Small CW-models for Eilenberg-Mac Lane spaces*

in honour of Prof. Dr. Hans-Joachim Baues
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Part 1. Simplicial sets.

The *simplex category* Δ is the category of finite non-empty ordinals $[n] = \{0, 1, \ldots, n\}$.
A *simplicial set* is a functor $X : \Delta^{\mathrm{op}} \to \mathrm{Sets}$.
The category of simplicial sets is denoted $\widehat{\Delta}$.

The functor $\Delta \to \mathrm{Top} : [n] \mapsto \Delta_n$ induces a *left exact* topological realisation functor

$$\widehat{\Delta} \to \mathrm{Top} : X \mapsto |X| = X \otimes_{\Delta} \Delta_{-}.$$

The latter is the left adjoint part of a *Quillen equivalence*, and thus induces an equivalence of homotopy categories $\mathbf{Ho}(\widehat{\Delta}) \simeq \mathbf{Ho}(\mathrm{Top})$.

The Quillen model structure on simplicial sets has the *monomorphisms* as cofibrations, the *realisation weak equivalences* as weak equivalences, and the *Kan fibrations* as fibrations.
The resulting homotopy theory of simplicial sets is in a strong sense equivalent to the homotopy theory of topological spaces.

Each simplicial set X defines a (normalised) chain complex $N_*(X;\mathbb{Z})$. The chain functor

$$N_* : \widehat{\Delta} \to \mathsf{Ch}(\mathbb{Z})$$

is obtained by left Kan extension from its restriction to Δ. $N_*(\Delta[n];\mathbb{Z})$ is isomorphic to the chain complex of the *CW-complex* Δ_n.

Thus, N_* has a right adjoint $K : \mathsf{Ch}(\mathbb{Z}) \to \widehat{\Delta}$. Actually, $N_* : \widehat{\Delta} \leftrightarrows \mathsf{Ch}(\mathbb{Z}) : K$ is a Quillen pair (for the projective model structure on $\mathsf{Ch}(\mathbb{Z})$).

In particular, for a simplicial set X, and a chain complex (A,n) concentrated in degree n:

$$\mathrm{Hom}_{\mathsf{Ch}(\mathbb{Z})}(N_*(X),(A,n)) \cong \mathrm{Hom}_{\widehat{\Delta}}(X,K(A,n)).$$

Passing to homotopy classes, we get:

$$H^n(X;A) \cong [X,K(A,n)].$$

Since K factors through the category of abelian groups in $\widehat{\Delta}$, the *Theorem of Dold-Kan* yields:

$$\pi_k(K(A,n)) = H_k((A,n);\mathbb{Z}) = \begin{cases} A & \text{if } k = n; \\ 0 & \text{if } k \neq n. \end{cases}$$

Thus, $K(A,n)$ is an Eilenberg-Mac Lane object of type (A,n) in $\widehat{\Delta}$.

They show that $K(A,n) = \overline{W}^n K(A,0)$ for a *simplicial* bar construction $\overline{W}$, and moreover $N_*(\overline{W}^n K(A,0)) \sim B^n N_*(K(A,0))$ for a "more perspicuous" *algebraic* bar construction B.

Purpose of this talk:

• construct a CW-complex whose chain complex is *isomorphic* to $B^n\mathbb{Z}[A]$;
• Serre's calculation of $H^*(K(\mathbb{Z}/2,n);\mathbb{Z}/2)$ and of $(H\mathbb{Z}/2)^*(H\mathbb{Z}/2)$ on cochain level;
• connections to stable homotopy theory.

Part 2. *Geometric Reedy categories.*

For any small category $\mathcal{A}$, $\widehat{\mathcal{A}}$ denotes the category of presheaves on $\mathcal{A}$, and $\mathcal{A}[a]$ denotes the presheaf represented by the object a of $\mathcal{A}$. The *subobjects* of a form the so-called *face-poset* F_a of a. The *retractive quotients* of a form the so-called degeneracy-poset of a.

Def. 1. *A* geometric Reedy category *is a small category $\mathcal{A}$ such that:*

(GR1) any morphism factors uniquely into a retraction followed by a monomorphism;

(GR2) the face-poset of any object is finite and realises to a cone on a sphere;

(GR3) the degeneracy-poset of any object is a lattice;

(GR4) $\widehat{\mathcal{A}}$ has a natural cylinder object.

