Algebraic surfaces

Lecture I: The Picard group, Riemann-Roch,...

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Divisors and line bundles

Surface = smooth, projective, over $\mathbb{C}$.

$\mathrm{Pic}(S) = \{\text{line bundles on } S\}/\sim$, (group for $\otimes$).

$\mathrm{Div}(S) = \{D = \sum n_i C_i\}$. $D \geqslant 0$ (effective) if $n_i \geqslant 0\ \forall i$.

$$\{D \geqslant 0\} \stackrel{\sim}{\leftrightsquigarrow} \{(L,s) \mid L \in \mathrm{Pic}(S), 0 \neq s \in H^0(L)\}$$

We put $L = \mathcal{O}_S(D)$. Map $D \mapsto \mathcal{O}_S(D)$ extends by linearity to homomorphism $\mathrm{Div}(S) \twoheadrightarrow \mathrm{Pic}(S)$. Then $\mathrm{Pic}(S) = \mathrm{Div}(S)/\equiv$ where $D \equiv D' \Leftrightarrow D - D' = \mathrm{div}(\varphi)$, φ rational function on S.

C irreducible curve, $s \in H^0(\mathcal{O}_S(C))$ defining C. $\mathcal{O}_S(-C) \xhookrightarrow{s} \mathcal{O}_S$ $\Rightarrow$ $\mathcal{O}_S(-C) \cong$ ideal sheaf of C in S.

$f : S \to T \rightsquigarrow f^* : \mathrm{Pic}(T) \to \mathrm{Pic}(S)$.

$D \in \mathrm{Div}(T)$; if $f(S) \not\subset D$, $f^*D \in \mathrm{Div}(S)$ and $\mathcal{O}_S(f^*D) = f^*\mathcal{O}_S(D)$.

The intersection form

$C \neq D$ irreducible, $p \in C \cap D$. f, g equations of C, D in $\mathcal{O}_p$.

Definition : $m_p(C \cap D) := \dim_{\mathbb{C}} \mathcal{O}_p/(f, g)$.

Example: $m_p(C \cap D) = 1 \iff (f, g) = \mathfrak{m}_p \iff f, g$ local coordinates at $p \overset{\text{def}}{\iff} C$ and D transverse.

Definition : $(C \cdot D) := \sum_{p \in C \cap D} m_p(C \cap D)$.

Theorem

$\exists$ *bilinear symmetric form* $(\ \cdot\ ) : \mathrm{Pic}(S) \times \mathrm{Pic}(S) \to \mathbb{Z}$ *such that* $(\mathcal{O}_S(C) \cdot \mathcal{O}_S(D)) = (C \cdot D)$ *for* C, D *irreducible.*

The intersection form: step 1

Proof : For $L, M \in \operatorname{Pic}(S)$, we put:

$$(L \cdot M) = \chi(\mathcal{O}_S) - \chi(L^{-1}) - \chi(M^{-1}) + \chi(L^{-1} \otimes M^{-1})$$

Step 1 : $(\mathcal{O}_S(C) \cdot \mathcal{O}_S(D)) = (C \cdot D)$.

Proof : $C = \operatorname{div}(s)$, $D = \operatorname{div}(t)$. Exact sequence:

$$0 \to \mathcal{O}_S(-C-D) \xrightarrow{(t,-s)} \mathcal{O}_S(-C) \oplus \mathcal{O}_S(-D) \xrightarrow{(s,t)} \mathcal{O}_S \twoheadrightarrow \mathcal{O}_{C\cap D}\,.$$

Proof: $p \in S$, $f, g \in \mathcal{O}_p$ local equations for C and D.

$$0 \to \mathcal{O}_p \xrightarrow{(g,-f)} \mathcal{O}_p^2 \xrightarrow{(f,g)} \mathcal{O}_p \to \mathcal{O}_p/(f,g) \to 0\,.$$

Means: in $\mathcal{O}_p$, $af = bg \iff \exists\, k, a = gk, b = fk$.

Holds because $\mathcal{O}_p$ factorial, f, g prime $\neq$. Then:

$\chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)) - \chi(\mathcal{O}_S(-D)) + \chi(\mathcal{O}_S(-C-D)) = \chi(\mathcal{O}_{C\cap D})$

$= h^0(\mathcal{O}_{C\cap D})) = \sum_{p \in C\cap D} \mathcal{O}_p/(f,g) \stackrel{\text{def}}{=\!=} (C \cdot D)$. ■

The intersection form (continued)

Step 2 : $(L \cdot \mathcal{O}_S(C)) = \deg L_{|C} \quad \forall\, L \in \mathrm{Pic}(S),\ C$ smooth.

