Comprehensive factorisation & non-commutative Stone duality

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[1]joint with Mai Gehrke and Ralph Kaufmann

Examples (notions of covering)

- topological covering/X $\rightsquigarrow$ $\Pi_1(X)$-set
- discrete fibration/$\mathcal{C}$ $\rightsquigarrow$ set-valued presheaf on $\mathcal{C}$

Purpose of the talk

- general notion of covering & associated factorisation system using Lawvere's comprehension schemes '70.
- apply to idempotent semigroups to get non-commutative versions of Stone duality '37.

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Definition (category of adjunctions)

objects of Adj_* are categories with a distinguished terminal object
morphisms of Adj_* are adjunctions $(f_!, f^*)$.

Definition (comprehension scheme)

A *comprehension scheme* on $\mathcal{E}$ is a pseudo-functor $P : \mathcal{E} \to \mathrm{Adj}_*$ such that for each object B of $\mathcal{E}$ the functor

$$\begin{array}{ccc} \mathcal{E}/B & \longrightarrow & PB \\ (f : A \to B) & \longmapsto & f_!(\star_{PA}) \end{array}$$

has a *fully faithful* right adjoint $\mathrm{el}_B : PB \to \mathcal{E}/B$.

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Definition

- A morphism $f : A \to B$ is a *P-covering* if it belongs to the essential image of el_B.
- A comprehension scheme is *consistent* if P-coverings *compose* and are *left cancellable*: $gf, g \in \mathrm{Cov}_B \implies f \in \mathrm{Cov}_B$.
- A morphism $f : A \to B$ is *P-connected* if $f_!(\star_{PA}) \cong \star_{PB}$.

Theorem (B-Kaufmann '17)

There is a 1-1 correspondence between consistent comprehension schemes and complete orthogonal factorisation systems.

Proof.

- *ccs* induces (P-connected, P-covering)-factorisation.
- $(\mathcal{L}, \mathcal{R})$-factorisation induces *ccs* with $\mathrm{el}_B = \mathcal{R}/B$.

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Remark (Frobenius)

A *ccs* satisfies *Frobenius reciprocity* (Lawvere '70) if and only if P-connected maps are stable under pullback along P-coverings.

Examples (comprehensive factorisation systems)

- $\mathrm{Sets} \to \mathrm{Adj}_* : X \mapsto (\mathcal{P}X, \subset)$ induces epi/mono-factorisation.
- $\mathrm{Cat} \to \mathrm{Adj}_* : C \mapsto PC = [C^{\mathrm{op}}, \mathrm{Sets}]$ induces the comprehensive factorisation of a functor (Street-Walters '73).
- PC restricts to $\mathrm{Posets} \subset \mathrm{Cat}$ and $\mathrm{Gpd} \subset \mathrm{Cat}$ (Bourn '87).
- $\exists$*ccs* $\mathrm{Multicat} \to \mathrm{Adj}_*$ and $\mathrm{Feyn} \to \mathrm{Adj}_*$ (B-Kaufmann '17).
- $\mathrm{Top}_{slsc} \to \mathrm{Adj}_* : X \mapsto \mathrm{Sh}_{loc}(X)$ yields a comprehensive factorisation of a continuous map of slsc spaces.

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Remark (espace étalé)

The equivalence $\mathrm{Sh}(X) \simeq \{\text{local homeomorphisms}/X\}$ restricts to an equivalence $\mathrm{Sh}_{loc}(X) \simeq \{\text{topological coverings}/X\}$.

Lawvere '70: *... we remark that although our discussion below of comprehension hinges on the operation Σ, there is one structure in which all features of hyperdoctrines except Σ exist ..., but in which there is clearly a kind of "extension", namely the espace étalé.*

Proposition ($f_!$ for locally constant sheaves on slsc spaces)

For any slsc space, monodromy induces an equivalence of categories $\mathrm{Sh}_{loc}(X) \simeq \Pi_1(X)\text{-sets}$. In particular for $f : X \to Y$,

$$\begin{array}{ccc} \mathrm{Sh}_{loc}(X) & \xrightarrow{\exists f_!} & \mathrm{Sh}_{loc}(Y) \\ \downarrow \simeq & & \downarrow \simeq \\ \Pi_1(X)\text{-sets} & \xrightarrow{\Pi_1(f)_!} & \Pi_1(Y)\text{-sets} \end{array}$$

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Proposition (homotopical characterisation of connected maps)

A map of slsc spaces $f : X \to Y$ is connected iff $\pi_0(f)$ is bijective and $\pi_1(f, x) : \pi_1(X, x) \to \pi_1(Y, f(x))$ is surjective $\forall x \in X$.