Remark. Baues uses a slightly weaker notion than (GR1-3) in his analysis of Adams' cobar construction (*DI-categories with retractions*).

(GR2) implies that for any a, F_a is the face-poset of a finite regular CW-complex C_a. In particular, F_a is *ranked*, and the objects a of $\mathcal{A}$ are *graded* by rk(F_a); monics (resp. retractions) rise (resp. lower) this degree. Thus, $\mathcal{A}$ is a *Reedy category* by (GR1-2).

(GR3) implies (i) $\mathcal{A}[a] \times \mathcal{A}[b]$ is the union of its representable subobjects; (ii) any "element" $\mathcal{A}[a] \to X$ factors uniquely as the "degeneracy" of a "non-degenerate" element (*Eilenberg*).

(GR1-3) imply the existence of a geometric realisation functor

$$|-|_{\mathcal{A}} : \widehat{\mathcal{A}} \to \mathsf{Top}$$

mapping $\mathcal{A}[a]$ to C_a, and mapping a general presheaf X to a CW-complex $|X|_{\mathcal{A}}$ with as many cells as there are non-degenerate elements in X.

A natural cylinder for $\widehat{\mathcal{A}}$ is a functorial factorisation of the codiagonal $X \sqcup X \to X$ into a monomorphism $X \sqcup X \rightarrowtail \mathrm{Cyl}(X)$ followed by a realisation weak equivalence $\mathrm{Cyl}(X) \xrightarrow{\sim} X$.

Recall that the *category of elements* $\mathcal{A}/X$ has as objects the elements $\mathcal{A}[-] \to X$, and as morphisms commuting triangles of elements of X.

Lemma 1. *If $\mathcal{A}$ fulfills (GR1-3), then $|f|_{\mathcal{A}} : |X|_{\mathcal{A}} \to |Y|_{\mathcal{A}}$ is a weak equivalence iff $\mathcal{A}/f : \mathcal{A}/X \to \mathcal{A}/Y$ is. Moreover, the nerve of $\mathcal{A}$ is weakly equivalent to $|*|_{\mathcal{A}}$.*

Theorem 1. *(Cisinski) The presheaf category $\widehat{\mathcal{A}}$ for a geometric Reedy category $\mathcal{A}$ is a cofibrantly generated model category with monomorphisms as cofibrations, and realisation weak equivalences as weak equivalences. The fibrations are characterised by horn-filler conditions.*

The realisation functor $|-|_{\mathcal{A}}$ is the left adjoint part of a Quillen equivalence $\widehat{\mathcal{A}} \leftrightarrows \mathrm{Top}/|*|_{\mathcal{A}}$.

Proposition 1. -
(i) Δ is a geometric Reedy category;
(ii) The product of two geometric Reedy categories is a geometric Reedy category;
(iii) For any presheaf X on a geometric Reedy category $\mathcal{A}$, the category of elements $\mathcal{A}/X$ is a geometric Reedy category.

Def. 2. *For a small category $\mathcal{A}$, the* wreath-product $\Delta \wr \mathcal{A}$ *is the category*

- *with objects the m-tupels $(a_1, \ldots, a_m) \in \mathcal{A}^m$ for varying $m \geq 0$;*
- *with morphisms all $(m+1)$-tupels*

$$(\phi; \phi_1, \ldots, \phi_m) : (a_1, \ldots, a_m) \to (b_1, \ldots, b_n)$$

consisting of a simplicial operator $\phi : [m] \to [n]$ and morphisms $(\phi_i)_{1 \leq i \leq m}$ in $\widehat{\mathcal{A}}$ of the form

$$\phi_i : \mathcal{A}[a_i] \to \prod_{\phi(i-1) < k \leq \phi(i)} \mathcal{A}[b_k].$$

A geometric Reedy category $\mathcal{A}$ is called *flat* if the realisation functor $|-|_\mathcal{A}$ is *left exact*; in particular this implies $|*|_\mathcal{A} = *$.

