On the profinite fundamental group of a connected Grothendieck topos

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Louvain-La-Neuve, July 2-8, CT2023

joint work with Victor Iwaniack
Université Côte d'Azur

Problem (finite objects)

- finite set = finite sum of singletons
- finite covering = covering map with finite fibres
- finite object in a topos = ?

Definition (an object X of a Grothendieck topos $\gamma : \mathscr{E} \to \mathscr{S}$ is)

- *locally finite* if there is a cover $(U_i)_{i \in I}$ of $1_{\mathscr{E}}$ such that $X \times U_i \cong \gamma^*(\{1, \ldots, n_i\}) \times U_i$ in $\mathscr{E}/U_i$ for each $i \in I$;
- *decomposition-finite* if it is a finite sum of connected objects;
- *finite* if it is locally finite and decomposition-finite.

Theorem

A connected Grothendieck topos is *finitely generated* if and only if it is equivalent to $\mathbb{B}G$ for a profinite group G.

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Proposition (TFAE for an object X in a Grothendieck topos $\mathscr{E}$)

- X locally finite
- there is a globally supported U such that $X \times U$ is a *finite cardinal* in $\mathscr{E}/U$ (Johnstone)
- X *decidable Kuratowski-finite* (Kock-Lecouturier-Mikkelsen)

Lemma (in an elementary topos)

X is decidable and Kuratowski-finite if and only if

- the singleton map $\{-\} : X \to \Omega^X$ factors through 2^X
- and the induced map $(X^*, \cdot) \to (2^X, \vee)$ is an epimorphism.

Corollary (a sheaf $\mathcal{F}$ on a topological space E is)

- locally finite if and only if $\mathrm{Et}(\mathcal{F}) \to E$ is a finite covering
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Lemma

Binary sums and binary products of finite objects are finite.

Lemma

Complemented subobjects of finite objects are finite. The image of a morphism between finite objects is complemented.

Definition

A *pretopos* is an exact and extensive category. A pretopos is *embedded* in a topos if the inclusion is full and exact.

Proposition (for any Grothendieck topos $\mathscr{E}$ with finite $1_{\mathscr{E}}$)

the full subcategory $\mathscr{E}_f$ of finite objects of $\mathscr{E}$ is an embedded pretopos in which all subobjects are complemented.

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Definition (Grothendieck SGA1)

A *Galois category* $\mathcal{C}$ is a pretopos with complemented subobjects and with an *exact conservative* fibre functor $F_{\mathcal{C}} : \mathcal{C} \to \mathscr{S}_f$.

Theorem (for any connected Grothendieck topos $\gamma : \mathscr{E} \to \mathscr{S}$)

the pretopos $\mathscr{E}_f$ of finite objects of $\mathscr{E}$ is a Galois category for an essentially unique fibre functor $F_{\mathscr{E}_f} : \mathscr{E}_f \to \mathscr{S}_f$.

Definition (splitting Galois objects)

A *Galois object* is a connected, globally supported object A such that $A \times \gamma^*(\mathrm{Aut}(A)) \cong A \times A$.
A Galois object A is said to *split* X if $X \times A$ is constant in $\mathscr{E}/A$.
$\mathrm{Spl}(A)$ is the full subcategory of objects of $\mathscr{E}$ split by A.

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Remark (intrinsic cardinality of finite connected objects)

The category $\mathrm{Spl}(A)$ is equivalent to the category $\mathbb{B}\mathrm{Aut}(A)$ of $\mathrm{Aut}(A)$-sets via $M \mapsto X = A \times_{\mathrm{Aut}(A)} M$.

Each finite connected object X is contained in a "smallest" category $\mathrm{Spl}(A)$. The $\mathrm{Aut}(A)$-set M may be identified with $\mathscr{E}(A, X)$ via a canonical isomorphism

$$\gamma^*(\mathscr{E}(A, X)) \times A = \gamma^*\gamma_*(X^A) \times A \xrightarrow{\cong} X \times A.$$

Remark (intrinsic cardinality of finite objects)

Each $B \twoheadrightarrow A$ induces $\mathrm{Spl}(A) \subset \mathrm{Spl}(B)$ and $\mathscr{E}(A, X) \cong \mathscr{E}(B, X)$.

If $X = \coprod_{i \in \pi_0(X)} X_i$ and A_i splits X_i then each A splitting a connected component of $\prod A_i$ splits all X_i and

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Each finite connected object X is contained in a "smallest" category $\mathrm{Spl}(A)$. The $\mathrm{Aut}(A)$-set M may be identified with $\mathscr{E}(A, X)$ via a canonical isomorphism

$$\gamma^*(\mathscr{E}(A, X)) \times A = \gamma^*\gamma_*(X^A) \times A \xrightarrow{\cong} X \times A.$$

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Each $B \twoheadrightarrow A$ induces $\mathrm{Spl}(A) \subset \mathrm{Spl}(B)$ and $\mathscr{E}(A, X) \cong \mathscr{E}(B, X)$.

If $X = \coprod_{i \in \pi_0(X)} X_i$ and A_i splits X_i then each A splitting a connected component of $\prod A_i$ splits all X_i and

$$\mathscr{E}(A, X) = \coprod_{i \in \pi_0(X)} \mathscr{E}(A_i, X_i).$$

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Definition

atomic = locally connected and Boolean

Lemma

In an atomic Grothendieck topos each object is a sum of "atoms".

Proposition (Leroy, Moerdijk, Bunge, Dubuc)

For a pretopos $\mathscr{P}$ embedded in a connected Grothendieck topos $\mathscr{E}$ sth. each object of $\mathscr{P}$ is a sum of atoms and the latter form a set, the subcategory $s\mathscr{P}$ of $\mathscr{E}$ is an atomic Grothendieck topos and $s\mathscr{P} \hookrightarrow \mathscr{E}$ is the inverse image of a geometric morphism.

Theorem (for any connected Grothendieck topos $\mathscr{E}$)

the subcategory $\mathscr{E}_{sf}$ is a pointed, atomic Grothendieck topos. The surjective "Galois point" $\mathscr{G}_{\mathscr{E}} : \mathscr{S}_{sf} \to \mathscr{E}_{sf}$ is right adjoint to the "fibre functor" $F_{\mathscr{E}} : \mathscr{E}_{sf} \to \mathscr{S}_{sf}$.

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Proposition

The automorphism group $\mathrm{Aut}(\mathscr{G}_{\mathscr{E}})$ carries a unique profinite topology such that $\mathscr{E}_{sf} \simeq \mathbb{B}\mathrm{Aut}(\mathscr{G}_{\mathscr{E}})$.

Proof.

$\mathscr{E}_{fc}$ is an atomic site: $\mathscr{E}_{sf} = \mathrm{Sh}(\mathscr{E}_{fc}) \simeq \mathbb{B}\mathrm{Aut}(\mathscr{G}_{\mathscr{E}})$. □

Definition (profinite fundamental group)

$$\hat{\pi}(\mathscr{E}) = \mathrm{Aut}(\mathscr{G}_{\mathscr{E}})$$

Examples

- $\mathscr{E}$ "finitely gen." iff $\mathscr{E} = \mathscr{E}_{sf}$ iff $\mathscr{E} = \mathbb{B}G$ for profinite group G
- $\hat{\pi}(\mathrm{Sh}(E)) = \hat{\pi}_1(E)$ (profinite completion)
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