Proof : Exact sequences $0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$,

$\otimes L^{-1}$: $\qquad 0 \to L^{-1} \otimes \mathcal{O}_S(-C) \to L^{-1} \to L^{-1}_{|C} \to 0.$

$$\chi(\mathcal{O}_C) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)),\ \chi(L^{-1}_{|C}) = \chi(L^{-1}) - \chi(L^{-1} \otimes \mathcal{O}_S(-C))$$

$\Rightarrow\ (L \cdot C) = \chi(\mathcal{O}_C) - \chi(L^{-1}_{|C}) = \deg L_{|C}$ (R-R on C). ■

Step 3 : $(\ \cdot\ )$ is bilinear.

Put $s(L, M, N) := (L \cdot M \otimes N) - (L \cdot M) - (L \cdot N)$.

- Symmetric in L, M, N.
- $= 0$ when $L = \mathcal{O}_S(C)$.

Fact (Serre): $\forall\, L \in \mathrm{Pic}(S),\ L \cong \mathcal{O}_S(C - D)$, with C, D smooth curves (In fact, hyperplane sections in appropriate embeddings).

The intersection form: end of proof

$L, M \in \mathrm{Pic}(S)$; $M = \mathcal{O}_S(C - D)$, C, D smooth curves. Then

$0 = s(L, M, \mathcal{O}_S(B)) = (L \cdot M \otimes \mathcal{O}_S(B)) - (L \cdot M) - (L \cdot \mathcal{O}_S(B))$

$\Rightarrow\ (L \cdot M) = (L \cdot \mathcal{O}_S(A)) - (L \cdot \mathcal{O}_S(B))$ **linear** in L, hence in M. ■

Examples

① $S = \mathbb{P}^2$

$C \subset \mathbb{P}^2$ defined by a form $F_d(X, Y, Z)$ of degree d. $\dfrac{F_d}{Z^d}$ rational function $\Rightarrow\ C \equiv dH$, H line in $\mathbb{P}^2$. Thus $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}[H]$,

$(C \cdot D)$: $\deg(C)\deg(D)$ (**Bézout theorem**).

Examples

② $S = \mathbb{P}^1 \times \mathbb{P}^1$

Put $A = \mathbb{P}^1 \times \{0\}$, $B = \{0\} \times \mathbb{P}^1$, $U = S \smallsetminus (A \cup B) \cong \mathbb{A}^2$.

$D \in \mathrm{Div}(S)$: $D_{|U} = \mathrm{div}(\varphi)$ for some rational function φ.

$D - \mathrm{div}\,\varphi = aA + bB$ for some $a, b \in \mathbb{Z} \implies$

$\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}[A] \oplus \mathbb{Z}[B]$. $(A \cdot B) = 1$ (transverse).

$A^2 = \big(A \cdot (\mathbb{P}^1 \times \{1\})\big) = 0$, $B^2 = 0$: intersection form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

③ $p : S \to C$, $F := p^{-1}(x)$. $\exists\, D \in \mathrm{Div}(C)$, $x \notin D$, $x \equiv D$; then

$F \equiv p^*D \;\Rightarrow\; F^2 = F \cdot p^*D = 0$.

④ $D \geqslant 0$, $D \cdot C < 0 \;\Rightarrow\; D = C + E$, $E \geqslant 0$.

(otherwise $D = \sum n_i C_i$, $C_i \neq C \;\Rightarrow\; C \cdot C_i \geqslant 0 \;\forall\, i$)

⑤ $C^2 < 0, C \equiv D \geqslant 0 \;\Rightarrow\; D = C$ $(\Leftrightarrow h^0(\mathcal{O}_S(C)) = 1)$.

Canonical line bundle and Riemann-Roch

Ω^1_S = sheaf of differential 1-forms, locally isomorphic to $\mathcal{O}_S^2$ (locally $a(x,y)dx + b(x,y)dy$).

$\mathcal{K}_S = \bigwedge^2 \Omega^1_S$ = sheaf of 2-forms = **canonical line bundle** (locally $\omega = f(x,y)dx \wedge dy, \mathrm{div}(\omega) = \mathrm{div}(f)$).

K_S or K = **canonical divisor** = divisor of any rational 2-form.

Example : $K_{\mathbb{P}^2} \equiv -3H$.

Indeed the 2-form $\dfrac{XdY \wedge dZ + YdZ \wedge dX + ZdX \wedge dY}{XYZ}$ is well-defined, does not vanish, and has a pole $\equiv 3H$.

Example : C_1, C_2 smooth projective curves, $S = C_1 \times C_2$, projections $p_i : S \to C_i$. Then $K_S \equiv p_1^* K_{C_1} + p_2^* K_{C_2}$.

Indeed if α_i is a 1-form on C_i (possibly rational), $p_1^*\alpha_1 \wedge p_2^*\alpha_2$ is a 2-form on S, with divisor $p_1^* \mathrm{div}(\alpha_1) + p_2^* \mathrm{div}(\alpha_2)$.

