), we mean that Q induces an isomorphism $\mathscr{E}_s \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathscr{X}_s}}(\mathscr{E}_s, \omega_{\mathscr{X}_s})$ for each s.

1) Define $\mathfrak{a} = \sum_{i=1}^{N} P_i$ where the P_i are non-singular points of $\mathscr{X}_s$. Since $E = \mathscr{E}_s$ is locally free outside the singular locus, one gets as in [M 1]: $\Gamma(E) = \Gamma(E(\mathfrak{a})) \cap \Gamma(E/E(-\mathfrak{a}))$ in $\Gamma(E(\mathfrak{a})/E(-\mathfrak{a}))$.

2) Use Grothendieck duality instead of Serre duality: since $\underline{\mathrm{Ext}}^1_{\mathcal{O}_{\mathscr{X}_s}}(E, \omega_{\mathscr{X}_s}) = 0$ by local duality, one still gets $\dim W_1 = \dim W_2 = Nr$, $\dim V = 2Nr$.

3) Replace ordinary residue by generalized residue ([A-K]). The function on $E(\mathfrak{a})/E(-\mathfrak{a})$ given by $q(\bar{a}) = \sum_{i=1}^{N} \mathrm{Res}_{P_i} Q(a_i)$ still defines a non-degenerate quadratic form on V.

The theorem follows as in [M 1].

Before stating the following corollary, we fix some notation: we denote by J_2 the set of line bundles L on C such that $L^2 \cong \mathcal{O}_C$; we define the pairing e_2 on J_2 by:

$$e_2(\alpha, \beta) = e_{2,N}(f^* \alpha, f^* \beta)$$

where $f: N \to C$ is the normalization of C, and $e_{2,N}: (JN)_2 \to \{\pm 1\}$ is the pairing induced on the group of points of order 2 by the Riemann form of JN ([M 3]); we could as well define e_2 directly as for abelian varieties, or by the cup product on $H^1(C, \mathbf{Z}/(2))$.

(1.2) **Corollary.** *Let* C *be a curve,* L_0 *a "theta-characteristic" on* C *(a torsion free, rank one* $\mathcal{O}_C$*-module such that* $L_0 \cong \underline{\mathrm{Hom}}(L_0, \omega_C)$*). Then the map from* J_2 *to* $\mathbf{Z}/(2)$ *defined by*

$$L \mapsto h^0(L_0 \otimes L) + h^0(L_0) \pmod 2$$

is a quadratic form on J_2 *whose associated bilinear form is* e_2*.*

Proof. Let L, M be two line bundles on C such that $L^2 \cong M^2 \cong \mathcal{O}_C$. We want to prove:

$$h^0(L_0) + h^0(L_0 \otimes L) + h^0(L_0 \otimes M) + h^0(L_0 \otimes L \otimes M) \equiv \ln e_2(L, M) \pmod 2$$

where $\ln(1)=0$, $\ln(-1)=1$. One defines a quaternion algebra structure on $A=\mathcal{O}_C+L+M+L\otimes M$ as in [M 1]. Then A is isomorphic to $\underline{\mathrm{Hom}}(E,E)$ for some rank 2 vector bundle E on C: and since $f^*A\cong\underline{\mathrm{Hom}}(f^*E, f^*E)$, Lemma 2 in [M 1] gives:

$$\deg E\equiv \ln e_2(L,M) \quad \bmod 2.$$

Now by Lemma 1.3 below the set of vector bundles of given rank and multidegree is connected, hence by the theorem:

$$h^0(L_0\otimes A)\equiv h^0(L_0\otimes \underline{\mathrm{Hom}}(E',E')) \quad \bmod 2$$

for any rank 2 vector bundle E' with $\underline{\deg}\, E'=\underline{\deg}\, E$. We pick a line bundle F on C with $\underline{\deg}\, F=\underline{\deg}\, E$, and take $E'=\mathcal{O}_C\oplus F$. Then:

$$\begin{aligned}
h^0(L_0\otimes \underline{\mathrm{Hom}}(E',E'))&=2h^0(L_0)+h^0(L_0\otimes F)+h^0(L_0\otimes F^{-1})\\
&\equiv h^0(L_0\otimes F)-h^0(L_0\otimes F^{-1}) \quad \bmod 2\\
&=\chi(L_0\otimes F) \quad \text{by Grothendieck duality}\\
&=\deg F \quad \text{by Riemann-Roch}\\
&=\ln e_2(L,M).
\end{aligned}$$

(1.3) **Lemma.** *Given integers $r\geq 1$; $d_1,\dots,d_c$, there exists an irreducible variety S and a vector bundle $\mathcal{E}$ on $C\times S$ of rank r and multidegree $(d_1,\dots,d_c)$ such that any vector bundle of rank r and multidegree $(d_1,\dots,d_c)$ on C is isomorphic to $\mathcal{E}_{|C\times\{s\}}$ for some $s\in S$.*

Proof. The lemma is well-known if C is non-singular ([S]). If C is singular, define the skyscraper sheaf δ by the exact sequence:

$$0\to\mathcal{O}_C\to f_*\mathcal{O}_N\to\delta\to 0.$$

Let E be a rank r vector bundle on C. Choosing an isomorphism $E\xrightarrow{\sim}\mathcal{O}_C^r$ near the singular points, we get:

$$0\to E\to f_*f^*E\to\delta^r\to 0.$$

Thus any vector bundle E on C can be given by a vector bundle $F=f^*E$ on N, plus a "descent data" morphism $h\colon f_*F\to\delta^r$ which must be surjective with locally free kernel.

Now let T be an irreducible variety parametrizing all vector bundles of rank r and multidegree $(d_1,\dots,d_c)$ on N, $\mathcal{E}$ the corresponding vector bundle on $N\times T$, p,q the projections from $C\times T$ onto C, T.

$$\begin{array}{ccccc}
N\times T & \xrightarrow{f_T} & & C\times T & \\
 & & \swarrow^{p} & & \searrow^{q} \\
 & & C & & T.
\end{array}$$

The sheaf $H = q_* \underline{\mathrm{Hom}}(f_{T_*}\mathscr{E}, p^* \delta^r)$ is locally free on T (since locally over T we may replace $\mathscr{E}$ by $\mathcal{O}^r_{C \times T}$; then H becomes isomorphic to $q_* p^* G$, with $G = \underline{\mathrm{Hom}}_C(f_* \mathcal{O}^r_N, \delta^r)$, but $q_* p^* G = \mathcal{O}_T \otimes_k H^0(C, G)$ is a free $\mathcal{O}_T$-module). We denote by S_1 the associated vector bundle ($S_1 = \mathbf{V}(\check{H})$ in EGA notation), by $k: C \times S_1 \to C \times T$ the projection; on $C \times S_1$ there is a canonical morphism:

$$h: k^* f_{T*} \mathscr{E} \to k^* p^* \delta^r$$

such that a point s in S_1 is given by a point $t \in T$, together with a morphism $h_{|C \times \{s\}}$: $f_*(\mathscr{E}_{|N \times \{t\}}) \to \delta^r$. We take the open set S in S_1 consisting of points $s \in S$ such that $h_{|C \times \{s\}}$ is surjective with locally free kernel. S is irreducible, and by what we have seen, S together with the vector bundle $\mathrm{Ker}\, h$ on $C \times S$ give a complete family of vector bundles on C.

2. Theta Divisor of a Generalized Jacobian

We consider a connected curve C of genus g, with only ordinary double points. We denote by JC the generalized Jacobian of C; recall that JC is a smooth commutative algebraic group, and the points of JC can be naturally identified with isomorphism classes of line bundles on C of multidegree $(0, \ldots, 0)$. The normalization $f: N \to C$ induces an epimorphism $f^*: JC \to JN$, whose kernel is a torus (i.e. a multiplicative group $(\mathbf{G}_m)^t$). For any line bundle L of degree $g-1$ on C, we define Θ_L as the locus of line bundles M in JC such that: $h^0(L \otimes M) \geqq 1$. We denote by Θ' the theta divisor on JN (defined up to translation).

Our aim is to prove that Θ_L is a divisor, algebraically equivalent to $(f^*)^{-1}(\Theta')$. However, if C is reducible, this will be true only for a good choice of $\underline{\deg}(L)$. To deal with this case, we associate to the curve C a graph Γ:

The set of vertices of Γ is the set $\{C_1, \ldots, C_c\}$ of irreducible components of C; an edge between two vertices C_i, C_j corresponds to a point of $C_i \cap C_j$.

(2.1) **Lemma.** *Let $\underline{d} = (d_1, \ldots, d_c)$ be a multidegree such that $\sum d_i = g-1$. The following conditions are equivalent:*

(i) *There exists a line bundle L on C, of multidegree $\underline{d}$, with $h^0(L) = 0$.*

(ii) *Given line bundles L_0 on C, and L_1 on N such that:* $\underline{\deg}(L_0) = \underline{d}$; $\underline{\deg}(L_1) = \frac{1}{2}\underline{\deg}(\omega_N)$, *there exists an irreducible non-singular variety S, a coherent sheaf $\mathscr{L}$ on $C \times S$, flat over S, and two points $s_0, s_1 \in S$ such that:*

- *$\mathscr{L}_s = \mathscr{L}_{|C \times \{s\}}$ is torsion-free of rank one, for each s in S;*
- *$\mathscr{L}_{s_0} \cong L_0$;*
- *$\mathscr{L}_{s_1} \cong f_*(L_1)$.*

(iii) *The graph Γ can be oriented in such a way that, if k_i denotes the number of edges starting from C_i, one has:*

$$d_i = p_a(C_i) - 1 + k_i \qquad (i = 1, \ldots, c).$$

Assume moreover that the restriction of ω_C to C_i has even degree for each i; then the multidegree $\underline{d} = \frac{1}{2}\underline{\deg}(\omega_C)$ satisfies conditions (i) *to* (iii).

Proof. (iii)$\Rightarrow$(ii). We use the same construction (and notation) as in Lemma 1.3. Here we take for T the variety of line bundles L on N with $\underline{\deg} L = \underline{d}$, and for $\mathscr{E}$ a Poincaré line bundle on $N \times T$; we define S_1 as in Lemma 1.3 and denote by S_0 the open set in S_1 consisting of points s such that $h_{|C\times\{s\}}$ is surjective; so a point s of S_0 corresponds to a line bundle L on N (of multidegree $\underline{d}$), together with a surjective morphism on C: $h_s: f_* L \to \delta$. Let z be a singular point, $f^{-1}(\{z\}) = \{x, y\}$, U a neighborhood of z in C which contains no other singular points and such that $L \cong \mathcal{O}_N$ on $f^{-1}(U)$. Choosing a generator of L on $f^{-1}(U)$, one checks easily that h_s is given by:

$$h_s(t) = \alpha\, t(x) + \beta\, t(y) \quad \text{with } \alpha, \beta \in k.$$

If α and $\beta \neq 0$, $\operatorname{Ker} h_s$ is an invertible sheaf on U; if for instance $\alpha \neq 0$, $\beta = 0$, then $\operatorname{Ker} h_s$ is isomorphic on U to $f_*(\mathcal{O}_N(-x))$.

Define $\mathscr{L} = \operatorname{Ker} h$ on S_0. Since $k^* \mathscr{E}$ and $k^* p^* \delta$ are flat on S_0 and h is surjective, $\mathscr{L}$ is flat on S_0. Let s be a point in S_0. If h_s is such that α and $\beta \neq 0$ at each singular point, then $\mathscr{L}_s = \mathscr{L}_{|C\times\{s\}}$ is an invertible sheaf of multidegree $\underline{d}$, and all the invertible sheaves of multidegree $\underline{d}$ are obtained that way. On the other hand, if we choose s such that $\alpha . \beta = 0$ at every singular point, then

$$\mathscr{L}_s = f_*(L(-\sum_i x_i))$$

where $\{f(x_1), \dots, f(x_m)\}$ is the set of singular points in C.

To achieve the proof that (iii) implies (ii), we must show that we can choose one point $x_i \in N$ above each double point z_i of C in such a way that: $2\, \underline{\deg}\, L(-\sum_i x_i) = \underline{\deg}\, \omega_N$. If z_i belongs to only one component of C, we choose x_i arbitrarily (among the two points of $f^{-1}(z_i)$). Suppose that z_i belongs to two components. We consider the graph Γ; we assume it is oriented so that property (iii) holds. The point z_i corresponds to an edge of Γ, and we choose the point $x_i \in f^{-1}(z_i)$ which lies in the component corresponding to the starting point of the edge.

For $1 \leqq j \leqq c$, let l_j be the number of double points of C which belong to C_j and not to C_k for $k \neq j$. Then:

$\deg L(-\sum_i x_i)_{|C_j} = d_j - k_j - l_j = p_a(C_j) - l_j - 1 = p_a(N_j) - 1$, where N_j is the component of N which dominates C_j. Hence, $2\, \underline{\deg}\, L(-\sum_i x_i) = \underline{\deg}\, \omega_N$, so we have proved that (iii) implies (ii).

(ii)$\Rightarrow$(i): Assume that property (ii) holds. We can choose L_1 such that $h^0(N, L_1) = 0$; then there is a neighbourhood U_1 of s_1 in S such that $h^0(C, \mathscr{L}_s) = 0$ for $s \in U_1$. On the other hand there is a neighbourhood U_0 of s_0 in S such that $\mathscr{L}_t$ is an invertible sheaf of multidegree $\underline{d}$ when $t \in U_0$. If $u \in U_0 \cap U_1$, the invertible sheaf $L = \mathscr{L}_u$ satisfies condition (i).

(i)$\Rightarrow$(iii): We prove this by induction on the number m of edges of Γ. If $m = 0$, there is nothing to prove. If $m > 0$, let us choose an edge of Γ, i.e. a point s of $C_i \cap C_j$ $(i \neq j)$. Let $f_s: N_s \to C$ be the normalization of C at the point s. We get as before an exact sequence:

$$0 \to L \to (f_s)_* f_s^* L \xrightarrow{h_s} \delta \to 0, \qquad \text{with } \delta = k(s).$$

Since $h^0(L)=0$ by hypothesis and $h^1(L)=0$ by Riemann-Roch, the mapping:

$$h_s\colon H^0(N_s, f_s^* L)\to\delta$$

is an isomorphism. Therefore $H^0(N_s, f_s^* L)$ is generated by a section t such that: $h_s(t)\neq 0$. If $f^{-1}(s)=\{s_i, s_j\}$, we have seen that: $h_s(t)=\alpha\, t(s_i)+\beta\, t(s_j)$, with α and $\beta\neq 0$. It follows that t cannot vanish identically both on C_i and C_j. Suppose that t does not vanish identically on C_i; then if x is a generic point of C_i, the sheaf $L'=f_s^* L(-x)$ on N_s verifies $h^0(L')=0$. If N_s is connected, one has $\deg(L')=p_a(N_s)-1$, so that one can apply the induction hypothesis to L'; if N_s has two connected components N_1, N_2, one checks that: $\deg L'_{|N_i}=p_a(N_i)-1$ $(i=1,2)$, so that we can apply the induction hypothesis to $L'_{|N_1}$ and $L'_{|N_2}$. In both cases, if e_s denotes the edge of Γ corresponding to s, we can find an orientation of $\Gamma-\{e_s\}$ such that (iii) holds (with respect to $\underline{\deg}(L')$); then we orient the edge e_s from C_i to C_j. It is immediate that the orientation obtained for Γ satisfies (iii).

The last assertion of the lemma follows from Euler's graph theorem: a graph Γ, such that the number of edges passing through each vertex is even, can be oriented in such a way that at each vertex p, the number of edges starting from p equals the number of edges abutting to p.

Recall that we denote by Θ' a theta divisor on JN, and by Θ_L the locus of line bundles M in JC such that $h^0(L\otimes M)\geqq 1$.

(2.2) **Proposition.** *Let L be a line bundle on C whose multidegree satisfies the equivalent conditions of Lemma 2.1 (for instance $2\,\underline{\deg}(L)=\underline{\deg}(\omega_C)$). Then Θ_L is a divisor, algebraically equivalent to $(f^*)^{-1}(\Theta')$.*

Proof. Choose $S, \mathscr{L}, s_0, s_1$ as in condition (ii) of the lemma, with $L=L_0$. We want to construct a divisor Z in $JC\times S$, flat over S, such that for each s in S, Z_s is the locus of line bundles L in JC with $h^0(\mathscr{L}_s\otimes L)\geqq 1$. We use Kempf's construction (see [Sz]). Let $\mathscr{P}$ be a Poincaré bundle on $C\times JC$, p, q, r, m the projections from $C\times JC\times S$ onto $C\times S$, $JC\times S$, $C\times JC$, C. We put $\mathscr{F}=r^*\mathscr{P}\otimes p^*\mathscr{L}$. Let $s\in S$, $\alpha\in JC$ (corresponding to a line bundle L_α on C); one has by definition:

$$\mathscr{F}_{|C\times\{\alpha\}\times\{s\}}=\mathscr{L}_s\otimes L_\alpha.$$

Let us choose g non-singular points $x_1, \ldots, x_g$ on C such that if $D=\sum x_i$, one has $\deg D_{|C_j}\geqq d_j$ for each j. Since $\mathscr{L}$ is invertible around $\{x_i\}\times S$, we get an exact sequence:

$$0\to\mathscr{F}\otimes m^*\mathcal{O}_C(-D)\to\mathscr{F}\to\mathscr{F}\otimes m^*\mathcal{O}_D\to 0.$$

Apply q_* to this exact sequence; we claim that:

- $q_*(\mathscr{F}\otimes m^*\mathcal{O}_D)$ is locally free of rank g.
- $R^1 q_*(\mathscr{F}\otimes m^*\mathcal{O}_D)=0$.
- $R^1 q_*(\mathscr{F}\otimes m^*\mathcal{O}_C(-D))$ is locally free of rank g.

To prove these assertions, it is enough (using [EGA III.7]) to show that for each $s \in S$, $\alpha \in JC$:

$$h^0(C, \mathcal{L}_s \otimes L_\alpha \otimes \mathcal{O}_D) = g,$$
$$h^1(C, \mathcal{L}_s \otimes L_\alpha \otimes \mathcal{O}_D) = 0,$$
$$h^1(C, \mathcal{L}_s \otimes L_\alpha(-D)) = g.$$

The first and the second equality are clear; the third one follows from the choice of the x_i and Riemann-Roch formula.

Therefore we obtain an exact sequence:

$$E_1 \xrightarrow{u} E_2 \longrightarrow R^1 q_*(\mathcal{F}) \longrightarrow 0$$

where E_1, E_2 are locally free sheaves of rank g on $JC \times S$.

Let $p = (\alpha, s)$ be a point in $JC \times S$. Since the formation of $R^1 q_*(\mathcal{F})$ commutes with base change, we get an exact sequence:

$$E_1(p) \xrightarrow{u(p)} E_2(p) \to H^1(C, \mathcal{L}_s \otimes L_\alpha) \to 0.$$

Since $h^1(\mathcal{L}_s \otimes L_\alpha) = h^0(\mathcal{L}_s \otimes L_\alpha)$ by Riemann-Roch, we conclude that $h^0(\mathcal{L}_s \otimes L_\alpha) \geqq 1$ if and only if $\det(u(p)) = 0$.

