The lattice path operad*

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Third Arolla Conference
on Algebraic Topology
August 19th, 2008

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Part 1. Iterated loop spaces and E_n-operads.

Let $(X, *)$ be a based topological space and $(S^n, *)$ be the n-sphere. Then

$$\Omega^n X = \underline{\mathbf{Top}}_*(S^n, X)$$

is an algebra over the *coendomorphism operad*

$$\mathrm{Coend}(S^n)(k) = \underline{\mathbf{Top}}_*(S^n, \overbrace{S^n \vee \cdots \vee S^n}^{k})$$

where the action is given by *composition* :

$$\begin{array}{ccc}
\mathrm{Coend}(S^n)(k) \times (\Omega^n X)^k & \longrightarrow & \Omega^n X \\
\cong \big\downarrow & & \big\downarrow = \\
\underline{\mathbf{Top}}_*(S^n, (S^n)^{\vee k}) \times \underline{\mathbf{Top}}_*((S^n)^{\vee k}, X) & \longrightarrow & \underline{\mathbf{Top}}_*(S^n, X)
\end{array}$$

The operad $(\mathcal{C}_n(k))_{k\geq 0}$ of little n-cubes is a suboperad of $(\text{Coend}(S^n)(k))_{k\geq 0}$. Therefore any n-fold loop space is a $\mathcal{C}_n$-algebra.

Theorem 1. *(Boardman-Vogt, May, Segal) Any $\mathcal{C}_n$-algebra is up to group completion an n-fold loop space. In particular, $\Omega^n S^n X$ is the group completion of the free C_n-algebra on X.*

Theorem 2. *(F. Cohen) $H_*(\Omega^n S^n X, \mathbb{Z}/p\mathbb{Z})$ is a $H_*(\mathcal{C}_n, \mathbb{Z}/p\mathbb{Z})$-algebra on $H_*(X, \mathbb{Z}/p\mathbb{Z})$ equipped with certain Dyer-Lashof operations.*

For any field k, a $H_*(\mathcal{C}_2, k)$-algebra is called a *Gerstenhaber k-algebra*. The Hochschild cohomology $HH^*(A; A)$ of an associative k-algebra A is a Gerstenhaber algebra, whence *Deligne's conjecture*: Is this structure induced by a dg-E_2-operad action on $CC^*(A; A)$?

Proofs of the Deligne conjecture have been given by Tamarkin, McClure-Smith, Kontsevich-Soibelman and Berger-Fresse.

Part 2. Enriched categories.

Let $\mathcal{E} = (\mathcal{E}, \otimes_{\mathcal{E}}, I_{\mathcal{E}}, \underline{\mathcal{E}}(-,-))$ be a closed symmetric monoidal category, for instance $(\mathbf{Top}, \times, *, \underline{\mathbf{Top}}(-,-))$ or $(\mathsf{Ch}(k), \otimes_k, k, \underline{\mathsf{Hom}}_k(-,-))$.

Def. 1. *An $\mathcal{E}$-category $\mathcal{A}$ consists of objects $A, A', \cdots \in \mathcal{A}_0$ and (for each pair of objects) hom-objects $\underline{\mathcal{A}}(A, A') \in \mathcal{E}_0$, together with*

- *units* $u_A : I_{\mathcal{E}} \to \underline{\mathcal{A}}(A, A), \quad A \in \mathcal{A}_0,$

- *compositions* $\underline{\mathcal{A}}(A', A'') \otimes_{\mathcal{E}} \underline{\mathcal{A}}(A, A') \to \underline{\mathcal{A}}(A, A'')$

fulfilling unit and associativity axioms.

Any closed symmetric monoidal category $\mathcal{E}$ is an $\mathcal{E}$-category. There is a 2-category of $\mathcal{E}$-categories, $\mathcal{E}$-functors and $\mathcal{E}$-natural transformations.

Lemma 1. *The iterated tensor-product* $\otimes_{\mathcal{E}}^k$: $\mathcal{E} \times \cdots \times \mathcal{E} \to \mathcal{E}$ *is an* $\mathcal{E}$*-functor, i.e. there are canonical maps* $\underline{\mathcal{E}}(X,Y)^{\otimes k} \to \underline{\mathcal{E}}(X^{\otimes k}, Y^{\otimes k})$.

The *coendomorphism operad* of an object X of $\mathcal{E}$ is given by

$$\mathrm{Coend}(X)(k) = \underline{\mathcal{E}}(X, X^{\otimes k}), \quad k \geq 0,$$

with the obvious structural maps.