Riemann-Roch

Recall: $L \in \mathrm{Pic}(S) \rightsquigarrow H^i(S, L) = H^i(L)$, $i = 0, 1, 2$.

$h^i(L) = \dim H^i(L)$. $\chi(L) := h^0(L) - h^1(L) + h^2(L)$.

If $L = \mathcal{O}_S(D)$, we write $H^i(D)$, $h^i(D)$, $\chi(D)$.

Theorem

- **Riemann-Roch :** $\chi(L) = \chi(\mathcal{O}_S) + \frac{1}{2}(L^2 - \mathcal{K}_S \cdot L)$.
- **Serre duality :** $h^i(L) = h^{2-i}(\mathcal{K}_S \otimes L^{-1})$.

Since the term h^1 is difficult to control, we will most often use R-R as an inequality, using Serre duality. In divisor form:

$$h^0(D) + h^0(K - D) \geqslant \chi(\mathcal{O}_S) + \frac{1}{2}(D^2 - K \cdot D)\,.$$

Proof of Riemann-Roch

We admit Serre duality. Riemann-Roch follows directly from the definition of the intersection form:

Proof : $L^{-1}\cdot(L\otimes\mathcal{K}_S^{-1}) = \chi(\mathcal{O}_S) - \chi(L) - \chi(\mathcal{K}_S\otimes L^{-1}) + \chi(\mathcal{K}_S)$

$= 2\chi(\mathcal{O}_S) - 2\chi(L)$ by Serre duality. Hence

$$\chi(L) = \chi(\mathcal{O}_S) - \frac{1}{2}L^{-1}\cdot(L\otimes\mathcal{K}_S^{-1}) = \chi(\mathcal{O}_S) + \frac{1}{2}(L^2 - L\cdot\mathcal{K}_S). \quad \blacksquare$$

The genus formula

Corollary (genus formula)

$$C \text{ irreducible} \subset S \;\Rightarrow\; g(C) := h^1(\mathcal{O}_C) = 1 + \frac{1}{2}(C^2 + K\cdot C).$$

Proof : Exact sequence $0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0 \implies$

$$\chi(\mathcal{O}_C) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)) \overset{\text{R-R}}{=\!=} -\frac{1}{2}(C^2 + K\cdot C)\,. \quad \blacksquare$$

Examples : • $C \subset \mathbb{P}^2$ of degree $d \;\Rightarrow$

$$g(C) = 1 + \frac{1}{2}(d^2 - 3d) = \frac{1}{2}(d-1)(d-2)\,.$$

• $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (p,q) (i.e. $C \equiv pA + qB$) $\Rightarrow$

$$g(C) = 1 + \frac{1}{2}(2pq - 2p - 2q) = (p-1)(q-1)\,.$$

The genus of a singular curve

Remark : *Let* $n : N \to C$ *be the normalization of* C*. Then* $g(C) \geqslant g(N)$*, with equality iff* C *is smooth.*

Proof : Exact sequence $\quad 0 \to \mathcal{O}_C \to n_*\mathcal{O}_N \to \mathcal{T} \to 0$ with $\mathcal{T}$ concentrated on the singular points of C.

Hence $H^i(\mathcal{T}) = 0$ for $i > 0$. Therefore $\chi(\mathcal{O}_C) = \chi(\mathcal{O}_N) - h^0(\mathcal{T})$, and $g(C) = g(N) + h^0(\mathcal{T}) \geqslant g(N)$, equality iff $C = N$ smooth. ∎

Corollary

$C^2 + K \cdot C \geqslant -2$; *equality* $\Rightarrow$ $C \cong \mathbb{P}^1$.

Indeed $C^2 + K \cdot C = 2g(C) - 2 \geqslant 2g(N) - 2 \geqslant -2$. ∎

Numerical invariants

Algebraic surfaces are distinguished by their numerical invariants:

- The most important: K^2, $\chi(\mathcal{O})$.

  Though we will not use this in the lectures, I want to mention:

Theorem

1. *(M. Noether)* $K^2 \geqslant 2\chi(\mathcal{O}) - 6$;
2. *(Miyaoka-Yau)* $K^2 \leqslant 9\chi(\mathcal{O})$.

The relation of $K^2/\chi(\mathcal{O})$ with the geometry of the surface is a long chapter of surface theory ("geography").

Refined invariants:

- $h^2(\mathcal{O}) = h^0(K)$ (Serre duality), the **geometric genus** p_g;
- $h^1(\mathcal{O}) = H^0(\Omega^1)$ (Hodge theory), the **irregularity** q;
- $h^0(nK)$ $(n \geqslant 1)$, the **plurigenera** P_n.