We define the divisor Z on $JC \times S$ by the equation $\det(u) = 0$. Then, by construction, for each s in S, Z_s is the locus of line bundles M in JC with $h^0(\mathcal{L}_s \otimes M) \geqq 1$. Since by 2.1(ii), for each s in S one has $Z_s \neq JC$, Z is flat on S. Since:

$$f_*(L_1) \otimes M \cong f_*(L_1 \otimes f^* M),$$

one has $Z_{s_1} = (f^*)^{-1}(\Theta')$, where $\Theta' = \{M \in JN, h^0(L_1 \otimes M) \geqq 1\}$; and also $Z_{s_0} = \Theta_L$. So Θ_L is algebraically equivalent to $(f^*)^{-1}(\Theta')$.

Fix an L as in the Proposition, and put $\Theta_L = \Theta$. The divisor Θ on JC defines a group homomorphism:

$$\lambda: \begin{cases} JC(k) \to \operatorname{Pic}(JC) \\ \quad a \mapsto \mathcal{O}_{JC}(\Theta_a - \Theta) \end{cases}$$

([L, p. 75]); here $JC(k)$ denotes the group of rational points of JC. Similarly the divisor Θ' on JN defines a morphism of algebraic groups:

$$\mu: JN \to \underline{\operatorname{Pic}}^0(JN) = \widehat{JN}.$$

(2.3) **Corollary.** *The diagram:*

$$\begin{array}{ccc} JC(k) & \xrightarrow{\lambda} & \operatorname{Pic}(JC) \\ {\scriptstyle f^*}\downarrow & & \uparrow{\scriptstyle \operatorname{Pic}(f^*)} \\ JN(k) & \xrightarrow{\mu} & \widehat{JN}(k) \end{array}$$

is commutative.

The corollary is an immediate consequence of Proposition 2.2 and Proposition 4 in [L, p. 75].

(2.4) *Remark.* Let $C \to S$ be a flat family of curves of genus g, with at most ordinary double points, and let $JC \to S$ be the corresponding flat family of Jacobian varieties (JC is an algebraic space: see [A]). Locally over S for the étale topology, one can find a line bundle L on C such that $\deg(L_s) = g-1$ and $h^0(C_s, L_s) = 0$ for each s in S. Then, for each s in S, the multidegree of L_s satisfies condition (i) of Lemma 2.1. One sees as in the proof of Proposition 2.2 that the divisors Θ_{L_s} on JC_s fit together to define a divisor Θ on JC, flat over S, such that $\Theta_{|JC_s} = \Theta_{L_s}$ for each s.

3. Generalized Prym Varieties: Definition

Throughout the rest of this paper, $\tilde{C}$ is a connected curve with only ordinary double points, $\tilde{f}: \tilde{N} \to \tilde{C}$ its normalization, $\iota: \tilde{C} \to \tilde{C}$ an involution ($\iota^2 = \mathrm{Id}$).

(3.1) **Lemma.** *The quotient curve $\tilde{C}/(\iota)$ has only ordinary double points.*

Proof. We have only to check what happens at a singular point s of $\tilde{C}$ fixed under the involution. Let $\mathcal{O}_s$ be the local ring of $\tilde{C}$ at s; the completion $\hat{\mathcal{O}}_s$ can be identified with $k[[u, v]]/(uv)$. If the involution exchanges the two branches of $\tilde{C}$ at s, one can choose u, v so that $\iota^* u = v$, $\iota^* v = u$; then the subring of invariants in $\hat{\mathcal{O}}_s$ is the ring of formal power series in $u+v$, which is regular. If the branches at s are not exchanged, one can choose u, v so that $\iota^* u = -u$, $\iota^* v = -v$; hence the subring of invariants is $k[[u^2, v^2]]$, which is isomorphic to $k[[x, y]]/(xy)$.

We now assume:

(*) The fixed points of ι are exactly the singular points, and at a singular point the two branches are not exchanged under ι.

We note $C = \tilde{C}/(\iota)$ the quotient curve, $\pi: \tilde{C} \to C$ the projection, $f: N \to C$ the normalization of C, $\pi': \tilde{N} \to N$ the morphism deduced from π (so that $f \circ \pi' = \pi \circ \tilde{f}$):

$$
\begin{array}{ccc}
\tilde{N} & \xrightarrow{\tilde{f}} & \tilde{C} \\
\downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\
N & \xrightarrow{f} & C
\end{array}
$$

π' is a two-sheeted covering, ramified at the points x_i, y_i of $\tilde{N}$ which lie over singular points z_i of $\tilde{C}$.

(3.2) **Lemma.** $\pi^* \omega_C \cong \omega_{\tilde{C}}$.

Proof. $\omega_{\tilde{C}}$ is the sheaf of forms $\tilde{\omega}$ on $\tilde{N}$, regular except for simples poles at the x_i and y_i, with $\mathrm{Res}_{x_i}(\tilde{\omega}) + \mathrm{Res}_{y_i}(\tilde{\omega}) = 0$. Since π is etale outside the singular points, and $\mathrm{Res}_{x_i} \pi^* \omega = 2 \, \mathrm{Res}_{\pi x_i} \omega$ for a form $\omega \in \omega_C$, one gets $\mathrm{div}(\pi^* \omega) = \pi^* \mathrm{div}(\omega)$, hence $\pi^* \omega_C \cong \omega_{\tilde{C}}$.

One should notice that π is not a two-sheeted covering in the usual sense; in fact, π is not flat at the singular points. From Lemma 3.2 we obtain: $p_a(\tilde{C}) = 2p_a(C) - 1$.

We shall need some facts about Cartier divisors on $\tilde{C}$ and C. Let $\tilde{K}$ (resp. K) be the ring of rational functions on $\tilde{C}$ (resp. C), that is the product of the fields of functions of the components. The group of Cartier divisors on $\tilde{C}$ is:

$$\operatorname{Div}(\tilde{C}) = \bigoplus_{x \in \tilde{C}_{\mathrm{reg}}} \mathbf{Z} \cdot x + \bigoplus_{s \text{ singular}} \tilde{K}_s^*/\mathcal{O}_s^*.$$

Let s be a singular point of $\tilde{C}$, s_1, s_2 the two points of $\tilde{N}$ lying over s, v_1, v_2 the corresponding valuations of $\tilde{K}$. One has an exact sequence:

$$0 \to k^* \to \tilde{K}_s^*/\mathcal{O}_s^* \xrightarrow{v_1, v_2} \mathbf{Z} \oplus \mathbf{Z} \to 0$$

$\left(\text{the kernel is identified with } k^* \text{ by } f \mapsto \dfrac{f(s_1)}{f(s_2)}\right)$.

It is convenient to split this exact sequence by choosing uniformizing parameters t_1 and t_2 at s_1 and s_2, thus getting an isomorphism: $\tilde{K}_s^*/\mathcal{O}_s^* \xrightarrow{\sim} k^* \times \mathbf{Z} \times \mathbf{Z}$. With this identification, assuming $\iota^* t_1 = -t_1$, $\iota^* t_2 = -t_2$, the action of ι on $\tilde{K}_s^*/\mathcal{O}_s^*$ is simply:

$$\iota^*(z, m, n)_s = ((-1)^{m+n} z, m, n)_s.$$

The norm of $\tilde{K}/K$ maps $\mathcal{O}_{\tilde{C}}$ into $\mathcal{O}_C$, thus gives a diagram of exact sequences:

$$\begin{array}{ccccc}
\tilde{K}^* & \longrightarrow & \operatorname{Div}(\tilde{C}) & \longrightarrow & \operatorname{Pic}(\tilde{C}) \to 0 \\
\downarrow{\scriptstyle N_{\tilde{K}/K}} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle Nm} \\
K^* & \longrightarrow & \operatorname{Div}(C) & \longrightarrow & \operatorname{Pic}(C) \to 0
\end{array}$$

where:

– π_* is the direct image under π: $\pi_*(\sum_i x_i) = \sum_i \pi x_i$ for $x_i \in C_{\mathrm{reg}}$,

$\pi_*((z, m, n)_s) = ((-1)^{m+n} z^2, m, n)_{\pi s}$ (follows from the formula for ι^*).

– Nm: $\operatorname{Pic}(\tilde{C}) \to \operatorname{Pic}(C)$ is the usual norm for line bundles ([EGA II.6.5]), which induces a morphism of algebraic groups $Nm: J\tilde{C} \to JC$.

(3.3) **Lemma.** *If L is a line bundle on $\tilde{C}$ such that $Nm\, L \cong \mathcal{O}_C$, then $L \cong M \otimes \iota^* M^{-1}$ for some line bundle M on $\tilde{C}$. Moreover M can be chosen of multidegree* $(0, \ldots, 0)$ *or* $(1, 0, \ldots, 0)$.

Proof. As in [M 1], Lemma 1, we get $L = \mathcal{O}(D)$ where $\pi_*(D) = 0$. Writing $D = \sum_i x_i + \sum_{s \text{ singular}} (z_s, m_s, n_s)$, we get that D is a linear combination of divisors $x - \iota x$, for $x \in \tilde{C}_{\mathrm{reg}}$, and $(-1, 0, 0)_s$ at singular points s; but $(-1, 0, 0) = (1, 0, 1) - \iota^*(1, 0, 1)$, hence $D = E - \iota^* E$ for some divisor E and $L \cong M \otimes \iota^* M^{-1}$ with $M = \mathcal{O}(E)$. We may replace M by $M \otimes \pi^* N$ for any line bundle N on C, hence assume $\underline{\deg}(M) = (\varepsilon_1, \ldots, \varepsilon_c)$ where $\varepsilon_i = 0$ or 1. Since $(\mathrm{Id} - \iota)^*(1, 1, -1) = 0$ and $\tilde{C}$ is connected, we can further normalize M as in the statement of the lemma.

We denote by P (resp. P_1) the variety of line bundles in $\mathrm{Ker}(Nm)$ of the form $M\otimes\iota^* M^{-1}$ with $\underline{\deg}(M)=(0,\dots,0)$ (resp. $(1,0,\dots,0)$). Note that P is a connected algebraic subgroup of $J\tilde{C}$.

(3.4) **Proposition.** *Fix a line bundle L_0 on C with $L_0^2\cong\omega_C$. The function:*

$$L\mapsto h^0(L\otimes\pi^* L_0)$$

is constant mod 2 *on P and on P_1, and takes opposite parity on P and P_1.*

Notice that such an L_0 always exists, since the hypothesis (*) insures that $\deg\omega_{C|C_i}$ is even for all i.

Proof. We first prove that $h^0(L\otimes\pi^* L_0)=h^0(L_0\otimes\pi_* L)$ is constant mod 2 when L varies in a connected algebraic family; for this we exhibit a non-degenerate quadratic form $S^2(L_0\otimes\pi_* L)\to\omega_C$ (or, what amounts to the same, $S^2\pi_* L\to\mathcal{O}_C$) and apply Theorem 1.1. The norm induces a quadratic form:

$$S^2\pi_* L\to Nm(L)\cong\mathcal{O}_C$$

which can be identified locally with:

$$Q:\begin{cases} S^2\pi_*\mathcal{O}_{\tilde{C}}\to\mathcal{O}_C\\ \qquad h\mapsto Nm(h)\end{cases}$$

and Q is easily seen to be non-degenerate by local computation. The proof in [M 1] that $h^0(L\otimes\pi^* L_0)$ takes opposite parity on P and P_1 applies in a straightforward manner to our case (use Lemma 3.2 in Step II).

(3.5) **Proposition.** *P is an abelian variety of dimension $p_a(C)-1$.*

Proof. We look at the diagram:

$$\begin{array}{ccccccc} 0\to\tilde{T} & \longrightarrow & J\tilde{C} & \longrightarrow & J\tilde{N} & \to 0\\ \downarrow{\scriptstyle Nm} & & \downarrow{\scriptstyle Nm} & & \downarrow{\scriptstyle Nm} & \\ 0\to T & \longrightarrow & JC & \longrightarrow & JN & \to 0\end{array}$$

where $\tilde{T}$ and T are the groups of classes of divisor of multidegree $(0,\dots,0)$ with singular support; π^* induces an isomorphism of T onto $\tilde{T}$, hence, since $Nm\circ\pi^*=2$, $Nm_{|\tilde{T}}$ is surjective and $\mathrm{Ker}\,Nm_{|\tilde{T}}=\tilde{T}_2=\{$points of order 2 in $\tilde{T}\}$.

Thus one gets an exact sequence:

$$0\to\tilde{T}_2\to P\times\mathbf{Z}/2\xrightarrow{g} R\to 0$$

where $R=\mathrm{Ker}\,Nm_{|J\tilde{N}}$ is a complete subvariety of $J\tilde{N}$; therefore P is complete, reduced and connected, hence an abelian variety.

(3.6) *Remark.* R is an abelian variety, called in [M 2] the Prym variety associated to the ramified two-sheeted covering $\tilde{N}\to N$. Notice that $g: P\to R$ is an isogeny, but not an isomorphism if the dimension t of T is greater than 2: in fact $\#\ker(g)=2^{t-1}$.

Using the line bundle $\pi^* L_0$ we define a divisor Θ on $J\tilde{C}$ (§2).

(3.7) **Theorem.** *Θ induces twice a principal polarization on P.*

The statement of the theorem means that $\Theta' = \Theta_{|P}$ is algebraically equivalent to 2Ξ, where Ξ is an ample divisor with $h^0(\mathcal{O}(\Xi)) = 1$; or, equivalently, that the morphism

$$\rho: \begin{cases} P \to \hat{P} \\ a \mapsto \mathcal{O}_P(\Theta'_a - \Theta') \end{cases}$$

is twice an isomorphism. By Corollary 2.3, we have a commutative diagram:

$$\begin{array}{ccccccc}
P & & \xrightarrow{\quad\rho\quad} & & & & \hat{P} \\
\downarrow{\scriptstyle g} & & & & & & \uparrow{\scriptstyle \hat{g}} \\
R & \hookrightarrow & J\tilde{N} & \xrightarrow{\mu} & \widehat{J\tilde{N}} & \longrightarrow & \hat{R}
\end{array}$$

We first need the following lemma:

(3.8) **Lemma.** *Let $h: A \to B$ be an isogeny of abelian varieties, $\beta: B \to \hat{B}$ a principal polarization on B, $\alpha = \hat{h} \circ \beta \circ h$ the pullback of β on A.*

Assume:

(i) $\operatorname{Ker} h \subset A_2$, *the set of points of order two in A;*

(ii) *$h(A_2)$ is totally isotropic maximal with respect to the symmetric pairing defined by β on B_2.*

Then $\operatorname{Ker} \alpha = A_2$.

Proof of the Lemma. Let us consider A_2 and B_2 as vector spaces over $\mathbf{F}_2$, and denote by A_2^* and B_2^* the dual spaces. Cartier duality provides us with a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & A_2 & \hookrightarrow & A & \xrightarrow{2} & A & \to & 0 \\
 & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & \\
0 & \to & B_2 & \hookrightarrow & B & \xrightarrow{2} & B & \to & 0 \\
 & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} & & \\
0 & \to & B_2^* & \longrightarrow & \hat{B} & \xrightarrow{2} & \hat{B} & \to & 0 \\
 & & \downarrow{\scriptstyle {}^t h_2} & & \downarrow{\scriptstyle \hat{h}} & & \downarrow{\scriptstyle \hat{h}} & & \\
0 & \to & A_2^* & \longrightarrow & \hat{A} & \xrightarrow{2} & \hat{A} & \to & 0
\end{array}$$

where the horizontal rows are exact.

It follows from (i) that $\operatorname{Ker} \hat{h} \subset (\hat{B})_2$, hence: $\beta^{-1}(\operatorname{Ker} \hat{h}) = \beta_2^{-1}(\operatorname{Ker} {}^t h_2) = \beta_2^{-1}((\operatorname{Im} h_2)^*) = (\operatorname{Im} h_2)^{\perp}$, where the sign $\perp$ means orthogonality with respect to the pairing on B_2 defined by β.

Now, (ii) implies that $(\operatorname{Im} h_2)^{\perp} = \operatorname{Im} h_2$, so that: $\operatorname{Ker}\alpha = h^{-1}(\operatorname{Im} h_2) = A_2 + \operatorname{Ker} h = A_2$.

Proof of Theorem (3.7). We apply the lemma to the isogeny $h: P \times JN \to J\tilde{N}$ and the principal polarization on $J\tilde{N}$. We check the conditions (i) and (ii):

(i) An element of $\operatorname{Ker}(h)$ is a pair (L, M) with $L \in P$, $M \in JN$ and $\tilde{f}^* L \otimes \pi'^* M \cong \mathcal{O}_{\tilde{N}}$. Write $M = f^* M'$ for some $M' \in JC$; then $L \otimes \pi^* M' \in \ker \tilde{f}^* = \tilde{T}$, hence since $\tilde{T} = \pi^*(T)$, $L \cong \pi^* M''$ for some $M'' \in JC$. Using $Nm(L) \cong \mathcal{O}_C$ one gets $M''^2 = \mathcal{O}_C$, hence $L^2 = \mathcal{O}_{\tilde{C}}$ and $M^2 = \mathcal{O}_N$.

(ii) Let $e_{2,\tilde{N}}$ (resp. $e_{2,N}$) the pairing defined on $J\tilde{N}_2$ (resp. JN_2) by the polarization. Nm and π'^* are dual with respect to the polarizations on $J\tilde{N}$ and JN, therefore:

$$e_{2,\tilde{N}}(\pi'^* a, b) = e_{2,N}(a, Nm(b)) \qquad \text{for } a \in JN_2,\ b \in J\tilde{N}_2.$$

In particular, $\pi'^* JN_2$ is orthogonal to both itself and $\tilde{f}^* P_2$. Now if $L, M \in P_2$, Corollary 1.2 gives:

$$\begin{aligned}\ln e_{2,\tilde{N}}(\tilde{f}^* L, \tilde{f}^* M) &= h^0(\pi^* L_0) + h^0(\pi^* L_0 \otimes L) + h^0(\pi^* L_0 \otimes M)\\ &\quad + h^0(\pi^* L_0 \otimes L \otimes M) \pmod 2\\ &\equiv 4 h^0(\pi^* L_0) \bmod 2 \text{ by Proposition 3.6,}\end{aligned}$$

hence the subspace $h(P_2 \times JN_2)$ is totally isotropic in $J\tilde{N}_2$. To prove that it is maximal, we have only to check that:

$$\dim_{\mathbf{F}_2}(h(P_2 \times JN_2)) = \tfrac{1}{2}\dim_{\mathbf{F}_2}(J\tilde{N}_2).$$

Since $\dim_{\mathbf{F}_2}(P_2 \times JN_2) = \dim_{\mathbf{F}_2}(J\tilde{N}_2)$, it is equivalent to show that:

$$\dim_{\mathbf{F}_2}(\operatorname{Ker} h) = \tfrac{1}{2}\dim_{\mathbf{F}_2}(J\tilde{N}_2).$$

We have shown in the proof of (i) that:

$\operatorname{Ker} h = \{(\pi^* a, f^* a) \text{ where } a \in JC_2 \text{ is such that } \pi^* a \in P\}$.

Since π^* (resp. f^*) is injective when C is singular (resp. non-singular), we conclude that $\operatorname{Ker} h$ is isomorphic to the group of points $a \in JC_2$ such that $\pi^* a \in P$; this group is the kernel of the linear form given by the composition:

$$\varphi: JC_2 \xrightarrow{\pi^*} \operatorname{Ker}(Nm) \longrightarrow \operatorname{Ker}(Nm)/P \xrightarrow{\sim} \mathbf{F}_2$$

(where the last isomorphism is given by Lemma 3.3).