Proposition 1. *Let* X, Y *be two objects of* $\mathcal{E}$*. Assume that* Y *is a commutative monoid in* $\mathcal{E}$*. Then* $\underline{\mathcal{E}}(X,Y)$ *is a* Coend(X)*-algebra.*

The Coend(X)-algebra structure is given by

$$\begin{array}{ccc}
\mathrm{Coend}(X)(k) \otimes \underline{\mathcal{E}}(X,Y)^{\otimes k} & \longrightarrow & \underline{\mathcal{E}}(X,Y) \\
\Big\downarrow \, enrichment & & multiplication \, \Big\uparrow \\
\underline{\mathcal{E}}(X, X^{\otimes k}) \otimes \underline{\mathcal{E}}(X^{\otimes k}, Y^{\otimes k}) & \xrightarrow[composition]{} & \underline{\mathcal{E}}(X, Y^{\otimes k}).
\end{array}$$

Part 3. *Condensation of coloured operads.*

An N*-coloured operad* in $\mathcal{E}$ is given by a collection of objects $\mathcal{O}(n_1,\ldots,n_k;n)$ of $\mathcal{E}$, where $(n_1,\ldots,n_k,n)\in N^{k+1}$, together with units, Σ_k-actions and composition maps

$$\mathcal{O}(n_1,\ldots,n_k;n)\otimes_{\mathcal{E}}\mathcal{O}(m_1,\ldots,m_l;n_i)\xrightarrow{\circ_i}$$
$$\mathcal{O}(n_1,\ldots,n_{i-1},m_1,\ldots,m_l,n_{i+1},\ldots,n_k;n),$$

which are unital, associative and equivariant.

If $N=\{*\}$ then $\mathcal{O}(k)=\mathcal{O}(\overbrace{*,\ldots,*}^{k};*)$ is a symmetric operad in $\mathcal{E}$.

Each N-coloured operad $\mathcal{O}$ defines a category $\mathcal{O}_u$ of *unary operations* with object-set N:

$$\mathcal{O}_u(n,n')=\mathcal{O}(n;n').$$

A coloured operad $\mathcal{O}$ in $\mathcal{E}$ can also be presented as a *multitensor* on $\mathcal{O}_u$ with values in $\mathcal{E}$:

$$\overbrace{\mathcal{O}_u^{\mathsf{op}} \times \cdots \times \mathcal{O}_u^{\mathsf{op}}}^{k} \times \mathcal{O}_u \xrightarrow{\mathcal{O}(-,\ldots,-;-)} \mathcal{E}$$

This defines a *lax symmetric monoidal structure* on $\mathcal{E}^{\mathcal{O}_u}$ by the coend formula:

$$(X_1 \otimes_{\mathcal{O}} \cdots \otimes_{\mathcal{O}} X_k)(n) =$$
$$\int^{n_1,\ldots,n_k} \mathcal{O}(-,\cdots,-;n) \otimes_{\mathcal{E}} X_1(-) \otimes_{\mathcal{E}} \cdots \otimes_{\mathcal{E}} X_k(-).$$

In particular, for each object $\delta \in \mathcal{E}^{\mathcal{O}_u}$, we get a *coendomorphism operad*

$$\mathrm{Coend}_{\mathcal{O}}(\delta)(k) = \underline{\mathrm{Hom}}_{\mathcal{O}_u}(\delta, \delta \otimes_{\mathcal{O}} \cdots \otimes_{\mathcal{O}} \delta).$$

Proposition 2. *Let X be an algebra over the coloured operad $\mathcal{O}$ in $\mathcal{E}$. Let $\delta \in \mathcal{E}^{\mathcal{O}_u}$. Then* $\underline{\mathrm{Hom}}_{\mathcal{O}_u}(\delta, X)$ *is a* $\mathrm{Coend}_{\mathcal{O}}(\delta)$*-algebra.*

$\mathcal{E} = \mathbf{Top}$ or $\mathcal{E} = \mathrm{Ch}(\mathbb{Z})$ contains Sets as the subcategory of *discrete objects* via the *strong monoidal* functor $S \mapsto \sqcup_S I_{\mathcal{E}}$ ($I_{\mathcal{E}} =$ unit of $\mathcal{E}$).

We shall construct a *coloured operad* $\mathcal{L}$ in *Sets*, parametrizing the combinatorial structure of *iterated loop spaces* in the following sense:

- $\mathcal{L} = \cup_{m \geq 0} \mathcal{L}_m$ and $\mathcal{L}_u = \Delta = (\mathcal{L}_m)_u$;

- For the standard object $\delta : \Delta \to \mathcal{E}$, $\mathrm{Coend}_{\mathcal{L}_m}(\delta)$ is an E_m-*operad* in $\mathcal{E}$.