Proposition 2. *For any flat geometric Reedy category $\mathcal{A}$, the wreath-product $\Delta \wr \mathcal{A}$ is again a flat geometric Reedy category.*

Def. 3. *The diagonal $\delta_\mathcal{A} : \Delta \times \mathcal{A} \to \Delta \wr \mathcal{A}$ is defined by*

$$([n], a) \mapsto \overbrace{(a, \ldots, a)}^{n}.$$

Proposition 3. *For any flat geometric Reedy category $\mathcal{A}$, the following diagram commutes:*

$$\begin{array}{ccc}
\widehat{\Delta \wr \mathcal{A}} & \xrightarrow{\;|-|_{\Delta \wr \mathcal{A}}\;} & \mathrm{Top} \\
{\scriptstyle (\delta_\mathcal{A})^*}\downarrow & \nearrow{\scriptstyle |-|_{\Delta \times \mathcal{A}}} & \\
\widehat{\Delta \times \mathcal{A}} & &
\end{array}$$

Part 3. *Iterated wreath-products.*

Let $\Theta_1 = \Delta$ and, inductively, $\Theta_n = \Delta \wr \Theta_{n-1}$. In particular, there is an iterated diagonal

$$\delta_n : \overbrace{\Delta \times \cdots \times \Delta}^{n} \to \Theta_n.$$

The objects of Θ_n can be identified with *finite level-trees* of height $\leq n$. Batanin associates to each such level-tree T an n-graph T_*.

Theorem 2. *Θ_n is isomorphic to the full subcategory of* nCat *spanned by the free n-categories on T_*, where T runs through the finite level-trees of height $\leq n$. The associated nerve functor $nCat \to \widehat{\Theta}_n$ is fully faithful.*

Remark. The composite nCat $\to \widehat{\Theta}_n \overset{\delta_n^*}{\to} \widehat{\Delta^n}$ is the classical n-simplicial nerve of an n-category. The latter is not a full functor for $n \geq 2$! Although the realisations are homeomorphic (see Prop. 3), the cellular structures are different.

Remark. The morphisms of Θ_n can be described in terms of the tree-structure of its objects. We get in particular a *shuffle-formula:*

$$\Theta_n[S] \times \Theta_n[T] = \bigcup_{U \in \mathsf{shuff}(S,T)} \Theta_n[U].$$

The realisation functor associates to each tree T a *convex subset* of a *cube* of dimension equal to the number of edges of T. The 1-level trees realise to *simplices*, the linear trees to *balls*.

Segal's category Γ has as objects the finite sets $\underline{n} = \{1, \ldots, n\}$ and as morphisms $\underline{m} \to \underline{n}$, the m-tupels of pairwise disjoint subsets of $\underline{n}$. Segal's functor $\gamma : \Delta \to \Gamma : [n] \mapsto \underline{n}$ extends to a functor of wreath-products $\gamma \wr \mathcal{A} : \Delta \wr \mathcal{A} \to \Gamma \wr \mathcal{A}$.

Recall that $\phi : [m] \to [n]$ maps to

$$\gamma(\phi) = (]\phi(0), \phi(1)], \ldots,]\phi(m-1), \phi(m)]).$$

Def. 4. *Define* $\gamma_1 : \Theta_1 \xrightarrow{\gamma} \Gamma$, *and inductively,*

$$\gamma_n : \Theta_n = \Delta \wr \Theta_{n-1} \xrightarrow{\gamma \wr \gamma_{n-1}} \Gamma \wr \Gamma \xrightarrow{\alpha} \Gamma$$

where α *is induced by disjoint sum.*

$\gamma^{\mathrm{op}} : \Delta^{\mathrm{op}} \to \Gamma^{\mathrm{op}} \subset \mathrm{Sets}_*$ represents a *circle*, namely $\Delta[1]/\partial\Delta[1]$.

$\gamma_n^{\mathrm{op}} : \Theta_n^{\mathrm{op}} \to \Gamma^{\mathrm{op}} \subset \mathrm{Sets}_*$ represents an n-*sphere*, namely $\Theta_n[1_n]/\partial\Theta_n[1_n]$ where 1_n is the linear tree of height n.

Each Γ-space $A : \Gamma^{\mathrm{op}} \to \mathrm{Top}$ induces a *Segal spectrum* $(\underline{A}(S^n))_{n\geq 0}$.

Proposition 4. *For each* $n \geq 0$, *there is a homeomorphism* $\underline{A}(S^n) \cong |\gamma_n^* A|_{\Theta_n}$.