We now prove that $\varphi \neq 0$. If C has a singular point s, let D_s be the divisor $(-1,0,0)_s$ and d the class of $\mathcal{O}_C(D)$ in JC; then $d \in JC_2$ and since $\pi^* D_s = (-1,0,0)_{\tilde{s}}$, one has $\pi^* d \in P_1$, hence $\varphi(d) = 1$. If C is non-singular, it follows from [M2] (or directly from Corollary 1.2) that:

$$\varphi(a) = (\varepsilon \cdot a) \qquad \text{for each } a \in JC_2,$$

where ε is the only non-zero element of $\operatorname{Ker}(\pi^*)$; in particular $\varphi \neq 0$. Therefore, in any case, if we put $t = \dim(T) = \dim(\tilde{T})$, we obtain:

$$\dim_{\mathbf{F}_2}(\operatorname{Ker} h) = \dim_{\mathbf{F}_2}(JC_2) - 1 = 2g - t - 1$$

while $\dim_{\mathbf{F}_2}(J\tilde{N}_2) = 2\dim(J\tilde{N}) = 2(2g - 1 - t)$, which achieves the proof of (ii).

Therefore the lemma applies and gives:

$$\operatorname{Ker}(\hat{h} \circ \mu \circ h) = P_2 \times JN_2.$$

But the polarization $\alpha = \hat{h} \circ \mu \circ h$ can be written as a sum of four morphisms:

$$\rho: P \to \hat{P}; \quad \sigma: P \to \widehat{JN}; \quad \tau: JN \to \hat{P}; \quad v: JN \to \widehat{JN}.$$

Since π'^* and Nm are dual to each other with respect to the principal polarizations of $J\tilde{N}$ and JN ([M2]), we find that $\sigma = 0$; then from $\hat{\alpha} = \alpha$ we deduce that $\tau = \hat{\sigma} = 0$. We conclude that $\operatorname{Ker}(\rho) = P_2$, which proves the proposition.

(3.9) **Definition.** *The abelian variety P, together with the principal polarization Ξ defined by $2\Xi \equiv \Theta_{|P}$ in $NS(P)$, is the (generalized) Prym variety associated to $(\tilde{C}, \iota)$.*

Actually, as in the non-singular case, the relation between Θ and Ξ can be made much more precise:

(3.10) **Proposition.** *Choose L_0 such that $h^0(\pi^* L_0)$ is even (and $L_0^2 \cong \omega_C$). Then, with the preceding notation: $\Theta_{|P} = 2\Xi$, where $\Xi \subset P$ is a divisor in the class of the principal polarization on P.*

Notice that one can always find such an L_0: if s is a singular point of C, D the divisor $(-1, 0, 0)_s$, then $h^0(\pi^*(L_0(D))) \equiv h^0(\pi^* L_0) + 1 \pmod 2$, by Lemma 3.3 and Proposition 3.4.

To prove the proposition, we need Riemann's singularity theorem for singular curves. We state a more general result:

(3.11) **Proposition.** *Let C be a curve, JC^* the variety of line bundles L on C such that $2\underline{\deg} L = \underline{\deg}\, \omega_C$, Θ the divisor of line bundles M in JC^* such that $h^0(M) \geqq 1$, $L \in \Theta$, φ the pairing:*

$$H^0(L) \otimes H^0(\omega_C \otimes L^{-1}) \to H^0(\omega_C).$$

Choose a basis (s_i) of $H^0(L)$, a basis (t_j) of $H^0(\omega_C \otimes L^{-1})$; identify $H^0(\omega_C)$ to the dual of the tangent space T to JC^* at L. Assume the function $\det(\varphi(s_i \otimes t_j))$ is not identically zero on T; then it defines a hypersurface in T which is equal to the tangent cone to Θ at L. In particular the multiplicity of Θ at L is $h^0(L)$.

Proof of Proposition (3.11). The proposition is proved in Kempf's thesis (see [Sz]) when C is irreducible; the argument can be adapted to the general case as follows: let $\mathscr{P}$ be a Poincaré bundle on $C \times JC^*$, q: $C \times JC^* \to JC^*$ the projection. Put $h^0(L) = m$; choosing m non-singular points $x_1, \ldots, x_m$ such that $H^0(C, L(-\sum x_i)) = (0)$, we get as in Proposition 2.1 an exact sequence in a neighborhood U of L in JC^*:

$$\mathcal{O}_U^m \xrightarrow{u} \mathcal{O}_U^m \longrightarrow R^1 q_*(\mathscr{P}) \to 0$$

and a local equation for Θ is $\det(u) = 0$.

Since $h^0(L) = m$, the coefficients u_{ij} of u are zero at L; the tangent cone to Θ at L is given by the determinant of the first-order terms of the u_{ij} — unless this determinant is identically zero. In other words, let t be a tangent vector to JC^*

at L; t corresponds to a morphism $\operatorname{Spec} k[\varepsilon] \to JC^*$ ($\varepsilon^2=0$), or equivalently to a line bundle L_ε on $C_\varepsilon = C \times \operatorname{Spec} k[\varepsilon]$; and t is tangent to Θ at L if $\dim_k \operatorname{Ker}(t^* u) \geqq m+1$. Now by construction, $\operatorname{Ker}(t^* u) \cong H^0(C_\varepsilon, L_\varepsilon)$; the tangent vector t, viewed as an element of $H^1(C, \mathcal{O}_C) \cong \operatorname{Ext}^1_{\mathcal{O}_C}(L, L)$, corresponds to an extension of sheaves:

$$0 \to L \to L_\varepsilon \to L \to 0$$

which gives:

$$0 \to H^0(L) \to H^0(L_\varepsilon) \to H^0(L) \xrightarrow{t} H^1(L).$$

Thus t belongs to the tangent cone if and only if the map $H^0(L) \to H^1(L)$ defined by cup-product with t is not an isomorphism. By choosing a basis for $H^0(L)$ and $H^0(\omega_C \otimes L^{-1})$ one finds the statement of the proposition.

Proof of Proposition (3.10). We must check that for line bundles L in $J\tilde{C}^*$ satisfying $\iota^* L \cong \omega_{\tilde{C}} \otimes L^{-1}$, the determinant given in Proposition 3.11 is not identically zero. But the vanishing of $\det(\varphi(s_i \otimes t_j))$ implies that there is a $s \in H^0(\tilde{C}, L)$ such that $\varphi(s \otimes t) = 0$ for all $t \in H^0(\tilde{C}, \omega_{\tilde{C}} \otimes L^{-1})$; since $\varphi(s \otimes \iota^* s)$ is non-zero, this is impossible. Thus we can use Riemann's singularity theorem, and the argument in [M2, p. 342] applies identically.

(3.12) *Remark.* To avoid the choice of a theta-characteristic L_0 as in Proposition 3.10, it is often convenient to look at the Prym variety in JC^*, after translation by $\pi^* L_0$: thus the Prym variety becomes the variety of line bundles L in JC^* such that $Nm(L) \cong \omega_C$ and $h^0(L)$ is even, Ξ is the divisor of effective line bundles in P, and $\Theta_{|P} = 2\Xi$.

4. Dimension of Sing Ξ

Keeping the notation of § 3, we denote by $\tilde{C}$ a connected curve of genus $2g-1$ with ordinary double points, ι an involution of $\tilde{C}$ satisfying condition (*), C the quotient curve (of genus g). Our aim is to extend Mumford's description of Sing Ξ ([M2]) to the Prym variety and its polarization defined in § 3. According to Remark 3.12, we look at the situation in $J\tilde{C}^*$; we denote:

$$P = \{\text{line bundles } L \text{ on } \tilde{C},\ Nm(L) \cong \omega_C,\ h^0(L) \text{ even}\},$$
$$\Xi = \{L \text{ in } P,\ h^0(L) \geqq 2\}.$$

(4.1) **Lemma.** *A line bundle L in P belongs to* SingΞ *if and only if:*

(i) *either* $h^0(L) \geqq 4$,

(ii) *or* $h^0(L) = 2$, *and for a basis* $\{s, t\}$ *of* $H^0(\tilde{C}, L)$ *one has:*

$$\iota^* s \otimes t = s \otimes \iota^* t \quad \textit{in} \quad H^0(\tilde{C}, L \otimes \iota^* L) = H^0(\tilde{C}, \omega_{\tilde{C}}).$$

Proof. By Proposition 3.10, a point L in P belongs to SingΞ either if it is of multiplicity $\geqq 3$, or if it is of multiplicity 2 and the tangent space to P is contained in the tangent cone to Θ at L. The first case gives (i); in the second case, we can apply the analysis in [M 2, p. 343], using Proposition 3.11, and find condition (ii).

We first get rid of case (i) following [M 2].

(4.2) **Proposition.** *Let Z be an irreducible component of* $\operatorname{Sing}\Xi$ *with* $\dim Z \geqq g-5$. *Then, a generic line bundle L in Z has the property:*

(P) *There exist two linearly independent sections s, t in $H^0(\tilde{C}, L)$ such that $\iota^* s \otimes t = s \otimes \iota^* t$.*

Proof. The proof in [M 2, p. 345] applies identically once one knows the following lemma:

(4.3) **Lemma.** *Let X be a curve of genus g. Let us denote by G_d^r the variety of line bundles L on X with $\deg L = d$, $h^0(L) \geqq r+1$. Let Z be an irreducible subvariety of G_d^r, L a line bundle in Z with $h^0(L) = r+1$, φ_L the pairing:*

$$\varphi_L \colon H^0(L) \otimes H^0(\omega_X \otimes L^{-1}) \to H^0(\omega_X).$$

Then

$$\dim Z \leqq g - \dim \operatorname{Im} \varphi_L.$$

Proof as in [S-D, p. 162].

We now begin the study of line bundles with property (P). We fix some notation. If L is a line bundle on a curve X, we denote by $|L|$ the set of effective divisors D such that $\mathcal{O}(D) \cong L$; this is an open set in $\mathbf{P}(H^0(X, L))$ (which may be different from $\mathbf{P}(H^0(X, L))$ if X is reducible). We shall say for convenience that L is non-singular if $|L|$ contains a divisor with non-singular support (or equivalently, if at each double point x of X, there is a global section s of L such that $s(x) \neq 0$).

(4.4) **Lemma.** *Let L be a line bundle on $\tilde{C}$ with property* (P); *assume that at each double point of C, either s or t do not vanish. Then $L \cong \pi^* M(E)$, where:*

- *M is a non-singular line bundle on C with $h^0(M) \geqq 2$.*
- *E is a divisor on $\tilde{C}$ with non-singular support.*
- *$\pi_* E \in |\omega_C \otimes M^{-2}|$; in particular $\omega_C \otimes M^{-1}$ and $\omega_C \otimes M^{-2}$ are non-singular line bundles.*

Proof. One can suppose that s and t are both non-zero at each double point of $\tilde{C}$. Put $\varphi = s/t$; since $\iota^* \varphi = \varphi$, one gets $\varphi = \pi^* \psi$, where ψ is a rational function on C. Let E be the divisor of common zeros of s and t, $Z(s)$ (resp. $Z(\varphi)$, $Z(\psi)$) the divisor of zeros of s (resp. φ, ψ); all these divisors have non-singular support and one has:

$$Z(s) = Z(\varphi) + E \quad \text{with} \quad Z(\varphi) = \pi^* Z(\psi).$$

This gives $L \cong \pi^* M(E)$, with $M = \mathcal{O}_C(Z(\psi))$; the last statement follows from the isomorphism $Nm(L) \cong \omega_C$.

(4.5) Notice that the argument is still valid if the involution ι has some non-singular fixed points.

We are thus led to study the dimension of the locus of line bundles M on a curve C with ordinary double points, such that $h^0(M) \geqq 2$ and M, $\omega_C \otimes M^{-1}$ and $\omega_C \otimes M^{-2}$ are non-singular. We need some preliminary lemmas:

(4.6) **Lemma.** *Let L, M be two non-singular line bundles on C, φ the pairing:*

$$H^0(L)\otimes H^0(M)\to H^0(L\otimes M).$$

Then $\dim \operatorname{Im}\varphi \geqq h^0(L)+h^0(M)-1$.

Proof. The lemma follows from the fact that $|L|$ and $|M|$ are non-empty, and the morphism:

$$|L|\times|M|\to|L\otimes M|$$

is generically finite.

(4.7) **Lemma.** *Let L be a non-singular line bundle on C; suppose $\omega_C\otimes L^{-1}$ is non-singular. Then:*

$$h^0(L)\leqq\frac{\deg L}{2}+1.$$

If equality holds, then either $L=\mathcal{O}_C$ or $L=\omega_C$ or there exists on C a non-singular line bundle M with $\deg M=h^0(M)=2$ *(we'll say for short that "C has a non-singular g_2^1").*

Proof. The first statement follows from Lemma 4.6 and the Riemann-Roch theorem; the second statement is proved as in [S-D, p. 159], noting that if one chooses $D'\in|\omega_C\otimes L^{-1}|$ with non-singular support, then $\mathcal{O}_C((D,D'))$ and $\omega_C(-(D,D'))$ are non-singular.

(4.8) **Lemma.** *Let Z be an irreducible subvariety of G_d^r, $(0<d<2g-2)$ such that for L generic in Z both L and $\omega_C\otimes L^{-1}$ are non-singular. Then* $\dim(Z)\leqq d-2r$. *Moreover if $\omega_C\otimes L^{-2}$ is non-singular, equality holds only if C has a non-singular g_2^1.*

Proof. The first part follows from Lemmas 4.3 and 4.6. If $\dim Z=d-2r$, a generic line bundle L in Z is generated by its global sections, and $h^0(L)=2$. From the exact sequence:

$$(4.8.1)\quad 0\to\omega_C\otimes L^{-2}\xrightarrow{(t,-s)}(\omega_C\otimes L^{-1})^2\xrightarrow{(s,t)}\omega_C\to 0$$

we get:

$$\dim\operatorname{Im}\varphi_L=2h^0(\omega_C\otimes L^{-1})-h^0(\omega_C\otimes L^{-2}),$$

hence by Lemma 4.3:

$$d-2r\leqq g-2(g+r-d)+h^0(\omega_C\otimes L^{-2})$$

or

$$h^0(\omega_C\otimes L^{-2})\geqq g-d=\tfrac{1}{2}\deg(\omega_C\otimes L^{-2})+1.$$

Since L^2 is clearly non-singular, C has a non-singular g_2^1 by Lemma 4.7.

(4.9) **Lemma.** *Let C be a curve of genus g with ordinary double points, such that for each component C_i of C, the intersection number of C_i with the rest of C is even. Suppose that C has no non-singular g_2^1. Let Z be an irreducible subvariety* *of* G_d^1, $0<d\leqq g-2$; *assume that for* L *generic in* Z, L, $\omega_C\otimes L^{-1}$ *and* $\omega_C\otimes L^{-2}$ *are non singular. Then,* $\dim Z\leqq d-3$.

If $\dim Z=d-3$, C *is one of the following:*

a) C *is trigonal* (=3*-sheeted covering of* $\mathbf{P}^1$);

b) C *is a two-sheeted covering of a curve of genus one, and* $g\geqq 6$;

c) C *is a plane quintic;*

d) C *is the union of two curves* C_1 *and* C_2, *with one of the following configurations:*

$\# C_1\cap C_2=2$.

$\# C_1\cap C_2=4$, *and neither* C_1 *nor* C_2 *is a rational curve.*

$\# C_1\cap C_2=4$, C_1 *is rational,* C_2 *has a non-singular* g_2^1 *and* $p_a(C_2)\geqq 4$.

$\# C_1\cap C_2=4$, C_1 *is rational and* $\omega_{C_2}\cong\mathcal{O}_{C_2}(\sum u_i)$, *where* $C_1\cap C_2=\{u_1,\dots,u_4\}$.

(4.9.1) *Proof.* The first statement is contained in Lemma 4.8. Assume $\dim Z=d-3$, and take the smallest d for which this happens (so that a generic L in Z is generated by its global sections). If $d=3$, we get a non-singular line bundle L with $h^0(L)=2$, $\deg L=3$. L defines a morphism $h\colon C\to\mathbf{P}^1$. If some union of components C_1 goes to a point under h, the intersection of C_1 with the rest of C consists of at most 2 points (since this number is even by hypothesis, and h is of degree 3), so we are in case d); if not, h is a 3-sheeted covering of $\mathbf{P}^1$: that is case a).

(4.9.2) Assume $d\geqq 4$ (hence $g\geqq 6$). The exact sequence (4.8.1) gives $h^0(L^2)=d$, so that we get a $(d-3)$-dimensional subvariety of G_{2d}^{d-1}. By Lemma 4.8, this is possible only if:

(i) $d=4$, $\dim Z=1$,

(ii) $d=5$, $\dim Z=2$, $g\geqq 7$.

Exclusing case d), we may assume that for any decomposition $C=C_1\cup C_2$ with $\# C_1\cap C_2=n$, the following holds:

(A) $n\geqq 4$, and $n\geqq 6$ except if C_1 or C_2 is a rational curve. Moreover since $\omega_C\otimes L^{-2}$ must be non-singular we get:

(B) $\deg L_{|C_i}\leqq p_a(C_i)-1+\dfrac{n}{2}$ $(i=1,2)$.

Furthermore we claim that one must have $\deg L_{|C_i}>0$ for any component C_i of C. Namely for L generic in Z, let us denote by $h_L\colon C\to\mathbf{P}^1$ the morphism defined by L. Let C_0 be the union of the components C_j of C such that $h_L(C_j)=\mathbf{P}^1$ (for L generic in Z). If $h_L^{-1}(\{z\})$ is one-dimensional for some $z\in\mathbf{P}^1$ and L generic, one must have $h_L^{-1}(\{z\})\cap C_0=\{u_1\dots u_4\}$ by (A), and $h^0(C_0,L_{|C_0}(-\sum u_i))\geqq 1$ for L generic in Z.

Let us put $L_{|C_0}=L_0$. If $\deg(L)=4$ (case (i)), we get:

$$L_0\cong\mathcal{O}_{C_0}(\textstyle\sum u_i)\qquad\text{for } L \text{ generic in } Z.$$

In order to get $\dim Z\geqq 1$, we must have $h^0(L_0)\geqq 3$.

Note that L_0 and $\omega_{C_0}\otimes L_0^{-1}$ are non-singular, and $p_a(C_0)\geqq 3$ by (B). Hence Lemma 4.7 gives:

- either $p_a(C_0)=3$ and $\omega_{C_0}\cong\mathcal{O}_{C_0}(\sum u_i)$
- or $p_a(C_0)\geqq 4$ and C_0 has a non-singular g_2^1.

Both cases are excluded (by d)).
If $\deg(L)=5$, we find:

$$L_0 \cong \mathcal{O}_{C_0}(\sum u_i + x); \qquad x \in C_0.$$

Since $\dim Z \geqq 2$, x must be a generic point of some component of C_0; but this contradicts the fact that L is generated by its global sections.

We conclude finally that $C_0 = C$, i.e. $\deg L_{|C_i} > 0$ for any component C_i.