In particular, any $\mathcal{L}_m$-algebra X in $\mathcal{E}$ gives rise to an E_m-algebra $\underline{\mathrm{Hom}}_{\Delta}(\delta, X)$. Being an $\mathcal{L}_m$-algebra in $\mathcal{E}$ is a combinatorial property !!

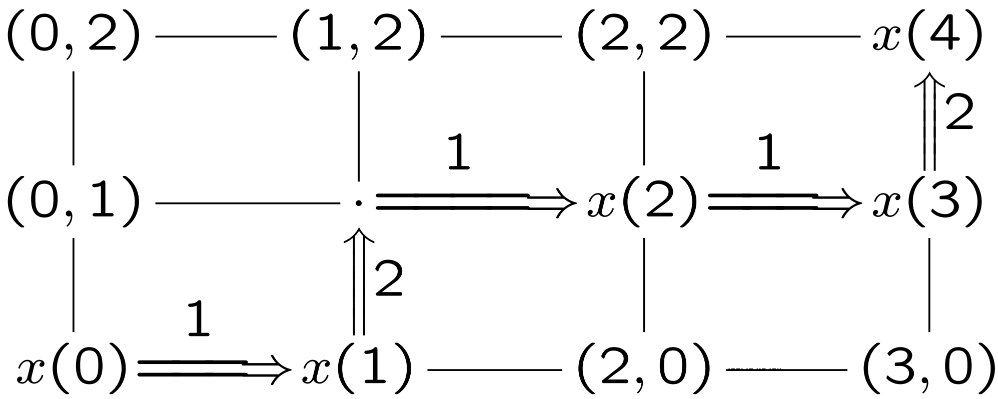
Part 4. *The lattice path operad.*

The *funny tensor product* of categories $\mathcal{A} \otimes \mathcal{B}$ has $(A, B) \in \mathcal{A}_0 \times \mathcal{B}_0$ as objects, and "free" compositions of $(f, 1_B) : (A, B) \to (A', B)$ and $(1_A, g) : (A, B) \to (A, B')$ as morphisms.

Def. 2. *The lattice path operad is the* $\mathbb{N}$-*coloured operad in sets defined by*

$$\mathcal{L}(n_1, .., n_k; n) = \mathrm{Cat}_{*,*}([n+1], [n_1+1] \otimes \cdots \otimes [n_k+1]).$$

Example. Let $x \in \mathcal{L}(2, 1; 3)$ be the lattice path:



The path is determined by the sequence of "directions" and "stops": $x = 1|21|1|2$.

$\mathcal{L}(n_1, \ldots, n_k; n)$ may be identified with the set of finite sequences containing $n_1 + 1$ times 1, $n_2 + 1$ times 2, ... ,$n_k + 1$ times k, and n (possibly multiple) stop's. Under this identification, the operad composition map is given by renumbering and substitution:

$$1||1\underline{2}|3|\underline{2} \circ_2 \underline{1}|\underline{32} = 1||1\underline{2}|5|\underline{43}.$$

Lemma 2. $\mathcal{L}_u = \Delta$. *(Joyal-duality)*

$$\mathcal{L}_u(n', n) = \mathsf{Cat}_{*,*}([n{+}1], [n'{+}1]) = \Delta([n'], [n]).$$

Let $\Delta\Sigma$ be the category of finite sets and finite set mappings *equipped with total orderings of the fibers*, cf. Feigin-Tsygan, Krasauskas and Fiedorowicz-Loday. (*Crossed simplicial group*).

Proposition 3. *(Extended Joyal-duality)*

$$\mathcal{L}(n_1, \ldots, n_k; n) = \{x \in \Delta\Sigma([n_1] * \cdots * [n_k], [n]) \quad \textit{sth. } \forall i : x|_{[n_i]} \in \Delta([n_i], [n])\},$$

where the operad composition is given by join and composition in $\Delta\Sigma$.

Def. 3. *(Filtration by complexity)*
For $1 \leq i < j \leq k$*, let* p_{ij} *be the projection*

$$[n_1 + 1] \otimes \cdots \otimes [n_k + 1] \to [n_i + 1] \otimes [n_j + 1].$$

Let $a_{ij}(x)$ *be the number of angles in the lattice path* $p_{ij} \circ x$*, and* $c(x) = \max_{i<j} a_{ij}(x)$*. Then,*

$$\mathcal{L}_m(n_1, .., n_k; n) = \{x \in \mathcal{L}(n_1, .., n_k; n) \,|\, c(x) \leq m\}$$

defines a suboperad $\mathcal{L}_m$ *of* $\mathcal{L}$ *with* $(\mathcal{L}_m)_u = \Delta$*.*

Proposition 4. *(Batanin) The category of* $\mathcal{L}_1$*-algebras is isomorphic to the category of cosimplicial* $\square$*-monoids (*$\square$ *is induced by ordinal sum).*

Proposition 5. *(Tamarkin) The category of* $\mathcal{L}_2$*-algebras in* $\mathcal{E}$ *is isomorphic to the category of multiplicative non-symmetric operads in* $\mathcal{E}$*.*

Example. The Hochschild cochain complex of an associative algebra is an $\mathcal{L}_2$-algebra.

