Central reflections and nilpotency in exact Mal'cev categories

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joint work with Dominique Bourn

CT2014
Cambridge, 29 Juin - 5 July, 2014

Proposition (Tierney)

A category $\mathbb{E}$ is abelian iff $\mathbb{E}$ is additive and exact.

$$\text{additive} \implies \exists \begin{cases} \text{pullbacks of split epis} \\ \text{pushouts of split monos} \\ \text{null-object} \end{cases} \implies \exists \begin{cases} \text{products} \\ \text{sums} \\ \text{null-object} \end{cases}$$

pre-additive $\implies$ *σ-pointed*

Lemma

For any σ-pointed category $\mathbb{E}$ there is $\theta_{X,Y} : X + Y \to X \times Y$.

- $\theta_{X,Y}$ is invertible iff $\mathbb{E}$ is *linear*;
- $\theta_{X,Y}$ is monic in the category of pointed objects of a topos;
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Definition

A pre-additive category is

- *protomodular* iff, for every split epi f, section and kernel of f strongly generate the domain of f;
- *Mal'cev* iff every reflexive relation is an equivalence relation;
- *semi-abelian* iff protomodular and exact.

Proposition (Bourn)

protomodular $\Longrightarrow$ Mal'cev $\Longrightarrow$ $\theta_{X,Y}$ strong epi for all X, Y

Corollary (for pre-additive categories)

$\mathbb{E}$ additive iff $\mathbb{E}$ and $\mathbb{E}^{op}$ protomodular iff $\mathbb{E}$ and $\mathbb{E}^{op}$ Mal'cev
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semi-abelian $\Longrightarrow$ σ-pointed exact Mal'cev $\Longrightarrow$ finitely cocomplete

Examples (of semi-abelian categories)

Groups, Lie algebras, cocommutative Hopf algebras over a field of characteristic zero, Heyting algebras, loops, ...

Purpose of the talk

A concept of *nilpotency* for σ-pointed *exact Mal'cev* categories based on the notion of *central extension*.

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A concept of *nilpotency* for σ-pointed *exact Mal'cev* categories based on the notion of *central extension*.

Fix an exact Mal'cev category $\mathbb{E}$.
The *discrete* equiv. relation Δ_X is the kernel pair of $1_X : X \to X$.
The *indiscrete* equiv. relation ∇_X is the kernel pair of $t_X : X \to \star_{\mathbb{E}}$.

Definition

Two equiv. relations R, S on X *centralize each other* iff there is a map $p : R \times_X S \to X$ such that $p(x, x, y) = y$ and $p(x, y, y) = x$.

For $R \subset X \times X$ and $S \subset X \times X$ we have $R \times_X S \subset X \times X \times X$.

There is a finest equiv. relation $[R, S]$ (the *Pedicchio-Smith commutator*) such that R and S centralize each other in $X/[R, S]$.

Definition

- An equiv. relation R on X is *central* iff $[R, \nabla_X] = \Delta_X$;
- A *central extension* is a regular epi with central kernel pair.

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- An *n-nilpotent object* of $\mathbb{E}$ is an n-fold central extension of $\star_{\mathbb{E}}$;
- $\mathrm{Nil}^n(\mathbb{E})$ is the subcategory spanned by the n-nilpotent objects;
- $\mathbb{E}$ is an *n-nilpotent category* iff $\mathrm{Nil}^n(\mathbb{E}) = \mathbb{E}$.

Proposition (for pointed exact Mal'cev categories)

- Central equiv. relation are one-to-one with central kernels;
- each central extension is the cokernel of its kernel.

Corollary

The abstract notion of n-nilpotent object yields for groups (Lie algebras) the classical notion of n-nilpotent group (Lie algebra).

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A reflective subcategory $\mathbb{D}$ of $\mathbb{E}$ is a *Birkhoff subcategory* iff $\mathbb{D}$ is closed under taking subobjects and quotients in $\mathbb{E}$.

Lemma (for Birkhoff subcategories of exact Mal'cev categories)

The associated reflection $I : \mathbb{E} \to \mathbb{D}$ is a *Birkhoff reflection*, i.e.

- for each X, the unit $\eta_X : X \to I(X)$ is a regular epi;
- for each regular epi $f : X \to Y$, the direct image under f of the kernel pair of η_X is the kernel pair of η_Y.