(4.9.3) Now if possibility i) holds, we apply the argument in [M 2, p. 349]; note that the line bundle M can be chosen non-singular. So we get a morphism $h: C \to \mathbf{P}^2$ such that for a generic L in Z, the morphism $C \to \mathbf{P}^1$ defined by L is the composition of h with a projection from $\mathbf{P}^2$ to $\mathbf{P}^1$. We denote by (C'_i) the irreducible components of $C' = h(C)$,

$$d_i = \deg C'_i, \qquad r_i = \deg h_{|h^{-1}(C'_i)}.$$

One has:

$$\deg L_{|h^{-1}(C'_i)} = r_i(d_i - e_i)$$

where $e_i = 1$ or 0 according to whether every center of projection lies on C'_i or not; and:

$$\sum_i r_i(d_i - e_i) = 4 \quad \text{with} \quad \sum_i e_i \leqq 1.$$

Now we examine the various possibilities:

If $\deg C' = 5$, h is birational: since $g \geqq 6$, C is a plane quintic.

If $\deg C' \leqq 4$ and C' is irreducible, one must have $r_1 = 2$, $d_1 = 3$: we get case b).

Suppose $\deg C' \leqq 4$, C' reducible. If $r_i = 1$ for some i, C'_i must have at least 4 intersection points with the rest of C', by (A): the only possibility is $d_1 = d_2 = 2$, $r_1 = 2$, $r_2 = 1$, $e_1 = 1$. But then $\deg L_{|h^{-1}(C'_2)} = 2$, which contradicts (B).

Thus one has $r_i \geqq 2$ for all i; the only possible case is $r_i = 2$ and $d_i - e_i = 1$ $(i = 1, 2)$. But then the intersection $h^{-1}(C'_1) \cap h^{-1}(C'_2)$ contains at most 4 points, and this contradicts either (A) or (B).

(4.9.4) The elimination of possibility (ii) in (4.9.2) is more tedious.

We first suppose $g \geqq 8$. We proceed as in (4.9.3): we fix a non-singular L_0 in Z; then we can choose $(g-8)$ points $P_1 \ldots P_{g-8}$ on C such that:

$$M = \omega_C \otimes L_0^{-1}(-\sum P_i) \quad \text{is non-singular and } h^0(M) = 4,$$
$$h^0(M \otimes L^{-1}) \geqq 1 \quad \text{for any } L \text{ in } Z.$$

We conclude that M defines a morphism $h: C \to \mathbf{P}^3$ such that for L generic in Z, the morphism $C \to \mathbf{P}^1$ defined by L is the composition of h with a projection from $\mathbf{P}^3$ to $\mathbf{P}^1$.

We use the same notation as in (4.9.3):

$$h(C) = C' = \bigcup_i C'_i, \qquad d_i = \deg C'_i, \qquad r_i = \deg h_{|h^{-1}(C'_i)}$$

e_i = intersection number of C_i' with a generic line of projection so that

$\deg L_{|h^{-1}(C_i')} = r_i(d_i - e_i)$ and $\sum r_i(d_i - e_i) = 5$ (C).

One has $e_i \leqq 2$ and $\sum_i e_i \leqq 2$ (if C' has a 2-dimensional family of trisecants, it must be the family of lines lying in a plane Π containing some component C_j'; but if the generic line of projection lay in Π, one would have $d_j - e_j = 0$, which is impossible (4.9.2)).

Now we look at the various possibilities:

i) If h was birational, the curve C' would have degree $\leqq 7$ and genus $\geqq 8$. Suppose C' does not contain any plane curve of degree $\geqq 3$. Then we can find a plane Π in $\mathbf{P}^3$ such that $\Pi \cap C'$ consists of distinct points $P_1, \ldots, P_d$ $(d \leqq 7)$, no 3 of them lying on a line. For $r \geqq 3$, we can always find a surface of degree r passing through $P_1 \ldots P_{k-1}$ and not P_k $(k \leqq d)$: namely, one can find a union of r planes with this property. This implies:

$$h^0(C', h^* \mathcal{O}_{\mathbf{P}}(r)) - h^0(C', h^* \mathcal{O}_{\mathbf{P}}(r-1)) = d \quad \text{for } r \geqq 3.$$

Since $h^1(C', h^* \mathcal{O}_{\mathbf{P}}(r)) = 0$ for r large enough, we get:

$$h^1(C', h^* \mathcal{O}_{\mathbf{P}}(2)) = 0.$$

Hence $h^0(C', h^* \mathcal{O}_{\mathbf{P}}(2)) = 2d + 1 - p_a(C') \leqq 7$.

Thus C' must be contained in 3 linearly independent quadrics, which is impossible.

If C' contains a plane curve of degree $\geqq 3$, one checks easily (using (A)) that $p_a(C') \leqq 6$. Thus h is not birational; C' must be reducible, of degree $\leqq 6$.

ii) *There cannot be any component of degree* 5 by (A) and (B).

iii) *There cannot be any component C_i' of degree* 4: by (C) and (A), one must have $r_i = 1$; by (B), this implies $e_i = 2$ and $n = 6$, i.e. C' is the union of a rational quartic and two trisecants; but this contradicts (A).

iv) *If C_i' is a conic, $e_i = 1$.*

Proof of iv. If $r_i = 2$, iv) follows from (C) and (A). If $r_i = 1$, we get from (B) $n = 6$; this is seen to be incompatible with (C).

v) *There cannot be any component C_i' of degree* 3.

Proof of v. If $e_i = 2$, C' is the union of C_i' and some lines (by iv)); one checks that this always contradicts (A). If $r_i = 1$, $p_a(C_i') = 0$, one gets by (B) $n \geqq 6$, which is impossible by (C). If $r_i = 1$, $p_a(C_i') = 1$, C' is a plane curve; by (A) C' must be the union of C_i' and a rational curve having 4 common points, which is impossible. If $r_i = 2$, $e_i = 1$, one gets a contradiction to (A) (using (C)).

vi) Therefore C' is a union of lines and conics; moreover $d_i - e_i = 1$ for all i, and $\sum r_i = 5$. One checks easily that every choice for the (r_i) leads to a contradiction with (A) or (B).

(4.9.5) We now suppose $g = 7$, C irreducible. Then we can modify the argument in [M2, p. 350] as follows: by Riemann-Roch, we get for L generic in Z: $\omega_C \otimes L^{-2} \cong \mathcal{O}_C(p+q)$, where p and q are non-singular points of C. Let W be the locus of effective divisors of degree 2 on C, C_{reg} the open set of non-singular points of C, $d: JC \to JC$ the multiplication by 2. Choosing base points, we get embeddings:

$$C_{\text{reg}} \subset W \subset JC$$

since the restriction to C_{reg} of any irreducible covering of JC is irreducible (see for instance [Se, § 6, Prop. 10]), we conclude that $d^{-1}(W)$ is irreducible. Consequently we get as in [M 2] $h^0(M) \geqq 3$ for any M with $M^2 \cong \omega_C$. But there is always such an M with $h^0(M)$ even. To see this, we can use Corollary 1.2 if the normalization N of C is not rational; if N is rational, we find an equisingular deformation of C into a hyperelliptic curve C_0 (i.e. such that there exists a two-sheeted covering $p: C_0 \to \mathbf{P}^1$), use Theorem 1.1 and the fact that if $M = p^* \mathcal{O}_{\mathbf{P}}(3)$, one has $M^{\otimes 2} = \omega_{C_0}$ and $h^0(M) = 4$.

Thus in any case we get an M of degree 6 with $h^0(M) \geqq 4$; one checks easily that M is non-singular, hence by Lemma 4.7 C is hyperelliptic.

(4.9.6) Suppose finally $g = 7$, C reducible. Again for L generic in Z we get $\omega_C \otimes L^{-2} \cong \mathcal{O}_C(p+q)$, where p, q are non-singular points of a component C_1. We put $C_0 = \bigcup_{i>1} C_i$, $n = \# C_0 \cap C_1$.

If $p_a(C_0) = 0$, it turns out that we can still apply the argument of (4.9.5). We first notice that the Jacobian of C is isomorphic to the Jacobian of the irreducible curve obtained from C_1 by identifying the points of $C_0 \cap C_1$; from this we deduce as in (4.9.5) that the set of line bundles L such that $\omega_C \otimes L^{-2} \cong \mathcal{O}_C(p+q)$, for some non-singular points $p, q \in C_1$, is irreducible. Thus we get $h^0(M(-p)) \geqq 2$ for any M with $M^2 \cong \omega_C$ and any p non-singular in C_1. Now we must rule out the possibility that every section of M vanishes on C_1; but this is impossible since the kernel of the restriction

$$H^0(C, \omega_C) \to H^0(C_1, \omega_{C|C_1})$$

is $H^0(C_0, \omega_{C_0})$, which is zero. So, we obtain $h^0(M) \geqq 3$, and we conclude as in (4.9.5).

Using (A) and (B), we find two cases with $p_a(C_0) \neq 0$:

a) $\deg L_{|C_1} = 1$, $p_a(C_1) = 0$, $n = 6$.

b) $\deg L_{|C_1} = 2$, $p_a(C_1) = p_a(C_0) = 1$, $n = 6$.

We notice that in both cases $L_{|C_0}$ is fixed (since $L^2_{|C_0} \cong \omega_{C|C_0}$) and $h^0(C_0, L_{|C_0}) = 3$, $h^0(C_1, L_{|C_1}) = 2$. Let $g: C_0 \to \mathbf{P}^2$ be the morphism defined by $L_{|C_0}$; put $C_0 \cap C_1 = \{x_1 \dots x_6\}$. A line bundle in Z corresponds to a morphism $h: C_1 \to \mathbf{P}^1$ (defined by $L_{|C_1}$), plus a projection φ from $\mathbf{P}^2$ to $\mathbf{P}^1$ such that: $\varphi \circ g(x_i) = h(x_i)$ $(1 \leqq i \leqq 6)$. In case a), $L_{|C_1}$ is fixed, so any projection from $\mathbf{P}^2$ to $\mathbf{P}^1$ should conserve the projective relations between the 6 distinct points $g(x_i)$, which is impossible.

In case b), for each degree 2 morphism h there must be a one-dimensional family of projections; this implies that the $g(x_i)$ lies on a conic Q, and the center of projection lies on Q. But then every morphism h should give the same projective relations between the $h(x_i)$, which is easily seen to be impossible.

We are now in position to prove the main theorem of this section. For simplicity of the statement, we assume that $\tilde{C}$ (or equivalently C) is a stable curve

([D-M]); in our situation, this means that we eliminate the case $C=C_1\cup C_2$, with $\# C_1\cap C_2=2$ and C_1 rational (see 4.11.3 below). We say that a curve is hyperelliptic if it can be realized as a two-sheeted covering of $\mathbf{P}^1$.

(4.10) **Theorem.** *Let $\tilde{C}$ be a stable curve of genus $2g-1$, ι an involution of $\tilde{C}$ satisfying (*), $C=\tilde{C}/(\iota)$ the quotient curve, (P,Ξ) the associated Prym variety. Recall that $p_a(C)=g$ and $\dim P=g-1$; we assume $g\geqq 5$. Then:*

If $\dim \operatorname{Sing}\Xi=g-3$, *$C$ is hyperelliptic, or $C=C_1\cup C_2$ with $\# C_1\cap C_2=2$; (P,Ξ) is a product of two principally polarized abelian varieties.*

If $\dim \operatorname{Sing}\Xi=g-4$, *$C$ is hyperelliptic or obtained from a hyperelliptic curve by identifying two points; (P,Ξ) is a hyperelliptic Jacobian.*

If $\dim \operatorname{Sing}\Xi=g-5$, *one of the following holds:*

a) *C is a 3-sheeted covering of $\mathbf{P}^1$; then (P,Ξ)* is a Jacobian.

b) *C is obtained from a hyperelliptic curve by identifying two points; then (P,Ξ) is a Jacobian.*

c) *C is a double cover of a stable curve of genus one and $g\geqq 6$.*

d) *C is a genus 5 curve with one vanishing thetanull (that is, a line bundle N such that $h^0(N)=2$, $N^2\cong\omega_C$) and $h^0(\tilde{C},\pi^*N)$ is even.*

e) *C is a plane quintic and $h^0(\tilde{C},\pi^*\mathcal{O}_C(1))$ is odd.*

f) *C is obtained from a hyperelliptic curve by identifying two pairs of points.*

g) *C is obtained from a genus 4 curve with one vanishing thetanull by identifying two points.*

h) *$C=C_1\cup C_2$ with $\# C_1\cap C_2=4$, and neither C_1 nor C_2 is a rational curve.*

i) *$C=C_1\cup C_2$ with $\# C_1\cap C_2=4$, C_1 is rational and C_2 is a hyperelliptic curve of genus $\geqq 3$.*

j) *$C=C_1\cup C_2$ with $C_1\cap C_2=\{u_1,\dots,u_4\}$, C_1 is rational and $\omega_{C_2}\cong\mathcal{O}_{C_2}(\sum u_i)$ (hence $p_a(C)=6$).*

Proof. Let Z be an irreducible component of $\operatorname{Sing}\Xi$ with $\dim Z\geqq g-5$, and L a generic line bundle in Z. According to Proposition 4.2 we can find two linearly independant sections s,t of L such that $s\otimes\iota^*t=\iota^*s\otimes t$.

(4.10.1) Assume first that the sections s,t have the property that at each singular point of $\tilde{C}$, either s or t is non-zero. Then, by Lemma 4.4, L is of the form $\pi^*M(E)$; that is, for each L in Z we get:

- a line bundle M on C with $h^0(M)\geqq 2$;
- an effective divisor $\pi_* E\in|\omega_C\otimes M^{-2}|$.

(Moreover if $E=0$, one has the supplementary condition $h^0(\pi^*M)$ even.)

Conversely for any such line bundle M and any effective divisor $D\in|\omega_C\otimes M^{-2}|$ we get as in [M 2] a finite number of points in $\operatorname{Sing}\Xi$.

Thus we can bound $\dim Z$ by:

(dimension of possible M's) $+\,h^0(\omega_C\otimes M^{-2})-1$.

Assume C has no non-singular g_2^1 and $\deg M\leqq g-2$; then Lemmas 4.7 and 4.9 give $\dim Z\leqq g-5$, and $\dim Z=g-5$ only in cases a) to d) of Lemma 4.9. This gives cases a), c), e), h), i), j) of the theorem (for plane quintics, the same study as in [M2, p. 347] gives the supplementary condition $h^0(\tilde{C}, \pi^* \mathcal{O}_C(1))$ odd). In case a), (P, Ξ) is a Jacobian by [R].

If $\deg M = g-1$, M is a theta-characteristic; this can give a $(g-5)$-dimensional $\operatorname{Sing} \Xi$ only in genus 5 (case d)).

If C has a non-singular g_2^1, this g_2^1 defines a morphism $g: C \to \mathbf{P}^1$; if at least one component of C is mapped to a point, then C is a union $C_1 \cup C_2$ with $\# C_1 \cap C_2 = 2$, and we conclude by Lemma 4.11 below. If not, g is a two-sheeted covering, and C is hyperelliptic; at this point one can either extend the analysis of two-sheeted coverings of hyperelliptic curves in [M2] to singular curves, or simply notice that a singular hyperelliptic curve is a specialization of a non-singular one, hence the Prym variety must be a hyperelliptic Jacobian or a product of two hyperelliptic Jacobians.

(4.10.2) Next suppose that the two sections s, t of L are such that s and t do not vanish simultaneously on any component of $\tilde{C}$, but vanish simultaneously at some singular points $z_1, \dots, z_n$. Let $f: \tilde{D} \to \tilde{C}$ be the normalization of $\tilde{C}$ at $z_1, \dots, z_n$, $D = \tilde{D}/(\iota)$ the quotient curve, $\pi': \tilde{D} \to D$ the projection; we define: $L_1 = f^* L(-\sum (x_i + y_i))$, where $\{x_i, y_i\} = f^{-1}(z_i)$. Then $Nm(L_1) \cong \omega_D$, and since s and t define global sections of L_1, L_1 has property (P).

First suppose that $\tilde{D}$ is connected. Then we can bound the dimension of possible L_1's as in 4.10.1; but to a line bundle L_1 on $\tilde{D}$ corresponds only a finite number of L's in the Prym variety (see Remark 3.6). Since $p_a(D) = g-n$, the only possibility in order to get $\dim Z \geqq g-5$ is $n=2$ and $D = D_1 \cup D_2$ with $\# D_1 \cap D_2 = 2$, which gives case h), or $n=2$ and D hyperelliptic, which gives case f), or $n=1$ and D hyperelliptic. In the last situation we can have $\dim Z = g-5$ or $g-4$; but both cases are specializations of case a). Namely, put $C' = D \cup R$, where R is a rational curve and $D \cap R = \{\pi'(x_1), \pi'(y_1)\}$. We can find a one-dimensional family of non-singular trigonal curves, with C' as special fibre; then we blow down R and get C as special fibre. It follows that the Prym variety is a Jacobian, which must be hyperelliptic if $\dim \operatorname{Sing} \Xi = g-4$.

Now suppose that $\tilde{D}$ is disconnected: say $\tilde{D} = \tilde{D}_1 \cup \tilde{D}_2$, with $\tilde{D}_1 \cap \tilde{D} = \emptyset$. The hypothesis on s, t implies $h^0(L_{1|\tilde{D}_1}) \geqq 1$, $h^0(L_{1|\tilde{D}_2}) \geqq 1$. Conversely, this is enough to insure that L_1 has property (P)! Namely take $a \in H^0(D_1, L_{1|\tilde{D}_1})$, $b \in H^0(D_2, L_{1|\tilde{D}_2})$, and define $s = (a, 0)$, $t = (0, b)$ via the identification:

$$H^0(\tilde{D}, L_1) = H^0(\tilde{D}_1, L_{1|\tilde{D}_1}) \oplus H^0(\tilde{D}_2, L_{1|\tilde{D}_2})$$

then clearly $\iota^* s \otimes t = s \otimes \iota^* t = 0$.

Let $D_i = \pi'(\tilde{D}_i)$; the locus of effective divisors E on $\tilde{D}_1$ such that $\pi'_* E \in |\omega_{D_1}|$ has dimension $p_a(D_1) - 1$. Thus we get:

$$\dim Z \leqq p_a(D_1) - 1 + p_a(D_2) - 1 = g - n - 1$$

provided neither D_1 nor D_2 is rational. To get $\dim Z \geqq g-5$, we must have either $n=2$, which gives the case $C = C_1 \cup C_2$, $\# C_1 \cap C_2 = 2$, to which we apply Lemma 4.11 below; or $n=4$, which gives case h).