Proposition 6. *For each simplicial set X, the norm. cochain complex $N^*(X)$ is an $\mathcal{L}$-algebra.*

The dual coaction is given by

$$\begin{aligned}\mathcal{L}(n_1,\cdots,n_k;n)\otimes N_n(X) &\to N_{n_1}(X)\otimes\cdots\otimes N_{n_k}(X)\\ x\otimes[\alpha] &\mapsto [x_1^*(\alpha)]\otimes\cdots\otimes[x_k^*(\alpha)]\end{aligned}$$

where $(x_1,\ldots,x_k)$ are the components of $x : [n_1]*\cdots*[n_k]\to[n]$.

Proposition 7. *Let S^m be $\Delta[m]/\partial\Delta[m]$ and X be a pointed object of $\mathcal{E}$. Then, the cosimplicial $\mathcal{E}$-object $(X,*)^{(S^m,*)}$ is an $\mathcal{L}_m$-algebra.*

There is an $\mathcal{L}$-coaction on S^m:

$$\begin{aligned}\mathcal{L}(n_1,\cdots,n_k;n)\times(S^m)_n &\to (S^m)_{n_1}\times\cdots\times(S^m)_{n_k}\\ x\times\alpha &\mapsto (x_1^*(\alpha),\ldots,x_k^*(\alpha)).\end{aligned}$$

If $c(x)\le m$, the image is in $(S^m)_{n_1}\vee\cdots\vee(S^m)_{n_k}$.

We now consider the case $\mathcal{E} = \mathbf{Top}$. Let $\delta : \Delta \to \mathbf{Top}$ be the *standard cosimplicial object*.

$\underline{\mathsf{Hom}}_\Delta(\delta, (X,*)^{(S^m,*)}) \cong \underline{\mathbf{Top}}_*(|S^m|, X) = \Omega^m X.$

Thus, any m-fold loop space is an algebra over the coendomorphism-operad

$$\begin{aligned}\mathcal{D}_m(k) &= \underline{\mathsf{Hom}}_\Delta(\delta, \delta \otimes_{\mathcal{L}_m} \cdots \otimes_{\mathcal{L}_m} \delta) \\ &= \mathsf{Tot}_\delta(Y_{m,k}),\ k \geq 0.\end{aligned}$$

Theorem 3. *(McClure-Smith) For $1 \leq m \leq \infty$, $\mathcal{D}_m$ is a topological E_m-operad.*

$\mathsf{Tot}_\delta(Y_{m,k}) \cong Y_{m,k}(0) \times \mathsf{Tot}_\delta(\delta) \simeq Y_{m,k}(0)$ and $Y_{m,k}(0)$ is the realization of the k-simplicial set $\mathcal{L}_m(-,\ldots,-;0)$ of *surjections* with codomain $\{1,\ldots,k\}$ and complexity $\leq m$.

We now turn to the case $\mathcal{E} = \mathsf{Ch}(\mathbb{Z})$ with $\delta : \Delta \to \mathsf{Ch}(\mathbb{Z}) : [n] \mapsto N_*(\Delta[n])$.

Totalization $\underline{\mathsf{Hom}}_\Delta(\delta, -)$ takes a cosimplicial module to the dg-module with differential $d = \sum(-1)^i \partial_i$. Thus the cochain complex $N^*(X)$ is a $\bar{\mathcal{X}}_\infty$-algebra, and the Hochschild cochain complex $CC^*(A; A)$ is a $\bar{\mathcal{X}}_2$-algebra, where $\bar{\mathcal{X}}_m$ is the coendomorphism operad

$$\bar{\mathcal{X}}_m(k) = \underline{\mathsf{Hom}}_\Delta(\delta, \delta \otimes_{\mathcal{L}_m} \cdots \otimes_{\mathcal{L}_m} \delta), \quad k \geq 0.$$

"Summing up the elements of the fibers" of

$$\mathcal{L}_m(-, \ldots, -; n) \to \mathcal{L}_m(-, \ldots, -; 0)$$

defines a cosimplicial dg-submodule of

$$|\mathcal{L}_m(-, \cdots, -; n)|_{\delta \otimes \cdots \otimes \delta},$$

and by *totalization* a dg-suboperad $\mathcal{X}_m$ of $\bar{\mathcal{X}}_m$:

$$\mathcal{X}_m(k) = |\mathcal{L}_m(-, \cdots, -; 0)|_{\delta \otimes \cdots \otimes \delta}, \quad k \geq 0.$$

This suboperad is the m-th filtration stage of the so-called *surjection operad* $\mathcal{X}$.

