The theta map for principal bundles on curves

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rational map defined by the sections of $\mathcal{L}$, $(|\mathcal{L}| = \mathbb{P}(H^0(\mathcal{M}_G, \mathcal{L})))$.

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Theme of the talk : What can we say about that map?

$G = SL(r)$

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- either $\Theta_E \in |r\Theta|$,
- or $\Theta_E = J^{g-1}$: *E has no Theta divisor.*

BNR

Theorem (Narasimhan, Ramanan, AB)

$$\begin{tikzcd} & {|\mathcal{L}|^*} \arrow[d, "\wr"] \\ \mathcal{M}_{SL(r)} \arrow[ur, dashed, "\theta"] \arrow[r, dashed, "E \mapsto \Theta_E"'] & {|r\Theta|} \end{tikzcd}$$

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Remark on the proof

Tautological lemma

X projective variety, $\varphi : X \dashrightarrow \mathbb{P}^N$, $\mathcal{L} := \varphi^*\mathcal{O}_{\mathbb{P}}(1)$. Assume:

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replaced by *ad hoc* argument with spectral curve.

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Example (Narasimhan-Ramanan)

$g = 3$, C non hyperelliptic: $\theta(\mathcal{M}_{SL(2)})$ quartic hypersurface

$\mathcal{Q} \subset |2\Theta| \cong \mathbb{P}^7$, singular along the Kummer variety $K(J)$ $\Longrightarrow$

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$\mathcal{S}^* \subset |3\Theta|^*$ is the Coble cubic, the unique cubic hypersurface in $|3\Theta|^*$ singular along the image of J^{g-1}.

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Much less evidence for 2.

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- For r even, $\mathcal{M}_{SO(r)} \xrightarrow{2:1} \mathcal{M}^{\mathcal{O}}_{O(r)}\ .$

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The map $\mathcal{M}^{\mathcal{O}}_{O(r)} \to \mathcal{M}_{SL(r)}$ *is an embedding.*

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   Thus $\Theta_E \in |r\Theta|^+$ or $|r\Theta|^-$, the eigenspaces of i^* in $|r\Theta|$.

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Idea of proof : Use tautological lemma. Non-degeneracy easy; Verlinde formula for $SO(r)$ gives $h^0(\mathcal{M}_{SO(r)}, \mathcal{L}) = r^g$. ∎

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Computation tricky; would be interesting to get [BNR]-type proof.

Base points

Theorem (Serman)

$\theta^{\pm} : \mathcal{M}^{\pm}_{SO(3)} \to |3\Theta|^{\pm}$ and $\theta^{+} : \mathcal{M}^{+}_{SO(4)} \to |4\Theta|^{+}$ are morphisms.

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Essentially sharp: for $g = 2$, 20 base points on $\mathcal{M}^{-}_{SO(4)}$.

Example

Example ($g = 2,\ r = 3$)

$$\begin{array}{c} \mathcal{M}_{SL(3)} \\ \Big\downarrow \theta \\ |3\Theta| \ \ (\cong \mathbb{P}^8) \\ \cup \\ \mathcal{S} \end{array}$$

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\begin{tikzcd}
\mathcal{M}^{-}_{SO(3)} \arrow[r, hook] \arrow[d, "\theta^{-}"] & \mathcal{M}_{SL(3)} \arrow[d, "\theta"] \\
|3\Theta|^{-}_{(\cong \mathbb{P}^3)} \arrow[r, hook] \arrow[d, phantom, "\cup" rotate=90] & |3\Theta|\ (\cong \mathbb{P}^8) \arrow[d, phantom, "\cup" rotate=90] \\
\mathcal{S}^{-} = \cup H_p & \mathcal{S}
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$$

$\mathcal{S} \cap |3\Theta|^{-} := \mathcal{S}^{-} =$ union of 6 planes

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\mathcal{M}^-_{SO(3)} \arrow[r, hook] \arrow[d, "\theta^-"] & \mathcal{M}_{SL(3)} \arrow[d, "\theta"] & \mathcal{M}^+_{SO(3)} \arrow[l, hook'] \arrow[d, "\theta^+"] \arrow[dr] & \\
|3\Theta|^-_{(\cong \mathbb{P}^3)} \arrow[r, hook] & |3\Theta| \ (\cong \mathbb{P}^8) & |3\Theta|^+_{(\cong \mathbb{P}^4)} \arrow[l, hook'] & \mathcal{Q} \arrow[l, hook'] \\
\cup & \cup & & \\
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\end{tikzcd}$$

$\mathcal{S} \cap |3\Theta|^- := \mathcal{S}^- =$ union of 6 planes

$\mathcal{S} \cap |3\Theta|^+ = \mathcal{Q} + 2H$, $\mathcal{Q} =$ Igusa quartic, $H = \Theta + |2\Theta| \subset |3\Theta|^+$.

The theta map for $Sp(2r)$

$$\mathcal{M}_{Sp(2r)} = \{(E,\varphi) \mid E \in \mathcal{M}_{SL(2r)}\ ,\ \varphi : \wedge^2 E \to \mathcal{O}_C \quad \text{non-deg.}\}$$

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$$E \cong E^* \ \Rightarrow\ \Theta_E \in |2r\Theta|^+$$

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[Subtlety: $h^0(E \otimes F)$ *even* (Mumford)]

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Theorem

$$\begin{tikzcd} & {|\mathcal{L}|^*} \arrow[d, "\wr"] \\ \mathcal{M}_{Sp(2r)} \arrow[ur, dashed, "\theta"] \arrow[r, dashed, "E \mapsto \Delta_E"'] & {|r\Delta|} \end{tikzcd}$$

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- The Θ_G span $|r\Delta|$: this is the *rank-level duality $SL(2)$-$GL(r)$* proved by Marian-Oprea and Belkale.

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Corollary ($r = 2$)

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For $r \geq 3$ j^* is surjective (but not injective) if C has no vanishing thetanull, so θ^+ is obtained from θ by projection.

Longo maï, Ramanan!

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