(4.10.3) Finally let us consider the general case. Then it may happen that s and t

vanish simultaneously on some components of $\tilde{C}$; let $\tilde{F}$ be the union of these components, $\tilde{G}$ the union of the other components, $F=\tilde{F}/(\iota)$, $G=\tilde{G}/(\iota)$, $\tilde{F}\cap\tilde{G}=\{x_1 \ldots x_r\}$. Put $L_1=L_{|\tilde{G}}(-\sum x_i)$; then, $Nm\,L_1 \cong \omega_G$ and again L_1 has property (P). Conversely given L_1 and any line bundle N on F such that $Nm\,N \cong \omega_{C|F}$, we get a finite number of line bundles L in Z such that:

$$L_{|\tilde{G}} \cong L_1(\textstyle\sum x_i), \qquad L_{|\tilde{F}} \cong N.$$

The dimension of possible N's is:

$$p_a(\tilde{F})-p_a(F)=p_a(F)-1+\frac{r}{2}$$

(recall that we use the convention that $p_a(F)=1-\chi(\mathcal{O}_F)$ for a not necessarily connected curve F). Assume first that $\tilde{G}$ is connected; then we can bound the dimension of possible L_1's as in (4.10.2) by $p_a(G)-e$, where $e \geqq 3$. Thus, we get:

$$\dim Z \leqq p_a(G)-e+p_a(F)-1+\frac{r}{2}=g-e-\frac{r}{2}.$$

The only new possibility is $r=4$, $e=3$ which gives case i), Finally if $\tilde{G}$ is disconnected, the dimension of possible L_1's is bounded by $p_a(G)-1$ (4.10.2). Therefore:

$$\dim Z \leqq p_a(G)-1+p_a(F)-1+\frac{r}{2}=g-1-\frac{r}{2}.$$

But, this gives no new case. Thus the proof of the theorem is complete, once we prove the following lemma:

(4.11) **Lemma.** *Suppose $\tilde{C}=\tilde{C}_1\cup\tilde{C}_2$, with $\tilde{C}_1\cap\tilde{C}_2=\{p,q\}$. Let $\tilde{C}_i'$ $(i=1,2)$ be the curve obtained from $\tilde{C}_i$ by identifying p and q, P_i the Prym variety associated to $\tilde{C}_i'$ with the involution induced by ι. Then $P\cong P_1\times P_2$ as principally polarized abelian varieties.*

(4.11.1) We first prove the following more general statement:

– *Let $\tilde{C}$ be a curve with ordinary double points, ι an involution of $\tilde{C}$ satisfying $(*)$, (P,Ξ) the associated Prym variety; let $\tilde{N}$ be a curve obtained from $\tilde{C}$ by blowing-up some singular points, such that* $\dim J\tilde{N}=\dim J\tilde{C}-1$. *Define* $R=\operatorname{Ker}(Nm)\subset J\tilde{N}$. *Then R is an abelian variety, the principal polarization on $J\tilde{N}$ induces twice a principal polarization Ψ on R, and (R,Ψ) is isomorphic to (P,Ξ).*

Proof of (4.11.1). As in Proposition 3.5, we get an exact sequence:

$$0\rightarrow \mathbf{Z}/(2)\rightarrow \mathbf{Z}/(2)\times P \xrightarrow{g} R \rightarrow 0$$

where $g\colon P\rightarrow R$ is an isomorphism. Then the proof of Theorem 3.7 applies to this situation and gives the statement about polarizations.

(4.11.2) *Proof of the Lemma.* Let $R_i=\operatorname{Ker}(Nm_{|J\tilde{C}_i})$ $(i=1,2)$. By (4.11.1), the principal polarization on $J\tilde{C}_i$ induces twice a principal polarization Ψ_i, and

(R_i, Ψ_i) is isomorphic to P_i with its principal polarization; then by (4.11.1) again, P is isomorphic to the product $(R_1, \Psi_1)\times(R_2, \Psi_2)$, hence the result.

(4.11.3) The proof of the lemma applies when $\tilde{C}_2$, say, is a rational curve: in that case one finds $P \cong P_1$. This allows to extend the theorem to non-stable curves (with ordinary double points).

5. Exchanged Components

We shall need to study the Prym variety under a more general assumption than hypothesis (*) in § 3. We start from a connected curve $\tilde{C}$ of genus $2g-1$, with ordinary double points, and an involution $\iota: \tilde{C} \to \tilde{C}$. If C is the quotient curve, the norm defines a morphism $Nm: J\tilde{C} \to JC$, and we define the Prym variety P by $P = (\operatorname{Ker} Nm)^0$. It is a group variety, extension of an abelian variety by a torus.

We now assume:

$$(**)\quad \begin{cases} - \ \iota \text{ is not the identity on any component of } \tilde{C}. \\ - \ p_a(C) = g. \\ - \ P \text{ is an abelian variety.} \end{cases}$$

Let us fix some notation:

$n_f =$ # nodes of $\tilde{C}$ fixed under ι, with the 2 branches not exchanged.
$n'_f =$ # nodes of $\tilde{C}$ fixed under ι, with the 2 branches exchanged.
$2n_e =$ # nodes exchanged under ι.
$c_f =$ # components fixed under ι.
$2c_e =$ # components exchanged under ι.
$r =$ # fixed non-singular points of ι.

(5.1) **Lemma.** *The assumptions* (**) *are equivalent to* $r = n'_f = 0$, $n_e = c_e$.

Proof. Let $\tilde{N}$ (resp. N) be the normalization of $\tilde{C}$ (resp. C). The covering $\pi': \tilde{N} \to N$ is ramified at the points of $\tilde{N}$ lying over fixed non-singular points, and fixed singular points with no exchanged branches; hence, by the Hurwitz formula:

$$p_a(\tilde{N}) - 1 = 2(p_a(N) - 1) + \frac{r}{2} + n_f$$

so

$$p_a(\tilde{C}) - 1 = p_a(\tilde{N}) - 1 + 2n_e + n_f + n'_f$$
$$= 2(p_a(N) - 1) + \frac{r}{2} + 2n_e + 2n_f + n'_f.$$

The singular points of C come from singular points on $\tilde{C}$ with no exchanged branches and from exchanged singular points, hence:

$$p_a(C) - 1 = p_a(N) - 1 + n_f + n_e.$$

Therefore

$$p_a(\tilde{C}) - 1 = 2(p_a(C) - 1) + \frac{r}{2} + n'_f.$$

Thus the condition $p_a(C)=g$ is equivalent to $r=n'_f=0$.

We now express the condition that P is an abelian variety; from the diagram:

$$\begin{array}{ccccccccc} 0 & \to & \tilde{T} & \longrightarrow & J\tilde{C} & \longrightarrow & J\tilde{N} & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle Nm} & & \downarrow{\scriptstyle Nm} & & \\ 0 & \to & T & \longrightarrow & JC & \longrightarrow & JN & \to & 0 \end{array}$$

and from the surjectivity of the norm it follows that P is an abelian variety if and only if $\dim \tilde{T} = \dim T$.

Here $\dim \tilde{T} = 2n_e + n_f - 2c_e - c_f + 1$

$\dim T = n_e + n_f - c_e - c_f + 1$

thus $\dim \tilde{T} - \dim T = n_e - c_e$, and the lemma is proved.

So the assumptions (**) are equivalent to (*) if $c_e=0$. When $c_e=n_e\neq 0$, let B be the union of the components of $\tilde{C}$ fixed under ι; we write $\tilde{C}=A\cup A'\cup B$, where $A'=\iota(A)$ and A and A' have no common component. Recall that to the curve C we associate a graph Γ: the vertices of Γ are the irreducible components of Γ, and the edges between two vertices are the intersection points of the two corresponding components. We say that C is tree-like if its graph is a tree and if each irreducible component of C is non-singular.

(5.2) **Proposition.** (i) *If $B=\emptyset$, then $\tilde{C}=A\cup A'$ where A can be chosen connected and tree-like, and $\# A\cap A'=2$.*

(ii) *If $B\neq\emptyset$, then $A\cap A'=\emptyset$; each connected component of A is tree-like and meets B at only one point. B is connected and satisfies condition (*).*

We shall use the following easy lemma:

(5.3) **Lemma.** *Let Γ be a connected graph, ι an involution of Γ without fixed points. There exists a connected subgraph S of Γ such that $S\cap\iota S=\emptyset$ and $S\cup\iota S$ contains every vertex of Γ.*

(Hint: take a connected subgraph S maximal for the property $S\cap\iota S=\emptyset$.)

Proof of the Proposition. If $B=\emptyset$, we write $\tilde{C}=A\cup A'$, where A corresponds to the subgraph S in Lemma 5.3. Let v be the number of vertices of S, e the number of edges, s the number of singular points on A which belong to only one component. The equality $c_e=n_e$ gives:

$$v=e+s+\tfrac{1}{2}(\# A\cap A').$$

Since $1-v+e\geq 0$, we get $s=0$, $\# A\cap A'=2$ and $1-v+e=0$, that is A is tree-like.

Assume $B\neq\emptyset$. By Lemma 5.1, the points of $A\cap A'$ are exchanged under ι; we put:

$$\# A\cap A'=2r,$$

$$\# A\cap B=\# A'\cap B=m.$$

$i_A=\#$ irreducible components of A.

$c_A = \#$ connected components of A.
$n_A = \#$ singular points of A.
$2b = \#$ singular points of B exchanged under ι.

Then $c_e = i_A$, $n_e = n_A + r + m + b$.

Recall that for any curve A, $n_A - i_A + c_A \geqq 0$ (this is the first Betti number of the graph of A, plus the number of double points of A which belong to only one component).

Thus, if $c_e = n_e$:

$$0 = n_A - i_A + r + m + b \geqq r + m + b - c_A.$$

Since $\tilde{C}$ is connected, any component of $A \cup A'$ meets B. But one can choose A in such a way that a connected component of $A \cup A'$ is either a connected component of A or A', or can be written $D \cup \iota D$ where D is connected (by Lemma 5.3). From this, we get $m \geqq c_A$, hence:

$$0 \geqq r + m + b - c_A \geqq r + b \geqq 0$$

and $r = 0$, $m = c_A$, $n_A - i_A + c_A = 0$, $b = 0$, which gives the proposition.

(5.4) **Theorem.** *Under the assumptions* (**), *any theta divisor of* $J\tilde{C}$ (§ 2) *induces on* P *twice a principal polarization* Ξ. *In case* (i) *of Proposition* 5.2, (P, Ξ) *is isomorphic to the Jacobian variety* JA *(with its principal polarization); in case* (ii), (P, Ξ) *is isomorphic to the product* $JA \times Q$ *(with the product polarization), where* Q *is the Prym variety associated to* (B, ι).

Proof. We recall that if (A, Θ_A), (B, Θ_B) are two principally polarized abelian varieties, the divisor $\Theta_A \times B + A \times \Theta_B$ defines a principal polarization on $A \times B$, which we call the product polarization.

In case (ii), $J\tilde{C}$ is isomorphic to $JA \times JA \times JB$, and $P = (\mathrm{Ker}\, Nm)^0$ to the subvariety $j(JA) \times Q$, where j is the embedding of JA in $JA \times JA$ defined by $j(x) = (x, -x)$ (note that since A is tree-like, JA is an abelian variety). If B' is the normalization of B, the polarization induced on P by a theta divisor of $J\tilde{C}$ is the pull-back of the product polarization on $JA \times JA \times JB'$ (by Proposition 2.2); therefore, it is the product of the polarization on $j(JA)$ induced by the product polarization of $JA \times JA$, and the polarization on Q induced by a theta divisor of JB. The result follows by noting that the product polarization on $JA \times JA$ induces twice the principal polarization on $j(JA)$.

In case (i), $\tilde{C}$ is obtained from the disjoint union $A \coprod A'$ $(A' = A)$ by identifying a point p (resp. q) in A with a point q (resp. p) in A'; $C = \tilde{C}/\iota$ is obtained from A by identifying p and q, and $\pi\colon \tilde{C} \to C$ is an unramified two-sheeted covering.

We look at the exact sequences:

$$\begin{array}{ccccccccc} 0 \to & \tilde{T} & \longrightarrow & J\tilde{C} & \longrightarrow & JA \times JA & \to 0 \\ & \downarrow{\scriptstyle Nm} & & \downarrow{\scriptstyle Nm} & & \downarrow{\scriptstyle m} & \\ 0 \to & T & \longrightarrow & JC & \longrightarrow & JA & \to 0 \end{array}$$

here m is the addition morphism; $\tilde{T}$ and T are isomorphic to the multiplicative group of k. One checks immediately that $Nm: \tilde{T} \to T$ is an isomorphism; hence $P \cong \operatorname{Ker} m = j(JA)$, and by the preceding remark the polarization induced on P is twice the principal polarization of the Jacobian variety JA.

6. Compactification of the Prym Mapping

In this section we apply the results of [D-M] to "compactify" the mapping which associates to a curve plus a 2-sheeted covering the corresponding Prym variety. The natural language here is the theory of stacks, used in [D-M]. However, to avoid excessive technicality, we first prove the main result over **C**, with a more down-to-earth language; then we consider the situation over **Z**, using algebraic stacks.

(6.1) **Construction.** *There exists an irreducible complete variety $\bar{S}$ over k, a family of stable curves $q: \tilde{\mathscr{C}} \to \bar{S}$, and a $\bar{S}$-involution $\iota: \tilde{\mathscr{C}} \to \tilde{\mathscr{C}}$ such that:*

a) *For each s in $\bar{S}$, the induced involution $\iota_s: \tilde{\mathscr{C}}_s \to \tilde{\mathscr{C}}_s$ is different from the identity on each component of $\tilde{\mathscr{C}}_s$.*

b) *$\tilde{\mathscr{C}}_s$ has genus $2g-1$, and the quotient curve $\tilde{\mathscr{C}}_s/(\iota_s)$ has genus g.*

c) *For any non-singular curve C of genus $2g-1$ with a fixed point free involution ι, the pair $(\tilde{C}, \iota)$ is isomorphic to $(\tilde{\mathscr{C}}_s, \iota_s)$ for some s in $\bar{S}$.*

We start with a complete family of stable curves of genus $2g-1$

$$q: \tilde{\mathscr{X}} \to T,$$

where T is a complete, irreducible variety ([D-M]). Using [D-M, p. 84], we see that the functor of T-involutions of $\tilde{\mathscr{X}}$ is representable by a scheme finite over T; this means that we can find a complete variety I, a finite morphism $r: I \to T$, and an I-involution σ of $\tilde{\mathscr{X}} \times_T I$, such that for any t in T, the correspondence:

$$r^{-1}(t) \to \{\text{involutions of } \tilde{\mathscr{X}}_t\},$$
$$x \mapsto \sigma_{|\tilde{\mathscr{X}}_t \times \{x\}}$$

is one-to-one.

Since the moduli space of non-singular curves of genus g with a two-sheeted covering is irreducible ([D-M]), we can find an irreducible component $\bar{S}$ of I such that property c) is satisfied, where we denote by $(\tilde{\mathscr{C}}, \iota)$ the pull-back of $(\tilde{\mathscr{X}} \times_T I, \sigma)$ over $\bar{S}$.

Suppose ι equals the identity on some component D of $\tilde{\mathscr{C}}_s$. One can find an open set $\tilde{U}$ in $\tilde{\mathscr{C}}$, stable under ι, smooth over $\bar{S}$, such that $\tilde{U} \cap D \neq \emptyset$. The quotient $U = \tilde{U}/(\iota)$ is smooth over $\bar{S}$, hence the finite morphism $\pi: \tilde{U} \to U$ is flat (use for instance [EGA IV 11.3.11 + 15.4.2]). Since π has degree 2, it must be ramified along $\tilde{U} \cap D$, which is impossible since D is reduced. This gives a); since the quotient curve $\tilde{\mathscr{C}}_s/(\iota_s)$ is reduced, its genus is independant of s, hence equal to g.

(6.2) We put $\mathscr{C} = \tilde{\mathscr{C}}/(\iota)$. By a result of M. Artin ([A]), the Jacobians of the curves $\tilde{\mathscr{C}}_s$ (resp. $\mathscr{C}_s$) fit together in an algebraic space over $\bar{S}$, denoted by $\underline{\operatorname{Pic}}^0(\tilde{\mathscr{C}}/\bar{S})$

(resp. $\underline{\mathrm{Pic}}^0(\mathscr{C}/\bar{S})$). The norm defines a morphism:

$$Nm: \underline{\mathrm{Pic}}^0(\tilde{\mathscr{C}}/\bar{S}) \to \underline{\mathrm{Pic}}^0(\mathscr{C}/\bar{S}).$$

Let $\bar{\mathscr{P}} = (\mathrm{Ker}\, Nm)^0$. It follows from the general theory of group schemes (e.g. [SGA 3 exp. VI_B Cor. 4.4]) that $\bar{\mathscr{P}}$ is a smooth algebraic space over $\bar{S}$, whose fibre over s is the Prym variety associated to $(\tilde{\mathscr{C}}_s, \iota)$.

The set S of points s in $\bar{S}$ such that $\bar{\mathscr{P}}_s$ is an abelian variety is open. By Lemma 1.3 in [FGA exp. 236], the restriction $\mathscr{P}$ of $\bar{\mathscr{P}}$ to S is proper over S.

Moreover, according to Remark 2.4, by choosing a line bundle L on $\tilde{\mathscr{C}}$ such that $\deg(L_s) = 2g-2$ and $h^0(L_s) = 0$ for each s, we can define (locally on S for the étale topology) a divisor Θ over $\underline{\mathrm{Pic}}^0(\tilde{\mathscr{C}}/S)$, and the restriction of Θ to $\mathscr{P}$ gives a S-morphism:

$$\rho: \mathscr{P} \to \hat{\mathscr{P}}$$

such that $\rho = 2\rho'$, where ρ' is an isomorphism from $\mathscr{P}$ onto $\hat{\mathscr{P}}$ (by Theorems 3.7 and 5.4).

Since ρ does not depend on the choice of L, the polarization ρ' is defined globally over S, so that we get a flat family of principally polarized abelian varieties over S, hence a morphism:

$$p: S \to \mathscr{A}_{g-1}$$

where $\mathscr{A}_{g-1}$ is the (coarse) moduli space of principally polarized abelian varieties of dimension $(g-1)$ over k (see [M 4]. Over $\mathbf{C}$, $\mathscr{A}_g$ is the quotient of the Siegel upper half-space H_g by the modular group $Sp(2g, \mathbf{Z})$).

(6.3) **Proposition.** *The mapping p is proper.*

Proof. Using the valuative criterion of properness ([EGA II, 7.3.8]) and the completeness of $\bar{S}$, we are reduced to prove the following:

Let T be the spectrum of a complete discrete valuation ring, η its generic point; also let $\tilde{\mathscr{C}} \to T$ be a family of stable curves with involution, such that $(\tilde{\mathscr{C}}_\eta, \iota)$ satisfies condition (**) and the Prym variety $\mathscr{P}_\eta$ has good reduction (i.e. extends to an abelian variety over T). Then $\mathscr{P}_s$ is an abelian variety.

But now since $\mathscr{P}_s$ is an extension of an abelian variety by a torus, it is isomorphic to the neutral component of the Neron model of $\mathscr{P}_\eta$ over T ([SGA 7 IX, 3.2]) which is abelian by hypothesis. Hence p is proper.

(6.4) **Proposition.** *Assume $k = \mathbf{C}$. Then every principally polarized abelian variety of dimension $g \leqq 5$ is a (generalized) Prym variety.*

Proof. In view of Proposition 6.3 and of the irreducibility of $\mathscr{A}_{g-1}$, it suffices to prove that p is generically surjective, and this is a classical result by Wirtinger ([W, § 59]). A different proof of Wirtinger's theorem can be given as follows: using Theorem 5.4 one sees easily that every Jacobian variety is a specialization of a Prym variety, hence $p(S)$ contains the Jacobians. This gives the result for $g \leqq 3$; and also for $g = 4$, since $p(S)$ is irreducible, contains the divisor of Jacobians and some other abelian varieties.

In genus 5, it is enough to prove that for a suitable choice of S, the morphism p is unramified at some point s of S. If $C=\mathscr{C}_s$ is non-singular, this is easily seen to be equivalent to the following: the natural map:

$$\varphi: S^2 H^0(C, \omega_C \otimes \eta) \to H^0(C, \omega_C^{\otimes 2})$$

(where η is the line bundle with $\eta^2 = \mathcal{O}_C$ associated to the 2-sheeted covering $\pi: \tilde{C} \to C$) is an isomorphism.