Theorem 4. *(McClure-Smith, Berger-Fresse) For* $1 \leq m \leq \infty$*,* $\mathcal{X}_m$ *is a dg-*E_m*-operad.*

This yields an E_∞-structure on $N^*(X)$ as well as an E_2-structure on $CC^*(A;A)$, solving the *Deligne conjecture*.

We finally consider the case $\mathcal{E} = \mathrm{Sets}^{\Delta^{\mathrm{op}}}$ with $\delta : \Delta \to \mathrm{Sets}^{\Delta^{\mathrm{op}}}$ the Yoneda-embedding.

Theorem 5. *(Berger-Fresse) The diagonal of the* k*-simplicial set* $\mathcal{L}(-,\cdots,-;0)$ *is the universal* $\Sigma(k)$*-bundle* $E\Sigma(k)$*. There is a weak equivalence of filtered dg-operads* $N_*(E_m\Sigma) \to \mathcal{X}_m$*, where* $E_m\Sigma$*,* $m \geq 1$*, denotes the Smith filtration of Barratt-Eccles'* E_∞*-operad* $E\Sigma$*.*

Theorem 6. *(Kashiwabara, Berger) For* $1 \leq m \leq \infty$*,* $E_m\Sigma$ *is a simplicial* E_m*-operad.*

The simplicial isomorphism

$$\alpha : E\Sigma(k)_d \cong \mathcal{L}(d, \ldots, d; 0)$$

is given by a "shuffle" which increases the filtration degree in a minimal way. For instance,

$$\alpha((123, 213, 231, 321)) = 122213333121.$$

For $k = 2$, this α is a filtration-preserving equivariant simplicial isomorphism.

The map of filtered dg-operads $N_*(E\Sigma) \to \mathcal{X}$ is induced by Alexander-Whitney maps

$$N_*(\Delta[n_1]\times..\times\Delta[n_k]) \to N_*(\Delta[n_1])\otimes\cdots\otimes N_*(\Delta[n_k])$$

via the identifications

$$\begin{aligned} N_*(E\Sigma(k)) &= |\mathcal{L}(-, \cdots, -; 0)|_{N_*(\delta\times\cdots\times\delta)} \\ \mathcal{X}(k) &= |\mathcal{L}(-, \cdots, -; 0)|_{N_*(\delta)\otimes\cdots\otimes N_*(\delta)} \end{aligned}$$

The compatibility with the operad structures and filtrations follows from a cellular decomposition of $E\Sigma(k)$ compatible with these data, which is induced by the *complete graph operad* $\mathcal{K}(k)$, $k \geq 0$.

Tamarkin's 2-operad action on $\mathcal{E}-$Cat.

Given two small $\mathcal{E}$-categories $\mathcal{A}$, $\mathcal{B}$ and two $\mathcal{E}$-functors $F, G : \mathcal{A} \to \mathcal{B}$, one defines a *cosimplicial object* of natural transformations $R^\bullet(F,G)$ where $R^n(F,G)$ is given by

$$\prod \underline{\mathcal{E}}(\underline{\mathcal{A}}(x_0,x_1)\otimes\cdots\otimes\underline{\mathcal{A}}(x_{n-1},x_n), \underline{\mathcal{B}}(F(x_0),G(x_n)))$$

where the product is over $(x_0,\ldots,x_n) \in \mathcal{A}_0^{n+1}$.

The *derived object* is then by definition

$$R(F,G) = \mathrm{Tot}_\delta(R^\bullet(F,G)).$$

If $\mathcal{A}$ is a one-object dg-category with $\underline{\mathcal{A}}(\star,\star) = A$, then $R(Id_\mathcal{A}) = CC^*(A;A)$.

Tamarkin constructs an $\mathbb{N}$-coloured 2-operad T_2 whose *symmetrization* is $\mathcal{L}_2$ and whose *totalization* is a contractible 2-operad in dgMod. He shows that T_2 acts on dgCat. This yields (by a theorem of Batanin) a "global" proof of the Deligne conjecture.