The structural maps $\underline{A}(S^n) \wedge S^1 \to \underline{A}(S^{n+1})$ are induced by *suspension functors*

$$\sigma_n : \Theta_n \to \Theta_{n+1} : T \mapsto (T).$$

Part 4. *Eilenberg-Mac Lane spaces.*

The Eilenberg-Mac Lane spectrum representing cohomology with coefficients in the abelian group A is induced by the Γ-set

$$HA : \Gamma^{\mathrm{op}} \to \mathrm{Sets} : \underline{n} \mapsto A^{\underline{n}}.$$

i.e. $\underline{\underline{HA}}(S^n) \cong |\gamma_n^*(HA)|_{\Theta_n}$ is an EM-space.

Therefore, the Θ_n-set $K(A,n) := \gamma_n^*(HA)$ is an EM-object of type (A,n) in $\widehat{\Theta}_n$.

$$K(A,n)(T) = A^{\gamma_n(T)} \text{ for } T \in \Theta_n.$$

Proposition 5. $N_*(K(A,n);\mathbb{Z}) \cong B^n\mathbb{Z}[A]$.

Since Θ_n is geometric Reedy, $|K(A,n)|_{\Theta_n}$ is a *CW-complex* with as many cells as there are non-degenerate elements in $K(A,n)$ resp. *pruned level-trees of height* n whose leaves are labelled by non-zero elements of A.

# cells in dim	0	1	2	3	4	5	6	7	8	9
$K(\mathbb{Z}/2\mathbb{Z},1)$	1	1	1	1	1	1	1	1	1	1
$K(\mathbb{Z}/2\mathbb{Z},2)$	1	0	1	1	2	3	5	8	13	21
$K(\mathbb{Z}/2\mathbb{Z},3)$	1	0	0	1	1	2	4	7	13	24

Remark. For finite A, the generating function $\sum_{d\geq 0} c_d t^d$ for the number c_d of cells of $K(A,n)$ in dimension d is a *rational function* of t, which yields for $t=-1$ an *Euler characteristic* of the CW-complex $|K(A,n)|_{\Theta_n}$. We get the "expected" value

$$\chi(K(A,n)) = (\#A)^{(-1)^n}.$$

Serre's calculation of $H^*(K(\mathbb{Z}/2\mathbb{Z},n);\mathbb{Z}/2\mathbb{Z})$.

One considers the path-fibration

$$\Omega K(\mathbb{Z}/2,n) \to PK(\mathbb{Z}/2,n) \to K(\mathbb{Z}/2,n).$$

$H^*(K(\mathbb{Z}/2,n-1);\mathbb{Z}/2))$ is an *abelian* Hopf algebra, i.e. (by Milnor-Moore) the enveloping algebra of a 2-restricted *abelian* Lie algebra. A *PBW-basis* of the latter has been constructed inductively by "saturating" a polynomial basis of $H^*(K(\mathbb{Z}/2,n-1);\mathbb{Z}/2)$ under the *cup-squaring* operation. Borel's Theorem yields then a polynomial basis of $H^*(K(\mathbb{Z}/2,n);\mathbb{Z}/2)$ as the image under *transgression* of the PBW-basis of $H^*(K(\mathbb{Z}/2,n-1);\mathbb{Z}/2)$.

This calculation can be carried out on cochain level as soon as cup squares and transgression can be represented on cochain level.

The cup product is deduced from

$$N^*(X) \otimes N^*(X) \xrightarrow{AW^*} N^*(X \times X) \xrightarrow{\Delta^*} N^*(X).$$

The cohomological transgression is deduced from the "homology-suspension"

$$H_*(K(A, n-1); \mathbb{Z}) \xrightarrow{\sigma_*} H_{*+1}(K(A, n); \mathbb{Z}).$$

A pruned level-tree is called 2-*admissible* if the root-vertex has valence 1, for each vertex the number of incoming edges is a power of 2, and vertices of same height have same number of incoming edges. A cocylce is *monogenic* if it belongs to the dual basis.

Proposition 6. $H^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$ *is a polynomial algebra generated by the monogenic cocycles on 2-admissible trees of height* n.
The latter represent the $Sq^I(e_n)$, $e(I) < n$.