We start from a non-singular curve X of genus 5, not trigonal, without vanishing theta-nulls. X is a complete intersection of 3 quadrics (P), (Q), (R) in $\mathbf{P}^4$; we identify the set of quadrics $(\lambda P + \mu Q + \nu R)$ containing X to a projective plane Π. Inside of Π we consider the discriminant locus C, which is a non-singular curve of degree 5 (hence of genus 6). The points of C correspond to quadrics of rank 4 through X; these quadrics contain two systems of generatrices, which define an unramified 2-sheeted covering of C, hence a line bundle η on C with $\eta^2 \simeq \mathcal{O}_C$.

Besides the embedding $i: C \to \Pi$, we consider an other embedding $j: C \to \mathbf{P}^4$ defined by:

$j(x)$ = focus of the singular quadric corresponding to x. One checks that: $j^* \mathcal{O}_{\mathbf{P}}(1) = i^* \mathcal{O}_\Pi(2) \otimes \eta$.

Now in the product embedding $C \xrightarrow{(i,j)} \Pi \times \mathbf{P}^4$, C is a complete intersection: in fact it is defined by the 5 equations:

$$\lambda P'_{X_i} + \mu Q'_{X_j} + \nu R'_{X_i} = 0; \qquad i = 0, \ldots, 4.$$

Therefore if p and q are the projections from $\Pi \times \mathbf{P}^4$ onto Π and $\mathbf{P}^4$, and $E = [p^* \mathcal{O}_\Pi(1) \otimes q^* \mathcal{O}_{\mathbf{P}}(1)]^5$, we get a resolution of $\mathcal{O}_C$ by the Koszul complex:

$$0 \to \Lambda^5 E \to \cdots \to E \to \mathcal{O}_{\Pi \times \mathbf{P}^4} \to \mathcal{O}_C \to 0.$$

Taking tensor products with $q^* \mathcal{O}_{\mathbf{P}}(2)$, we conclude by standard arguments that the restriction map:

$$j^*: H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}}(2)) \to H^0(C, \omega_C^2)$$

is an isomorphism; but this map can be identified with φ, hence the result.

Now we return to the case of an algebraically closed field k of arbitrary characteristic $\neq 2$.

(6.5) **Theorem.** (i) *Every principally polarized abelian variety of dimension $g \leqq 5$ is a (generalized) Prym variety.*

(ii) *The moduli space $\mathscr{A}_g$ of principally polarized abelian varieties over k is irreducible for $g \leqq 5$.*

Proof. We refer to [D-M] for the definition and the properties of algebraic stacks. All the schemes and stacks we consider are defined over $B = \mathrm{Spec}(\mathbf{Z}[\frac{1}{2}])$.

(6.5.1) We start with the algebraic stack $\mathscr{M}_{2g-1}[\frac{1}{2}]$ classifying stable curves of genus $(2g-1)$. The stack $\mathscr{M}_{2g-1}[\frac{1}{2}]$ is proper over B.

According to [D-M, Th. 1.11], the stack $\mathscr{I}'$ which classifies stable curves of genus $(2g-1)$ with a non trivial involution is finite and unramified over $\mathcal{M}_{2g-1}[\frac{1}{2}]$, hence proper over B. We consider the subset $\bar{\mathcal{I}}$ of $\mathcal{I}'$ which classifies curves with involution $(\tilde{C}, \iota) \to T$ such that for each t in T:

– The involution ι_t induced on $\tilde{C}_t$ is different from the identity on each component of $\tilde{C}_t$;

– $p_a(\tilde{C}_t/(\iota_t)) = g$.

The argument of (6.1) shows that $\bar{\mathcal{I}}$ is an open and closed subset of $\mathcal{I}'$.

We define as in (6.2) the open subset $\mathcal{I}$ of $\bar{\mathcal{I}}$ classifying stable curves with involution such that the associated Prym variety is an abelian variety. We get a morphism of stacks over B:

$$p: \mathcal{I} \to \mathcal{A}'_{g-1}$$

where $\mathcal{A}'_{g-1}$ denotes the algebraic stack classifying principally polarized abelian varieties of dimension $(g-1)$ over B.

(6.5.2) **Lemma.** *p is proper, and surjective if $g \leqq 6$.*

Proof. The proof of Proposition 6.3 gives the properness of p. Since the restriction of p in characteristic zero is surjective (6.4), p is surjective.

This proves part (i) of the Theorem; it remains to prove that the fibres of $\mathcal{I}$ over B are irreducible. By associating to a curve with involution $(\tilde{C}, \iota)$ the quotient curve $\tilde{C}/(\iota)$ we get a morphism of algebraic stacks over B:

$$q: \mathcal{I} \to \mathcal{M}_g[\tfrac{1}{2}].$$

The morphism q is not representable since the involution ι is an automorphism of $(\tilde{C}, \iota)$ which induces the identity on $\tilde{C}/(\iota)$. We introduce the universal curve $\tilde{\mathcal{C}}$ (resp. $\mathcal{C}$) over $\bar{\mathcal{I}}$ (resp. $\mathcal{M}_g[\frac{1}{2}]$), and its smooth open subset $\tilde{\mathcal{C}}_{\text{reg}}$ (resp. $\mathcal{C}_{\text{reg}}$); the stack $\tilde{\mathcal{C}}_{\text{reg}}$ classifies curves with involution $(\tilde{C}, \iota) \to T$, plus a section $\tilde{e}: T \to \tilde{C}_{\text{reg}}$ (we denote by $\tilde{C}_{\text{reg}}$ the open subset of $\tilde{C}$ consisting of points smooth over T). The stack $\mathcal{C}_{\text{reg}}$ classifies stable curves of genus g: $C \to T$, plus a section $e: T \to C_{\text{reg}}$. There is a morphism:

$$r: \tilde{\mathcal{C}}_{\text{reg}} \to \mathcal{C}_{\text{reg}}$$

such that the diagram:

$$\begin{array}{ccc} \tilde{\mathcal{C}}_{\text{reg}} & \xrightarrow{r} & \mathcal{C}_{\text{reg}} \\ \downarrow & & \downarrow \\ \bar{\mathcal{I}} & \xrightarrow{q} & \mathcal{M}_g[\frac{1}{2}] \end{array}$$

is commutative.

(6.5.3) **Lemma.** *The morphism r is representable and finite.*

Proof. This means the following:

a) Let $(\tilde{C}, \iota) \to T$ be a curve with involution (classified by $\bar{\mathcal{I}}$), $\tilde{e}: T \to \tilde{C}_{\text{reg}}$ a section. Then, any T-automorphism of $\tilde{C}$ which commutes with ι, induces the identity on $\tilde{C}/(\iota)$ and leaves $\tilde{e}$ fixed is the identity.

b) Let $C \to T$ be a stable curve of genus g, $e: T \to C_{\text{reg}}$ a section. We consider the functor F which associates to each T-scheme U the set:

$$F(U) = \text{set of isomorphism classes of} \left\{ \begin{array}{l} -\text{ stable curves with} \\ \quad \text{involution } (\tilde{C}, \iota) \to U \\ +\text{ section } \tilde{e}: U \to \tilde{C}_{\text{reg}} \\ +\ U\text{-morphism } \pi: \tilde{C} \to C_U \end{array} \right\}$$

$$\text{such that:} \left\{ \begin{array}{l} -(\tilde{C}, \iota) \text{ is classified by } \bar{\mathscr{I}}; \\ -\pi \circ \tilde{e} = e \\ -\pi \circ \iota = \pi \\ -\pi \text{ induces an isomorphism } \tilde{C}/(\iota) \xrightarrow{\sim} C_U. \end{array} \right.$$

Then, F is representable by a scheme finite over T.

To prove a), we note that the only non-trivial T-automorphism of $\tilde{C}$ which commutes with ι and induces the identity on $\tilde{C}/(\iota)$ is ι itself; but by Lemma 5.1, any fixed point of ι is singular in its fibre, so ι cannot leave $\tilde{e}$ fixed.

Let us prove b). We associate to the data $(\tilde{C}, \iota, \tilde{e}, \pi)$ the coherent sheaf $\mathscr{L}$ on C_U defined by:

$$\pi_* \mathcal{O}_{\tilde{C}} = \mathcal{O}_{C_U} \oplus \mathscr{L}.$$

$\mathscr{L}$ induces on each fibre $(C_U)_u$ a torsion-free rank one coherent sheaf. Note that the algebra structure on $\pi_* \mathcal{O}_{\tilde{C}}$ gives an isomorphism:

$$\lambda: \mathscr{L} \xrightarrow{\sim} \underline{\mathrm{Hom}}(\mathscr{L}, \mathcal{O}_{C_U}).$$

The section $\tilde{e}$ corresponds to an isomorphism:

$$\alpha: e^* \mathscr{L} \xrightarrow{\sim} \mathcal{O}_U$$

such that $(e^* \lambda)(\alpha^{-1}(1)) = \alpha$.

Conversely given $(\mathscr{L}, \alpha)$ such that $\mathscr{L}$ is isomorphic to $\underline{\mathrm{Hom}}(\mathscr{L}, \mathcal{O}_{C_U})$, there is exactly one λ satisfying the preceding condition; from $(\mathscr{L}, \alpha, \lambda)$ we can reconstruct the data $(\tilde{C}, \iota, \tilde{e}, \pi)$. Therefore, our functor F is isomorphic to the functor:

F': $U \rightsquigarrow$ {set of isomorphism classes of pairs $(\mathscr{L}, \alpha)$ on C_U where:

(i) $\mathscr{L}$ is a coherent sheaf whose restriction on each fibre is torsion-free of rank 1;

(ii) $\mathscr{L}$ is isomorphic to $\underline{\mathrm{Hom}}(\mathscr{L}, \mathcal{O}_{C_U})$;

(iii) α is an isomorphism $e^* \mathscr{L} \xrightarrow{\sim} \mathcal{O}_U$.

If we replace condition (ii) by a certain condition on $\deg \mathscr{L}$, then it follows from the work of Oda and Seshadri ([O-S]) that the functor F'' we obtain is representable by a "compactification of $\underline{\mathrm{Pic}}^0(C/T)$" K. The functor F' is a closed subfunctor of F'', hence is representable by a closed subscheme K_2 of K, proper over T; since the fibres of K_2 over T are finite, K_2 is finite over T. This achieves the proof of the lemma.

Now, we consider the "Teichmüller stacks" ${}_G\mathcal{M}_g^0$ and ${}_G\mathcal{M}_g$ ([D-M] p. 106) with $G=\mathbf{Z}/2$. Recall that ${}_G\mathcal{M}_g^0$ classifies smooth curves of genus g: $p: C\to T$ with an element of $H^0(T, R^1 p_*(\mathbf{Z}/2))$, and ${}_G\mathcal{M}_g$ is the normalization of $\mathcal{M}_g[\frac{1}{2}]$ with respect to ${}_G\mathcal{M}_g^0$. Let ${}_G\mathscr{C}_{\mathrm{reg}}$ (resp. ${}_G\mathscr{C}^0$) be the pull-back of $\mathscr{C}_{\mathrm{reg}}$ over ${}_G\mathcal{M}_g$ (resp. ${}_G\mathcal{M}_g^0$). These stacks are normal, irreducible, and their fibres over $B=\mathrm{Spec}(\mathbf{Z}[\frac{1}{2}])$ are normal and irreducible: this follows from the same result for ${}_G\mathcal{M}_g$ and ${}_G\mathcal{M}_g^0$ (proved in [D-M]) and the fact that the morphism ${}_G\mathscr{C}_{\mathrm{reg}}\to {}_G\mathcal{M}_g$ and ${}_G\mathscr{C}^0\to {}_G\mathcal{M}_g^0$ are flat, with normal fibres and with an irreducible generic fibre. In particular, ${}_G\mathscr{C}_{\mathrm{reg}}$ is the normalization of $\mathscr{C}_{\mathrm{reg}}$ with respect to ${}_G\mathscr{C}^0$.

Let $\tilde{\mathscr{C}}^0$ be the open subset of $\tilde{\mathscr{C}}$ which classifies smooth curves. It follows from the proof of Lemma 6.5.3 that $\tilde{\mathscr{C}}^0$ is isomorphic to ${}_G\mathscr{C}^0$. By the universal property of the normalization and Lemma 6.5.3, there is a morphism:

$$s: {}_G\mathscr{C}_{\mathrm{reg}}\to \tilde{\mathscr{C}}_{\mathrm{reg}}$$

which is finite and surjective. Therefore, there is a surjective morphism ${}_G\mathscr{C}_{\mathrm{reg}}\to\bar{\mathscr{I}}$; it follows that the fibres of $\bar{\mathscr{I}}$ over B are irreducible, and so are the fibres of $\mathscr{I}$. Since p is surjective when $g\leqq 6$ (Lemma 6.5.2), we conclude that the fibres of $\mathscr{A}'_{g-1}$, or equivalently the coarse moduli spaces $\mathscr{A}_{g-1}$ over k, are irreducible for $g\leqq 6$.

Let us mention a first (and immediate) consequence of Theorems 6.5, 5.4 and 4.10.

(6.6) **Proposition.** *Let (A,Θ) be a principally polarized abelian variety of dimension g $(2\leqq g\leqq 5)$.*

1) *If $\dim \mathrm{Sing}\,\Theta=g-2$, (A,Θ) is a product of two principally polarized abelian varieties.*

2) *If $\dim \mathrm{Sing}\,\Theta=g-3$, (A,Θ) is a hyperelliptic Jacobian.*

7. Schottky Problem in Genus 4

(7.1) In the moduli space $\mathscr{A}_4$, we look at the following subvarieties:

$N_0=$ locus of principally polarized abelian varieties with $\mathrm{Sing}\,\Theta\neq\emptyset$.

$\bar{J}_4=$ locus of Jacobian varieties and products of Jacobians.

$\theta_{\mathrm{null}}=$ locus of principally polarized abelian varieties with (at least) one vanishing theta-null, i.e. such that $\mathrm{Sing}\,\Theta$ contains a point of order two.

It is clear that N_0 is a closed algebraic subvariety of $\mathscr{A}_4$; $\bar{J}_4$ is an irreducible divisor in $\mathscr{A}_4$ ([Ho], [D-M]). The subvariety θ_{null} is a divisor: to see this, we can find a finite covering $g:\mathscr{A}'\to\mathscr{A}_4$ such that there exists on $\mathscr{A}'$:

– a complete family of principally polarized abelian varieties $q:\mathscr{X}\to\mathscr{A}'$, corresponding to g;

– a divisor Ψ in $\mathscr{X}$, flat over $\mathscr{A}'$, such that Ψ_t is a symmetric Θ divisor on $\mathscr{X}_t$ for any t in $\mathscr{A}'$;

– a set of sections of q $(e_\sigma)_{\sigma\in\Sigma}$ such that for any t in $\mathscr{A}'$, $\{e_\sigma(t)\}_{\sigma\in\Sigma}$ is the set of points of order two in $\mathscr{X}_t$.

By [M5] (or, over $\mathbf{C}$, by the classical theory of theta functions), the subset: $\Sigma^+ = \{\sigma \in \Sigma,\ e_\sigma(\mathscr{A}') \not\subset \Psi\}$ is non empty; therefore $\theta_{\text{null}} = g(\bigcup_{\sigma\in\Sigma^+} e_\sigma^{-1}(\Psi))$ is a divisor in $\mathscr{A}_4$.

(7.2) **Theorem.** $N_0 = \bar{J} \cup \theta_{\text{null}}$.

Proof. Let (A, Θ) be a principally polarized abelian variety with $\operatorname{Sing}\Theta \neq \emptyset$, which is neither a Jacobian nor a product; we must prove that (A, Θ) has a vanishing theta-null. By Theorems 6.5, 5.4 and 4.10, (A, Θ) is isomorphic to a Prym variety satisfying condition d), f), g) or h) in Theorem 4.10.

We will prove that the line bundles L which give singular points on Ξ are theta-characteristic ($L^2 \cong \omega_{\tilde{C}}$). This is obvious in case d). In the other cases, we start from a curve N which is:

- a hyperelliptic curve of genus 3 in case f);
- a genus 4 curve with one vanishing theta-null in case g);
- the disjoint union of two elliptic curves in case h).

In any case there is a line bundle H on N such that $H^{\otimes 2} \cong \omega_N$, $h^0(H) = 2$. C is obtained from N by identifying points p_1 to p_2, p_3 to p_4, etc. ... We get a diagram:

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{\tilde{f}} & \tilde{C} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{f} & C \end{array}$$

where π' is a two-sheeted covering ramified at $p_1, p_2, \ldots$.

We put

$$(\pi')^{-1}(p_i) = \{\tilde{p}_i\}.$$

The proof of Theorem 4.10 shows that the possible singularities of Ξ arise from line bundles L on $\tilde{C}$ such that:

$$\tilde{f}^* L = \pi'^* H(\textstyle\sum \tilde{p}_i).$$

The ramified covering π' defines a line bundle η on N with:

$$\eta^{\otimes 2} = \mathcal{O}_N(\textstyle\sum p_i) \qquad \pi'^*\eta = \mathcal{O}_{\tilde{N}}(\textstyle\sum \tilde{p}_i)$$

hence

$$\tilde{f}^* L = \pi'^*(H \otimes \eta).$$

Choose M on C such that $f^* M = H \otimes \eta$ and $M^{\otimes 2} = \omega_C$; then: $\tilde{f}^* L = \pi'^* f^* M = \tilde{f}^* \pi^* M$ and $Nm(\pi^* M) = \omega_C$.

Thus after suitable modification of M we get $L = \pi^* M$, hence $L^{\otimes 2} \cong \omega_{\tilde{C}}$.

Using Proposition 3.4, we see that exactly half of the line bundles M on C with $f^* M = H \otimes \eta$, $M^{\otimes 2} = \omega_C$ are such that $\pi^* M$ belongs to P (hence to $\operatorname{Sing}\Xi$). Thus we find:

2 singular points of Ξ in case f);

1 singular point in case g);
3 singular points in case h).

All these points correspond to vanishing theta-nulls.

(7.3) **Proposition.** *The divisor θ_{null} is the closure of the locus of Prym varieties associated to a non-singular curve C of genus five with one vanishing theta-null N and $h^0(\tilde{C}, \pi^* N)$ even.*

Proof. We have to prove that cases f), g), h) are specializations of case d). We keep the notation used in Theorem 7.2.

(7.3.1) In the canonical embedding (defined by ω_C), C is a complete intersection of 3 quadrics, except in case g) if $h^0(H(-p_1-p_2)) \geqq 1$. This is seen as for non-singular curves: C is contained in 3 linearly independant quadrics (by Riemann-Roch); if the intersection of these quadrics is bigger than C, it must be a cubic scroll, and C must be trigonal – thus the only possibility is the exceptional case mentioned above. This case is a specialization of the generic case g), so we may always assume that C is a complete intersection of 3 quadrics.

(7.3.2) Furthermore C is contained in a quadric Q of rank $\leqq 3$. This is because of the diagram:

$$\begin{array}{ccccc}
0 \to \operatorname{Ker} \varphi_C & \longrightarrow & S^2 H^0(C, \omega_C) & \xrightarrow{\varphi_C} & H^0(C, \omega_C^{\otimes 2}) \\
\uparrow & & \uparrow & & \uparrow \\
0 \to \operatorname{Ker} \varphi_N & \longrightarrow & S^2 H^0(N, \omega_N) & \xrightarrow{\varphi_N} & H^0(N, \omega_N^{\otimes 2})
\end{array}$$

which implies that $\operatorname{Ker} \varphi_C$ contains a quadratic relation of rank 3 in case f) and g), and of rank 2 in case h).