Corollary (existence of universal coverings)

For any based slsc space (X, x) the comprehensive factorisation

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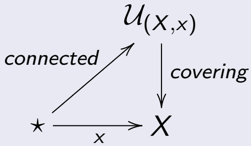

produces the *universal covering* of X at x.

Definition

A *band* (=idempotent semigroup) is a set $(X, \cdot)$ with an associative multiplication such that $x^2 = x$ for all $x \in X$.

Lemma (meet-semilattices)

Commutative bands are the same as posets with binary meets.

Lemma (Green's $\mathcal{D}$-relation)

Each band is partially ordered by $x \leq y \overset{dfn}{\iff} x = yxy$. The commutative bands form a reflective subcategory. The reflection is given by $X \to X/\mathcal{D}$ where $x\mathcal{D}y \overset{dfn}{\iff} x = xyx$ and $y = yxy$.

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Definition (Schützenberger '47)

A band is *left* (resp. *right*) *regular* if $xy = xyx$ (resp. $yx = xyx$).

Proposition (B-Gehrke '18)

The category of right regular bands admits a comprehensive factorisation system lifted along the functor $(X, \cdot) \mapsto (X, \leq)$.

Lemma (discrete objects)

For a right regular band X tfae:

- $(X, \leq)$ is order-discrete;
- $(X, \cdot)$ is a right zero band (i.e. $yx = x$);
- the terminal map $X \to \star_{RRB}$ is a covering.

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A band is *left* (resp. *right*) *regular* if $xy = xyx$ (resp. $yx = xyx$).

Proposition (B-Gehrke '18)

The category of right regular bands admits a comprehensive factorisation system lifted along the functor $(X, \cdot) \mapsto (X, \leq)$.

Lemma (discrete objects)

For a right regular band X tfae:

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Definition

A band X is called right *distributive* if

(i) X is right normal;

(ii) $X/\mathcal{D}$ is a (bounded) distributive lattice;

(iii) for any finite subset S of X consisting of pairwise commuting elements the join $\bigvee S$ in $(X, \leq)$ exists.

Example (the local sections of a sheaf form a distributive band)

We define $(U,\sigma)(V,\tau) = (U \cap V, \tau\rceil^{V}_{U\cap V})$. Local sections commute iff they glue. $(U,\sigma) \leq (V,\tau)$ iff $U \subset V$ and $\sigma = \tau\rceil_U$. (iii) expresses sheaf condition w/to *finite* open covers.

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Definition (skew-lattice, Leech '89)

A *skew lattice* $(S, \curlywedge, \curlyvee)$ consists of two bands $(S, \curlywedge)$ and $(S, \curlyvee)$ such that the following four absorption laws hold:

(i) $(y \curlywedge x) \curlyvee x = x = x \curlywedge (x \curlyvee y)$;

(ii) $x \curlyvee (x \curlywedge y) = x = (y \curlyvee x) \curlywedge x$.

Remark (lattice reflection)

The order relation of $(S, \curlywedge)$ is *dual* to the order relation of $(S, \curlyvee)$. Green's $\mathcal{D}$-relation yields a lattice $S/\mathcal{D}$, the *lattice reflection* of S. $(S, \curlywedge)$ is right regular iff $(S, \curlyvee)$ is left regular.

Definition (variety of distributive skew-lattices)

A skew-lattice is *symmetric* if $x \curlywedge y = y \curlywedge x \iff x \curlyvee y = y \curlyvee x$. A skew-lattice is *right distributive* if it is symmetric, right normal and its lattice reflection is distributive.

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Theorem (Stone '37)

There is a duality between the category of distributive lattices and the category of spectral spaces.

Theorem (B-Gehrke '18)

There is a duality between the category of right distributive bands and the category of sheaves over spectral spaces.

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