Proposition (for finitely cocomplete exact Mal'cev categories)

For each n, the subcategory $\mathrm{Nil}^n(\mathbb{E})$ is a Birkhoff subcategory of $\mathbb{E}$.

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The first Birkhoff reflection $I^1 : \mathbb{E} \to \mathrm{Nil}^1(\mathbb{E})$ is *abelianization*, in particular $\mathrm{Nil}^1(\mathbb{E}) = \mathrm{Ab}(\mathbb{E})$ is an *abelian* subcategory of $\mathbb{E}$.

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The relative Birkhoff reflection $I^{n,n+1} : \mathrm{Nil}^{n+1}(\mathbb{E}) \to \mathrm{Nil}^n(\mathbb{E})$ defined by $\mathrm{Nil}^n(\mathrm{Nil}^{n+1}(\mathbb{E})) = \mathrm{Nil}^n(\mathbb{E})$ is a *central* reflection.

Theorem

The unit of a central reflection is pointwise *affine*.

Corollary

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The first Birkhoff reflection $I^1 : \mathbb{E} \to \mathrm{Nil}^1(\mathbb{E})$ is *abelianization*, in particular $\mathrm{Nil}^1(\mathbb{E}) = \mathrm{Ab}(\mathbb{E})$ is an *abelian* subcategory of $\mathbb{E}$.

Lemma

The relative Birkhoff reflection $I^{n,n+1} : \mathrm{Nil}^{n+1}(\mathbb{E}) \to \mathrm{Nil}^n(\mathbb{E})$ defined by $\mathrm{Nil}^n(\mathrm{Nil}^{n+1}(\mathbb{E})) = \mathrm{Nil}^n(\mathbb{E})$ is a *central* reflection.

Theorem

The unit of a central reflection is pointwise *affine*.

Corollary

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Definition (Fibration of points and affine maps)

Let $\mathrm{Pt}_X(\mathbb{E})$ be the category of split epis of $\mathbb{E}$ with codomain X.
Each $f : X \to Y$ induces an adjunction $f_! : \mathrm{Pt}_X(\mathbb{E}) \leftrightarrows \mathrm{Pt}_Y(\mathbb{E}) : f^*$.
A map f is said to be *affine* if $(f_!, f^*)$ is an adjoint equivalence.

Remark (on affine maps and fibration of points)

- Every affine map has a central kernel relation;
- Affine maps fulfill two-out-of-three;
- If f regular epi then f^* fully faithful;
- A regular epi f is affine iff f^* is essentially surjective;
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Fix a σ-pointed exact Mal'cev category $\mathbb{E}$.

Remark

- Limits in $\mathrm{Nil}^n(\mathbb{E})$ are computed in $\mathbb{E}$;
- Colimits in $\mathrm{Nil}^n(\mathbb{E})$ are computed by applying the reflection I^n to the corresponding colimit in $\mathbb{E}$.

Theorem

$\mathbb{E}$ is n-nilpotent iff for all X, Y the map $\theta_{X,Y} : X + Y \to X \times Y$ exhibits $X + Y$ as an $(n-1)$-fold central extension of $X \times Y$.

The nilpotency degree of $\mathbb{E}$ "*measures*" the difference between sum and product in $\mathbb{E}$.

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$\mathbb{E}$ 2-nilpotent iff $\theta_{X,Y}$ central extension iff $\theta_{X,Y}$ affine extension.

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Definition ($\Xi_{X_1,\ldots,X_n}$ for $n = 2, 3$)

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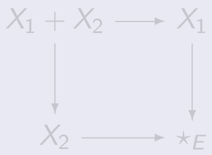

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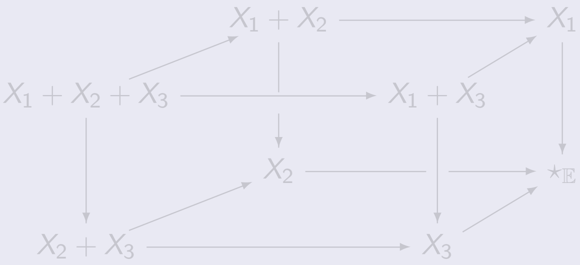

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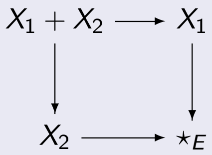