(7.3.3) In cases f) and g) we fix the quadric Q and a two-dimensional linear subspace $\Pi \subset Q$, and deform the other quadrics; in case h), we let Π be the singular locus of Q and deform Q in a rank 3 quadric containing Π. Thus we get a family of curves in $\mathbf{P}^4$:

$$\begin{array}{ccc}
\mathscr{C} & \hookrightarrow & \mathbf{P}^4 \times T \\
\downarrow & \swarrow & \\
T & &
\end{array}$$

such that $\mathscr{C}_0 = C$, and $\mathscr{C}_t$ is non-singular for $t \neq 0$.

Let $\mathscr{I}$ = ideal of $(\Pi \times T) \cap \mathscr{C}$ in $\mathscr{C}$, $\mathscr{L} = \underline{\operatorname{Hom}}_{\mathcal{O}_{\mathscr{C}}}(\mathscr{I}, \mathcal{O}_{\mathscr{C}})$. It is easy to check that $\mathscr{L}_{|C} \cong f_*(H)$, while $\mathscr{L}_{|\mathscr{C}_t}$ is a "vanishing theta-null".

Thus we have found a family of curves $\mathscr{C} \to T$ and a coherent sheaf $\mathscr{L}$ on $\mathscr{C}$ such that:

– For $t \neq 0$, $\mathscr{C}_t$ is a non-singular genus 5 curve, $\mathscr{L}_t$ is a line bundle with $\mathscr{L}_t^{\otimes 2} = \omega_{\mathscr{C}_t}$ and $h^0(\mathscr{L}_t) = 2$;

– $\mathscr{C}_0 = C$ and $\mathscr{L}_0 \cong f_*(H)$.

Moreover, from the multiplication $\mathscr{I}\otimes\mathscr{I}\to\omega_{\mathscr{C}/T}^{-1}\subset\mathcal{O}_{\mathscr{C}}$ we get a duality $\mathscr{L}\otimes\mathscr{L}\to\omega_{\mathscr{C}/T}$.

(7.3.4) There exists a line bundle L_0 on C such that:

$$L_0^{\otimes 2}\cong\omega_C, \quad f^*L_0=H\otimes\eta, \quad h^0(L_0) \text{ even}$$

(see Theorem 7.2). The variety of line bundles M on $\mathscr{C}_t$ with $M^{\otimes 2}\cong\omega_{\mathscr{C}_t}$ is étale over T, hence by passing to an étale covering of T, we may extend L_0 to a line bundle L on $\mathscr{C}$ with $L^{\otimes 2}\cong\omega_{\mathscr{C}/T}$. Then by Theorem 1.1, $h^0(\mathscr{C}_t, L_t)$ is even for any t in T.

(7.3.5) Define $\tilde{\mathscr{C}}=\operatorname{Spec}(\mathcal{O}_{\mathscr{C}}\oplus(\mathscr{L}\otimes L^{-1}))$, where the algebra structure is given by the morphism:

$$(\mathscr{L}\otimes L^{-1})\otimes(\mathscr{L}\otimes L^{-1})\to\mathcal{O}_{\mathscr{C}}$$

deduced from the duality $\mathscr{L}\otimes\mathscr{L}\to\omega_{\mathscr{C}/T}$. Then:
- for $t\neq 0$, $\pi_t\colon\tilde{\mathscr{C}}_t\to\mathscr{C}_t$ is an unramified two-sheeted covering, and $h^0(\pi_t^*\mathscr{L}_t)=h^0(\mathscr{L}_t)+h^0(L_t)$ is even;
- $\pi_0\colon\tilde{\mathscr{C}}_0\to\mathscr{C}_0$ can be identified to the morphism $\pi\colon\tilde{C}\to C$.

This proves that situations f), g), h) are specializations of situation d).

(7.3.6) One sees easily that a morphism $\tilde{C}\to C_0$ in case d) with C_0 singular is a specialization of an unramified two-sheeted covering $\tilde{C}\to C$ in case d) with C non-singular. This achieves the proof of Proposition 7.3.

(7.4) **Theorem.** *In characteristic zero, the divisor θ_{null} is irreducible.*

(7.4.1) *Proof.* We can suppose $k=\mathbf{C}$.

Let $f\colon X\to S$ be a complete family of curves of genus 5 with one vanishing theta-null; this means that there is a line bundle N on X whose restriction N_s to X_s satisfies:

$$N_s^2\cong\omega_{X_s}; \quad h^0(N_s)=2 \quad \text{for any } s \text{ in } S.$$

We can take for S an irreducible smooth variety (fix a rank 3 quadric Q_0 in $\mathbf{P}^4$; a generic curve of genus 5 with one vanishing theta-null can be obtained as the intersection of Q_0 with two arbitrary quadrics).

Let us denote by $(JX_s)_2$ the group of points of order two in JX_s, isomorphic to $H^1(X_s,\mathbf{Z}/2)$. We consider the covering Z of S whose fibre over a point s in S is $(JX_s)_2$. The quadratic form q on $(JX_s)_2$ given by:

$$q(\eta)=h^0(X_s, N_s\otimes\eta) \quad (\operatorname{mod} 2)$$

splits the covering Z into two parts: we denote by Z_0 the subcovering corresponding to $q=0$. We have to prove that Z_0 is irreducible; or equivalently, fixing a point s in S, that $\Pi_1(S,s)$ acts transitively on the set of $\eta\in(JX_s)_2$ with $q(\eta)=0$. Observe that it is enough to prove this assertion for *one* family $X\to S$ (not necessarily complete) of curves of genus 5 having a vanishing theta-null. We will reduce this problem to a statement about plane curves by using the following known facts about curves of genus five.

(7.4.2) Let X be a non-singular curve of genus 5, which is neither hyperelliptic nor trigonal. Then X is canonically embedded in $\mathbf{P}^4$ as a complete intersection of three quadrics. Hence the linear system of quadrics in $\mathbf{P}^4$ containing X is a projective plane Π. The discriminant locus (curve of singular quadrics in Π) is a plane quintic with ordinary double points; such a double point appears if and only if the corresponding quadric is of rank 3, which means that X has a vanishing theta-null. The rank 4 quadrics contain two systems of generatrices, which define a double covering $\pi: \tilde{C} \to C$; if C is singular, π is a degree two morphism with property (*) of § 3. In any case the Prym variety associated to $(\tilde{C}, C)$ is isomorphic to the Jacobian JX. Note also that $h^0(\pi^* \mathcal{O}_C(1))$ is odd.

Conversely, given a stable plane quintic C, a degree two morphism $\pi: \tilde{C} \to C$ with property (*) and such that $h^0(\pi^* \mathcal{O}_C(1))$ is odd, there exists a unique curve X of genus 5 such that $JX \cong \mathrm{Prym}(\tilde{C}, C)$.

Furthermore, this construction can be done locally over any base variety. Let $\mathscr{C} \to S$ be a family of stable plane quintics (with $\mathscr{C} \subset \mathbf{P}_S^2$) and $\pi: \tilde{\mathscr{C}} \to \mathscr{C}$ a degree 2 morphism such that for each s in S, the induced morphism $\pi_s: \tilde{\mathscr{C}}_s \to \mathscr{C}_s$ has property (*) and $h^0(\pi_s^* \mathcal{O}_{\mathscr{C}_s}(1))$ is odd. Then locally over S, there exists a family $X \to S$ of non-singular curves of genus 5 such that $JX_s \cong \mathrm{Prym}(\tilde{\mathscr{C}}_s, \mathscr{C}_s)$ for each s.

In the sequel of the proof, we are going to translate the assertion about the action of $\Pi_1(S)$ on $(JX_s)_2$ as an assertion about $\mathscr{C}$ and $\tilde{\mathscr{C}}$, and then we shall prove by monodromy methods that this last assertion is true for a good choice of the family $\mathscr{C} \to S$.

(7.4.3) Now let $g: C' \to T$ be a family of plane quintics with exactly one ordinary double point. We denote by R_t $(t \in T)$ the set of isomorphism classes of degree 2 morphisms $\pi: \tilde{C}' \to C'_t$ with property (*) and such that $h^0(\pi^* \mathcal{O}_{C'_t}(1))$ is odd. Let $p: S \to T$ be the covering of T such that $p^{-1}(t) = R_t$; we put $C = C' \times_T S$. We get a degree 2 morphism over S:

$$\pi: \tilde{C} \to C.$$

To this family of plane quintics with covering we associate (7.4.2) a family $X \to S$ of non-singular curves of genus 5 having one vanishing theta-null. Recall that there is a canonical isomorphism:

$$JX_s \xrightarrow{\sim} \mathrm{Prym}(\tilde{C}_s, C_s).$$

Let N_s be the normalization of C_s. By [M 2, p. 332], there is an isomorphism:

$$(JX_s)_2 \xrightarrow{\sim} H^1(N_s, \mathbf{Z}/2)$$

which is equivariant with respect to the action of $\pi_1(S, s)$.

To the quadratic form q on $(JX_s)_2$ corresponds a form q' on $H^1(N_s, \mathbf{Z}/2)$, invariant under the action of $\Pi_1(S, s)$. We wish to prove that for a certain choice of the family $C' \to T$, the group $\Pi_1(S, s)$ acts transitively on the set of $\eta \in H^1(N_s, \mathbf{Z}/2)$ with $q'(\eta) = 0$, $\eta \neq 0$.

Put $t = p(s)$. The group $\Pi_1(T, t)$ acts on R_t and $\Pi_1(S, s)$ is isomorphic to the isotropy subgroup of $s \in R_t$.

A covering $\pi: \tilde{C} \to C_s$ with property (*) is given by a cycle in $H_1(C_s, \mathbf{Z}/2)$

which does not belong to $\operatorname{Im} H_1(N_s, \mathbf{Z}/2)$; or in terms of cohomology, by a linear form:

$$\varphi: H^1(C_s, \mathbf{Z}/2) \to \mathbf{Z}/2$$

such that

$$\varphi(\omega) = 1,$$

where ω denotes the only non-zero element of the kernel of the cup product on $H^1(C_s, \mathbf{Z}/2)$. The composition:

$$\operatorname{Ker} \varphi \hookrightarrow H^1(C_s, \mathbf{Z}/2) \to H^1(N_s, \mathbf{Z}/2)$$

is an isomorphism respecting the symplectic structure and the action of $\Pi_1(S, s)$.

We now wish to express the condition that the degree 2 morphism $\pi: \tilde{C} \to C_s$ corresponding to φ belongs to R_t, that is $h^0(\pi^* \mathcal{O}_{\mathscr{C}_s}(1))$ is odd. We can embed the family $C \to S$ in a larger family of plane quintics $h: D \to U$ such that the fibres of h over $U - S$ are non-singular. Let us choose a path in U from s to a point u in $U - S$. By Lefschetz theory we get an injective map:

$$H^1(C_s, \mathbf{Z}/2) \hookrightarrow H^1(D_u, \mathbf{Z}/2)$$

which identifies $H^1(C_s, \mathbf{Z}/2)$ to the orthogonal of the "vanishing cycle" ω. The form φ determines an element $\gamma \in H^1(D_u, \mathbf{Z}/2)$ such that:

$$\varphi(x) = (x \cdot \gamma) \quad \text{for } x \in H^1(C_s, \mathbf{Z}/2).$$

There is a unique deformation invariant quadratic form r on $H^1(D_u, \mathbf{Z}/2)$ whose associated bilinear form is the cup-product (see [M 1]); algebraically, it is given by:

$$r(\eta) = h^0(D_u, \mathcal{O}_{D_u}(1) \otimes \eta) + 1 \quad \text{for } \eta \in (JD_u)_2 \cong H^1(D_u, \mathbf{Z}/2)$$

and its Arf invariant is equal to 1.

The condition $\varphi \in R_t$ is equivalent to $r(\gamma) = 1$. Notice that $\operatorname{Ker}(\varphi)$ is the subspace of $H^1(D_u, \mathbf{Z}/2)$ orthogonal to the hyperbolic plane generated by ω and γ. Also, since ω is a vanishing cycle and r is deformation invariant, one must have $r(\omega) = 1$. It follows that the condition $q(\gamma) = 1$ is equivalent to the following:

(A I) The Arf Invariant of the Restriction of r to $\operatorname{Ker}(\varphi)$ *is* 0

Note that this is consistent with the fact that the Arf invariant of the form q (7.4.1) is 0.

(7.4.4) We are now reduced to a monodromy computation. We will use the method of [A'C]. We consider the plane singularity:

$$x^5 + x y^3 = 0.$$

A versal deformation of this singularity is given by:

$$x^5+xy^3+t_1x^4y+t_2x^4+t_3x^3y+t_4x^3+t_5x^2y+t_6x^2+t_7xy$$
$$+t_8y^2+t_9x+t_{10}y+t_{11}=0.$$

Note that the projective completion of this curve has always an ordinary double point at infinity.

Recall the notation of [A'C]:

$\mathscr{D}_\eta$ = a small ball in $\mathbf{C}^{11}$,

$\Delta\subset\mathscr{D}_\eta$ = discriminant locus,

B_ε = a small ball in $\mathbf{C}^2$.

We take $T=\mathscr{D}_\eta-\Delta$, the family $C'\to T$ being given by the restriction of the versal deformation.

Fix $t\in T$; put $C'_t=C$, $F=C\cap B_\varepsilon$. A'Campo's method allows to compute the local monodromy, that is the action of $\Pi_1(T,t)$ on $H_1(F,\mathbf{Z})$, or equivalently on $H^1_c(F,\mathbf{Z})$. But now there is an equivariant exact sequence:

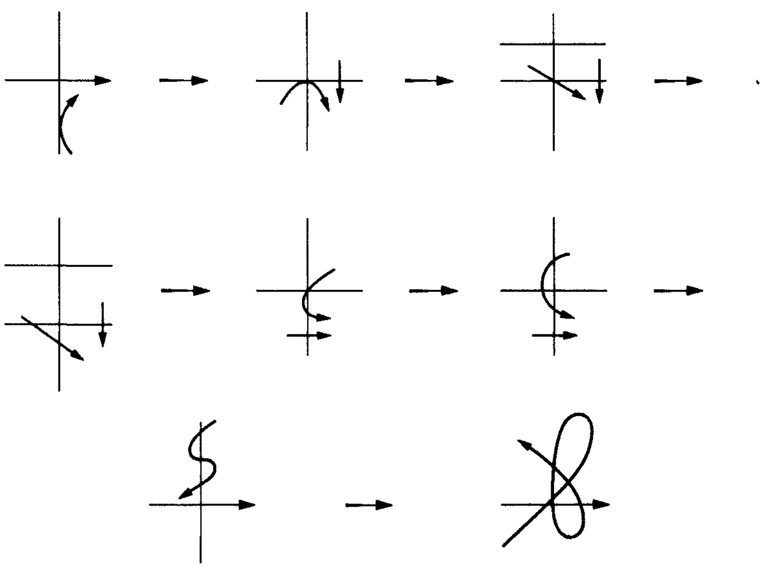
$$0\to H^0(C,\mathbf{Z})\to H^0(C-F,\mathbf{Z})\to H^1_c(F,\mathbf{Z})\to H^1(C,\mathbf{Z}).$$

Since $C-F$ is connected, and $\dim H^1_c(F)=\dim H^1(C)=11$, we get an isomorphism:

$$H^1_c(F,\mathbf{Z}/2)\xrightarrow{\sim}H^1(C,\mathbf{Z}/2)$$

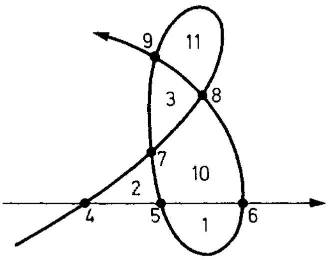
which respects the symplectic structure and the action of $\Pi_1(T)$.

We are going to choose a linear form φ on $H^1(C,\mathbf{Z}/2)$ as in (7.4.3), and prove that the subgroup of $\Pi_1(T,t)$ which leaves φ invariant acts transitively on the set of $\eta\in\operatorname{Ker}\varphi$ with $q(\eta)=0$, $\eta\neq 0$; by what we have seen, this will prove the Theorem. Now we apply A'Campo's method. To get a "confluence de Morse", we use the following succession of blowing up, deformation and blowing down:

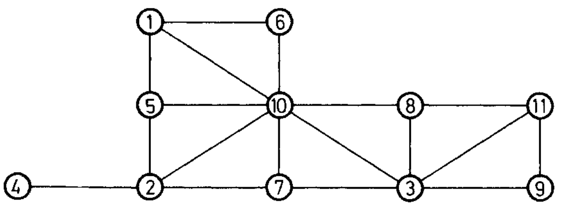


One checks immediately that these operations can be done without altering the double point at infinity.

We obtain finally the following "partage":



which gives the Dynkin diagram:



Recall that this means that there is a basis $\delta_1, \ldots, \delta_{11}$ of $H^1(C, \mathbf{Z}/2)$ with:

$(\delta_i \cdot \delta_j) = 1$ if ⓘ and ⓙ are connected by a line in the diagram $(i \neq j)$;
$= 0$ if they are not.

With the notation of (7.4.3), we have:

$$\omega = \delta_4 + \delta_5 + \delta_6.$$

The quadratic form r is characterized by $r(\delta_i) = 1$ for all i.

We choose the linear form φ on $H^1(C, \mathbf{Z}/2)$ defined by:

$$\varphi(\delta_6) = 1, \quad \varphi(\delta_i) = 0 \quad \text{for } i \neq 6.$$

We must check condition *(A1)*. It is convenient to use a symplectic basis for $\mathrm{Ker}(\varphi)$; we take:

$$\begin{aligned}
\alpha_1 &= \delta_4; & \beta_1 &= \delta_2,\\
\alpha_2 &= \delta_9; & \beta_2 &= \delta_{11},\\
\alpha_3 &= \delta_1; & \beta_3 &= \delta_4 + \delta_5,\\
\alpha_4 &= \delta_4 + \delta_7; & \beta_4 &= \delta_3 + \delta_9 + \delta_{11},\\
\alpha_5 &= \delta_8 + \delta_9; & \beta_5 &= \delta_1 + \delta_3 + \delta_4 + \delta_5 + \delta_7 + \delta_8 + \delta_{10} + \delta_{11}.
\end{aligned}$$

Then:

$$(\alpha_i \cdot \alpha_j)=0, \quad (\beta_i \cdot \beta_j)=0, \quad (\alpha_i \cdot \beta_j)=0, \quad \text{if } i \neq j,$$
$$(\alpha_i \cdot \beta_i)=1.$$

One checks immediately that:

$$\sum_{i=1}^{5} r(\alpha_i) \cdot r(\beta_i)=0,$$

i.e. condition *(A I)* is verified.

The monodromy group is generated by the transvections T_i:

$$T_i(x)=x+(x \cdot \delta_i)\,\delta_i, \qquad x \in H^1(C, \mathbf{Z}/2).$$

We will study the orbits in $\mathrm{Ker}(\varphi)$ of the group G generated by the transvections T_i which leave φ invariant, that is the T_i's for $i \neq 6$. We set up some notation:

If $x=\sum p_i \alpha_i+\sum q_j \beta_j$, we note:

$$x=\begin{bmatrix} p_1 \cdots p_5 \\ q_1 \cdots q_5 \end{bmatrix}.$$

We write $x \equiv y$ if $y=g \cdot x$ for some g in G.

The action of T_4, T_2, T_9. T_{11}, T_1 is very simple; for instance:

$$T_4(x)=\begin{bmatrix} p_1+q_1 & \cdots \\ q_1 & \cdots \end{bmatrix}$$

(which means that only the coordinate p_1 is modified).

Then we have:

$$T_5(x)=\begin{bmatrix} p_1+(q_1+p_3) & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & q_3+(q_1+p_3) & \cdot & \cdot \end{bmatrix},$$

$$T_7(x)=\begin{bmatrix} p_1+(q_1+q_4) & \cdot & \cdot & p_4+(q_1+q_4) & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

$$T_8(x)=\begin{bmatrix} \cdot & p_2+(q_2+q_5) & \cdot & \cdot & p_5+(q_2+q_5) \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

$$T_3(x)=\begin{bmatrix} \cdot & p_2+(p_4+p_2+q_2) & \cdot & \cdot & \cdot \\ \cdot & q_2+(p_4+p_2+q_2) & \cdot & q_4+(p_4+p_2+q_2) & \cdot \end{bmatrix},$$

$$T_{10}(x)=\begin{bmatrix} p_1+r & p_2 & p_3+r & p_4+r & p_5+r \\ q_1 & q_2 & q_3+r & q_4+r & q_5+r \end{bmatrix}$$

with $r=q_1+p_3+q_3+p_4+q_4+p_5+q_5$.

a) *There exists in* $G \cdot x$ *an element with* $p_1=q_1=0$.

Proof. If x does not have this property, we can suppose (using T_4, T_2) that:

$$x=\begin{bmatrix}1 & . & . & . & \cdot\\ 0 & . & . & . & .\end{bmatrix}.$$

Using T_7 and T_4, T_2, we can always suppose $p_4=1$. Then using T_3 and T_9, T_{11} we can suppose $q_4=1$. Then we apply T_7.

b) *There exists in $G\cdot x$ an element with $p_1=q_1=p_2=q_2=0$.*

Proof. Using a) and T_9, T_{11}, we can suppose:

$$x=\begin{bmatrix}0 & 1 & . & . & .\\ 0 & 0 & . & . & .\end{bmatrix}.$$

If $q_5=1$, we apply T_8.

If $q_5=0$, $q_4=0$, note that using eventually T_8, T_{11} we can suppose $q_1+p_3+q_3+p_4+q_4+p_5+q_5=1$. Then, we apply $T_7\,T_8\,T_{10}$.

If $q_5=0$, $q_4=1$, we manage in the same way to obtain $q_1+p_3+q_3+p_4+q_4+p_5+q_5=0$. Then we apply $T_8\,T_{10}\,T_7$.

c) *There exists in $G\cdot x$ an element with $p_i=q_i=0$ $(i=1,2,3)$.*

Proof. Applying T_1 we can suppose:

$$\text{either}\quad x=\begin{bmatrix}0 & 0 & 1 & . & .\\ 0 & 0 & 1 & . & .\end{bmatrix}\quad\text{or}\quad x=\begin{bmatrix}0 & 0 & 1 & . & .\\ 0 & 0 & 0 & . & .\end{bmatrix}.$$

We put $r=p_4+q_4+p_5+q_5$.

c$_1$) $$x=\begin{bmatrix}0 & 0 & 1 & . & .\\ 0 & 0 & 1 & . & .\end{bmatrix}.$$

If $r=1$, we apply $T_{10}\,T_1\,T_2\,T_4\,T_{10}\,T_2\,T_5$.

If $r=0$, using possibly $T_1\,T_{10}\,T_5$ we can suppose $q_4=1$. Then we apply $T_{10}\,T_7$.

c$_2$) $$x=\begin{bmatrix}0 & 0 & 1 & . & .\\ 0 & 0 & 0 & . & .\end{bmatrix}.$$

If $r=1$, we apply $T_{10}\,T_5$.

If $r=0$, using $T_5\,T_1\,T_{10}$ we can suppose $q_4=1$. Then applying $T_5\,T_7$ we find again situation c$_1$).

d) Let us now write:

$$\begin{bmatrix}0 & 0 & 0 & p_4 & p_5\\ 0 & 0 & 0 & q_4 & q_5\end{bmatrix}=\begin{pmatrix}p_4 & p_5\\ q_4 & q_5\end{pmatrix}.$$

Then we have:

$$\begin{pmatrix}1 & p_5\\ q_4 & 0\end{pmatrix}\equiv\begin{pmatrix}1 & p_5+1\\ q_4 & 0\end{pmatrix}\qquad\text{by } T_3\,T_8\,T_9\,T_3,$$

$$\begin{pmatrix}1 & p_5\\ q_4 & 0\end{pmatrix} \equiv \begin{pmatrix}1 & p_5+1\\ q_4+1 & 1\end{pmatrix} \quad \text{by } T_3 T_{11} T_8,$$

$$\begin{pmatrix}0 & p_5\\ q_4 & 1\end{pmatrix} \equiv \begin{pmatrix}0 & p_5\\ q_4+1 & 1\end{pmatrix} \quad \text{by } T_8 T_{11} T_9 T_3 T_8,$$

$$\begin{pmatrix}1 & p_5\\ 0 & q_5\end{pmatrix} \equiv \begin{pmatrix}0 & p_5\\ 1 & q_5\end{pmatrix} \quad \text{by } T_7 T_3 T_2 T_7 T_4 T_2 T_{11} T_3 T_7 T_9 T_3,$$

$$\begin{pmatrix}p_4 & p_5\\ 0 & q_5\end{pmatrix} \equiv \begin{pmatrix}p_4 & p_5+1\\ 1 & q_5+1\end{pmatrix} \quad \text{if } p_4+p_5+q_5=1, \text{ by } T_7 T_2 T_4 T_5 T_1 T_2 T_{10}.$$

From this, it is immediate to conclude that the non-zero elements of $H^1(N_s, \mathbf{Z}/2)$ fall into two orbits under G, according to the value of the quadratic form r. This achieves the proof of Theorem 7.4.

Let us mention the following corollary of Theorem 7.4:

(7.5) **Proposition.** *Assume* $\operatorname{char}(k)=0$. *For* (A, Θ) *generic in* θ_{null}, *the divisor* Θ *has only one ordinary double point (corresponding to the vanishing theta-null).*

Proof. It is clearly enough to prove the result for *one* principally polarized abelian variety (A, Θ) in θ_{null}. Let C be a non-singular curve of genus 5, not trigonal, having precisely two vanishing theta-nulls; i.e. two line bundles, H and H', with:

$$H^{\otimes 2} \cong H'^{\otimes 2} \cong \omega_C \quad \text{and} \quad h^0(H)=h^0(H')=2.$$

Then $\eta = H' \otimes H^{-1}$ satisfies $\eta^{\otimes 2} = \mathcal{O}_C$, thus determines a two-sheeted covering $\pi\colon \tilde{C} \to C$. The only singularity in the divisor Ξ of the associated Prym variety (P, Ξ) corresponds to the line bundle $\pi^* H$; we wish to compute the tangent cone to Ξ at this singular point. According to [M2, p. 343], we must take a basis $s_1 \dots s_4$ of $H^0(\tilde{C}, \pi^* H)$; if we identify the tangent space to P to $H^0(C, \omega_C \otimes \eta)^*$, the equation of the tangent cone is given by the Pfaffian of the matrix:

$$M = (s_i \otimes \iota^* s_j - s_j \otimes \iota^* s_i)_{1 \leqq i, j \leqq 4}.$$

Using the decomposition:

$$H^0(\tilde{C}, \pi^* H) = H^0(C, H) \oplus H^0(C, H \otimes \eta)$$

we get a basis s_1, s_2, t_1, t_2 of $H^0(\tilde{C}, \pi^* H)$ with:

$$\iota^* s_i = s_i \qquad \iota^* t_j = -t_j.$$

Then, the above matrix becomes:

$$M = 2 \left(\begin{array}{c:c} 0 & (a_{ij}) \\ \hdashline -(a_{ij}) & 0 \end{array}\right)$$

with $a_{ij} = s_i \otimes t_j \in H^0(C, \omega_C \otimes \eta)$ $(1 \leqq i, j \leqq 2)$.

Thus, the tangent cone is given by the equation:

$$a_{11} a_{22} - a_{12} a_{21} = 0.$$

Now from the exact sequence:

$$0 \to H^{-1} \xrightarrow{(s_2, -s_1)} \mathcal{O}_C^2 \xrightarrow{(s_1, s_2)} H \to 0$$

we deduce after tensorization with $H \otimes \eta$ that the natural map:

$$H^0(C, H) \otimes H^0(C, H \otimes \eta) \to H^0(C, \omega_C \otimes \eta)$$

is injective. Hence the (a_{ij})'s are linearly independant in $H^0(C, \omega_C \otimes \eta)$. Therefore the tangent cone to Ξ at the singular point is of rank 4, which means that this point is an ordinary double point.

(7.6) Note that this is in contrast with the situation for Jacobians: the Θ-divisor of a Jacobian variety has two ordinary double points, which can collapse in one non-ordinary double point.

(7.7) *Remark.* Let us denote by p the projection from the Siegel upper half-space H_4 onto $\mathscr{A}_4$. Theorem 7.2 and the work of Androtti-Mayer ([A-M]) give a theoretical way to write an equation for the divisor $p^{-1}(\bar{J}_4)$ in terms of theta-nulls. Notice first that the procedure given in [A-M] can be slightly modified to get only one equation for $p^{-1}(N_0)$. Namely the fact that a point z in H_4 belongs to $p^{-1}(N_0)$ is equivalent to the following (here we use freely the notation and results of [A-M], p. 227]):

the linear space of codimension 5 *in* $\mathbf{P}^{15}$ *given by:*

$$\sum_{\mu} \lambda_\mu \cdot \theta_2[\mu](0, z) = 0$$

$$(L) \quad \sum_{\mu} \lambda_\mu \cdot \frac{\partial^2 \theta_2[\mu]}{\partial u_\alpha^2}(0, z) = 0 \qquad \alpha = 1, \ldots, 4$$

has a non-empty intersection with the Kummer-Wirtinger variety $K(z) \subset \mathbf{P}^{15}$.

This gives one condition on z; it is expressed by the vanishing of a function $F(z)$ which is a homogeneous polynomial with rational coefficients in the "theta-nulls" $C(r, \mu, z)$. Note that $F(z)$ is the Chow form of $K(z)$ applied to the Plücker coordinates of the linear system (L).

We don't know whether the divisor given by $\theta(0, z) = 0$ in H_4 is irreducible; however we know that each of its components has multiplicity 1: this follows from the "heat equations" ([A-M]) and Theorem 7.4. Let us denote by $\varphi(z)$ the product of all the theta-null functions corresponding to even characteristics; in classical notation:

$$\varphi(z) = \prod_{\varepsilon \cdot \varepsilon' = 0} \theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (0, z).$$

Then it follows from what we have seen that a certain power of $\varphi(z)$, say $\varphi(z)^s$, divides $F(z)$, and the holomorphic function:

$$\varphi(z)^{-s} \cdot F(z)$$

gives an equation for $p^{-1}(\bar{J}_4)$.

8. Schottky Problem in Genus 5

In genus 5, Theorem 4.10 (together with Theorem 6.5) gives a complete description of the locus N_1 of principally polarized abelian varieties with $\dim \operatorname{Sing} \Theta \geqq 1$. N_1 has already many components. However, if we still denote by θ_{null} the divisor in $\mathscr{A}_5$ of principally polarized abelian varieties with (at least) one vanishing theta-null, we have:

(8.1) **Proposition.** $\bar{J}_5$ is the only component of N_1 not contained in θ_{null}.

Proof. It is well-known that a generic Jacobian has no vanishing theta-null ([F]). Clearly any product of principally polarized abelian varieties belongs to θ_{null}. Thus we must prove that every Prym variety in case c), f), h), i) or j) of Theorem 4.10 has a vanishing theta-null.

(8.1.1) *Case c.* We may assume that C is non-singular, since this is the generic case and θ_{null} is closed.

Let E be an elliptic curve, $h: C \to E$ a two-sheeted covering ramified at the points $\tilde{r}_1, \ldots, \tilde{r}_{10}$ of C, $\sigma: C \to C$ the corresponding involution; to this covering is associated a line bundle δ such that:

$$\delta^2 \cong \mathcal{O}_E(\textstyle\sum r_i) \quad \text{with} \quad r_i = h(\tilde{r}_i).$$

Let $\pi: \tilde{C} \to C$ be an unramified two-sheeted covering, given by a line bundle η on C with $\eta^2 = \mathcal{O}_C$; we consider the Prym variety (P, Ξ) associated to this covering. According to § 4, the singular points of Ξ correspond to line bundles:

$$L = \pi^* h^* M(x+y)$$

where:

- M is a line bundle of degree 2 on E;
- $\pi x + \pi y \equiv h^* p$ where $\mathcal{O}_E(p) = \delta \otimes M^{-2}$;
- $h^0(L)$ is even.

Such a line bundle gives a vanishing theta-null when $L^{\otimes 2} \simeq \omega_{\tilde{C}}$, that is:

$$p = r_i \quad \text{and} \quad L = \pi^*(h^* M(\tilde{r}_i)) \qquad (i = 1, \ldots, 10).$$

We wish to find such an L with $h^0(L)$ even.

Let us fix i and consider the quadratic form q on $(JC)_2$ associated to the "theta-characteristic" $h^* M(\tilde{r}_i)$ (see [M 1] or § 1). Notice that $h^0(L) \equiv q(\eta)$. If we replace M by $M \otimes \alpha$ ($\alpha \in (JE)_2$), we get a new line bundle L' with:

$$h^0(L') = q(h^* \alpha) + q(h^* \alpha \otimes \eta) = q(\eta) + (h^* \alpha \cdot \eta) = q(\eta) + (\alpha \cdot Nm\, \eta) \quad (\text{mod}). \quad (1.2)$$

Now we distinguish two cases:

c_1) $\sigma^* \eta \neq \eta$. Then $Nm\, \eta \neq 0$, so we can always find a theta-characteristic L with $h^0(L)$ even—which gives a vanishing theta-null.

c_2) $\sigma^* \eta = \eta$. Then $\tilde{C}$ is a Galois covering of E, with group $\mathbf{Z}/2 \times \mathbf{Z}/2$. Such a covering can be constructed in the following way: let $R = r_1 + \cdots + r_{10}$ be the ramification divisor of h; we take a partition $R=R_1+R_2$ with $R_i \geqq 0$, $\deg(R_i)$ even; we choose two line bundles δ_1, δ_2 on E such that:

$$\delta_i^2 \cong \mathcal{O}_E(R_i) \quad (i=1,2) \quad \text{and} \quad \delta_1+\delta_2=\delta.$$

Let $h_i\colon C_i \to E$ be the associated 2-sheeted coverings, ramified along R_i $(i=1,2)$; we put $\tilde{C}=C_1 \times_E C_2$, and denote by π_i the projection $\tilde{C} \to C_i$. The morphism $\tilde{C} \to E$ factors through h and thus defines a 2-sheeted covering $\pi\colon \tilde{C} \to C$.

The line bundle L can be written:

$$L=\pi_i^*(h_i^* M(q)) \quad \text{where } i=1 \text{ or } 2 \text{ and } q \in h_i^{-1}(p).$$

According to Proposition 3.4, one value of i gives an L with $h^0(L)$ even and the other gives $h^0(L)$ odd; we fix i such that $h^0(L)$ is even. When q is a ramification point of h_i, we have $L^2 \cong \omega_{\tilde{C}}$, so we get a vanishing theta-null. Now h_i is always ramified, except possibly when $\deg(R_1)=0$, $\deg(R_2)=10$. But in that case we find:

$$h^0(\pi_1^*(h_1^* M(q)))=h^0(h_1^* M(q))+h^0(h_1^*(M \otimes \delta_2^{-1})(q))=5+0=5$$
$$h^0(\pi_2^*(h_2^* M(q)))=h^0(h_2^* M(q))+h^0(h_2^*(M \otimes \delta_1^{-1})(q))=2+2=4.$$

Hence, $h_i=h_2$ is ramified.

(8.1.2) *Case f.* We use the notation of (7.2): so C is obtained from a hyperelliptic curve N by identifying p_1 to p_2, p_3 to p_4;

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{\tilde{f}} & \tilde{C} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{f} & C \end{array}$$

we denote by H the line bundle on N with $\deg(H)=h^0(H)=2$, by $\tilde{p}_i$ the point of $\tilde{N}$ lying over p_i. Here again we may assume that N is non-singular. The points of $\operatorname{Sing}\Xi$ correspond to line bundles L such that:

$$\tilde{f}^* L=\pi'^* H\left(\sum \tilde{p}_i+x+y\right) \quad \text{with } (\pi' x+\pi' y) \in |H|.$$

In order to get $L^{\otimes 2} \cong \omega_{\tilde{C}}$ we must have

$$\tilde{f}^* L^{\otimes 2} \cong \tilde{f}^* \omega_{\tilde{C}} \cong \omega_{\tilde{N}}\left(\sum \tilde{p}_i\right),$$

that is $(2x+2y) \in |\pi'^* H|$.

This happens in particular when $x+y=\pi'^* r$ and r is one of the 10 Weierstrass points of N. In this case, one has:

$$\tilde{f}^* L \simeq \pi'^*(H \otimes \delta(r))$$

where δ is the line bundle on N associated to the covering π' $(\delta^2=\mathcal{O}_N(\sum p_i))$. Then we conclude as in (7.2) that $L^{\otimes 2} \cong \omega_{\tilde{C}}$.

(8.1.3) The proof in cases h), i), j) is identical to (8.1.2); details are left to the reader.

(8.2) *Remarks.*

(8.2.1) One can prove by the same method that every Prym variety of dimension h with $\dim \operatorname{Sing} \Theta \geqq h-4$ and which is not a Jacobian has a vanishing theta-null.

(8.2.2) In genus 5, a careful study of the curve $\operatorname{Sing} \Xi$ described in Theorem 4.10 gives the following result:

Let (A, Θ) be a principally polarized abelian varieties of dimension 5. Then (A, Θ) is a non-hyperelliptic Jacobian if and only if $(\operatorname{Sing} \Theta)/\{\pm 1\}$ is isomorphic to a stable plane quintic.

(8.2.3) Here we use the notation of (7.7).

In [A-M], Andreotti and Mayer give a set of equations

$$f_\alpha(z)=0$$

defining $p^{-1}(\bar{J}_5)$ in the open set $\mathscr{E} \subset H_5$:

$$\mathscr{E}=\{z \in H_g, \theta(u, z) \text{ is irreducible}\}.$$

From Proposition 8.1, we conclude that the f_α's define $p^{-1}(J_5-\theta_{\text{null}})$ in the open set $\varphi(z) \neq 0$. The ideal of $p^{-1}(J_5)$ in $\mathscr{E}$ consists of all functions g holomorphic on $\mathscr{E}$ such that for some $r \geqq 0$:

$$g(z) \cdot \varphi^r(z)=\sum_\alpha a_\alpha(z) \cdot f_\alpha(z) \qquad a_\alpha \text{ holomorphic in } \mathscr{E}.$$

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