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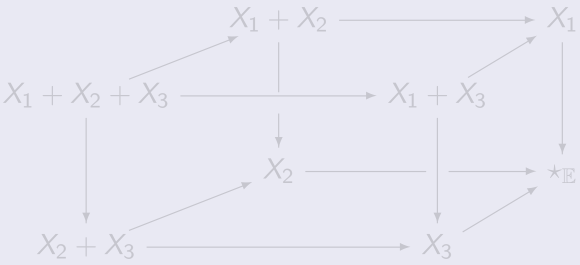

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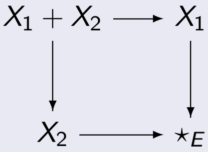

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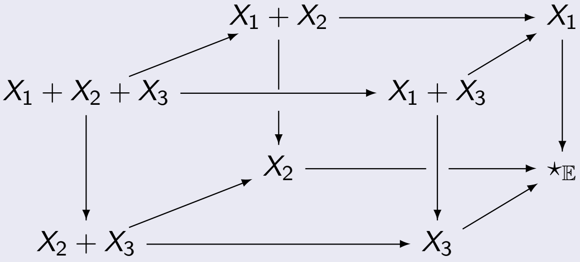

Definition (Identity functors of degree n and cross-effects)

- $P_{X_1,\dots,X_n} = \varprojlim_{[0,1]^n\setminus\{(0,\dots,0)\}} \check{\Xi}$ (limit of the punctured cube);
- Comparison map $\theta_{X_1,\dots,X_n} : X_1 + \cdots + X_n \to P_{X_1,\dots,X_n}$;
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Examples (n=2,3)

- $P_{X,Y} = X \times Y$ and $\theta_{X,Y} : X + Y \to X \times Y$ so that $\diamond(X,Y) = X \diamond Y$ co-smash product (Carboni-Janelidze) resp. second cross-effect (Hartl-van der Linden);
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If the identity functor of $\mathbb{E}$ has degree n then $\mathbb{E}$ is n-nilpotent.

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$\mathbb{E}$ has a quadratic identity functor iff $\mathbb{E}$ is 2-nilpotent and moreover one of the following two conditions is satisfied for all Z:

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If the identity functor of $\mathbb{E}$ has degree n then $\mathbb{E}$ is n-nilpotent.

Theorem (for σ-pointed exact Mal'cev categories)

$\mathbb{E}$ has a quadratic identity functor iff $\mathbb{E}$ is 2-nilpotent and moreover one of the following two conditions is satisfied for all Z:

- cobase change $(i_Z)_! : \mathbb{E} \to \mathrm{Pt}_Z(\mathbb{E})$ along initial maps $i_Z : \star_{\mathbb{E}} \to Z$ preserves binary products;
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Definition

A pre-additive category has *semi-exact sums* iff for all Z, cobase change $(i_Z)_! : \mathbb{E} \to \mathrm{Pt}_Z(\mathbb{E})$ preserves binary products and monos.

Theorem

For any pointed exact Mal'cev category $\mathbb{E}$ *with semi-exact sums* the subcategory $\mathrm{Nil}^n(\mathbb{E})$ has an identity functor of degree n.

Remark

- The category of groups (Lie algebras) has semi-exact sums. It is unclear whether this is preserved under Birkhoff reflection.
- The category $\mathbb{E}$ of groups (Lie algebras) has *centralizers for subobjects.* This implies [Bourn-Gray] that base-change $(i_Z)^* : \mathbb{E} \to \mathrm{Pt}_Z(\mathbb{E})$ preserves *all* colimits, whence quadraticity of $\mathrm{Nil}^2(\mathbb{E})$ without computing sums!

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Definition (Higgins-commutator, cf. Hartl-van der Linden)

$$\begin{array}{ccccc} \diamond(X,\dots,X) & \longrightarrow & X+\cdots+X & \longrightarrow & P_{X,\dots,X} \\ \downarrow & & \downarrow & & \downarrow \\ [X,\dots,X] & \longrightarrow & X & \longrightarrow & X/[X,\dots,X] \end{array}$$

Lemma (Iterated binary commutator)

$\mathrm{Ker}(\eta^n : X \to I^n(X)) = [X,[X,\cdots,[X,X]\cdots]]$

Lemma (Hartl-van der Linden)

$[X,[X,\cdots,[X,X]\cdots]] \subset [X,\dots,X]$

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