Exercises

1) Let C be an irreducible curve in $\mathbb{P}^2$, $p \in C$. We choose affine coordinates (x, y) with $p = (0, 0)$, and write the equation of C as $0 = f_m(x, y) + f_{m+1}(x, y) + \ldots$, where f_q is homogeneous of degree q. We have $f_m = \ell_1 \ldots \ell_m$, where the ℓ_i are linear forms; the lines $\ell_i = 0$ are the *tangent* to C at p. Show that a line ℓ passing through p is tangent to C if and only if $(C \cdot \ell)_p > m$.

2) Let C be a curve of genus g. Let $\Delta \subset C \times C$ be the diagonal $(\Delta = \{(x, x) \,|\, x \in C\}$.

a) Using the genus formula, prove that $\Delta^2 = 2 - 2g$.

b) Let $p, q : C \times C \to C$ be the two projections. Show that if $g > 0$, $\operatorname{Pic}(S \times S) \supset p^* \operatorname{Pic}(C) \oplus q^* \operatorname{Pic}(C) \oplus \mathbb{Z}[\Delta]$. What happens for $g = 0$?

Exercises

3) a) Let S_0 be a smooth surface in the affine space A^3, defined by an equation $f=0$. Prove that $\dfrac{dx\wedge dy}{f'_z}=\dfrac{dy\wedge dz}{f'_x}=\dfrac{dz\wedge dx}{f'_y}$ on S_0, so that this expression defines a non-vanishing 2-form on S_0.

b) Let S be a smooth surface in $\mathbb{P}^3$, defined by an equation $F=0$ of degree d. Prove that the expression

$$T^{d-4}\frac{TdY\wedge dZ+YdZ\wedge dT+ZdT\wedge dY}{F'_X}$$

defines a 2-form on S with divisor $(d-4)H$.

4) (Hodge index theorem) Let H be a divisor on S such that $H\cdot C>0$ for every curve $C\subset S$ (for instance a hyperplane section). Let D be a divisor such that $H\cdot D=0$. We will prove that $D^2\leqslant 0$.

Exercises

a) Show that $h^0(nD) = 0$ for all $n \in \mathbb{Z}$, $n \neq 0$.

b) If $D^2 > 0$, deduce from Riemann-Roch that $h^0(K - nD)$ and $h^0(K + nD) \to \infty$ when $n \to \infty$; conclude that $D^2 \leqslant 0$.

5) Let C, C' be two curves, D a divisor on $C \times C'$. Let $p \in C$, $p' \in C'$; put $A = p \times C$, $B = C \times p'$, $a = D \cdot A$ and $b = D \cdot B$. Prove the Castelnuovo-Severi inequality $D^2 \leqslant 2ab$ (apply the previous exercise to $H = A + B$, and the divisor $D - bA - aB$).

[Note: This inequality was the essential step in Weil's proof of his conjectures for curves.]

Algebraic surfaces

Lecture II: Rational and birational maps

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Blowing up

Proposition

$p \in S$. $\exists\, b : \hat{S} \to S$, unique up to isomorphism, such that

1. $b^{-1}(p) = E \cong \mathbb{P}^1$;
2. $b : S \smallsetminus E \xrightarrow{\sim} S \smallsetminus p$.

Sketch of proof: coordinates x, y in $U \ni p$

$\hat{U} \subset U \times \mathbb{P}^1 : xY - yX = 0$.

$b : \hat{U} \to U$ projection, satisfies ① and ②.

Then glue $S \smallsetminus p$ and $\hat{U}$ along $U \smallsetminus p$. ∎

In $\hat{U}' \subset \hat{U} : \{X \neq 0\}$, $y = xt$ with $t = \dfrac{Y}{X}$:

(x, t) local coordinates, $b(x, t) = (x, tx)$,

E given by $x = 0$.

The strict transform

We say that E is the **exceptional curve** of the blowing up.

$E \xrightarrow{\sim} \mathbb{P}(T_p(S))$: $(X,Y) \in E \leftrightarrow$ tangent direction $xY - yX = 0$.

For $C \subset S$, **strict transform** $\hat{C} :=$ closure of $C \smallsetminus p$ in $\hat{S}$.

$\hat{C} \cap E = \{$tangent directions to C at $p\}$.

Lemma

$b^*C = \hat{C} + mE$ in $\mathrm{Div}(\hat{S})$, where $m := m_p(C)$.

Proof : Eqn. of C in U: $0 = f(x,y) = f_m(x,y) + f_{m+1}(x,y) + ..$

Choose (x,y) such that $f_m(x,0) \neq 0$, i.e. C not tangent to $y = 0$.

$b^*f = f(x,tx) = x^m\big(f_m(1,t) + xf_{m+1}(1,t) + \dots\big)$, $f_m(1,0) \neq 0$

$\Rightarrow$ multiplicity of E in $\mathrm{div}(b^*f) = m$. ■

The Picard group of $\hat{S}$

Proposition

1. $\operatorname{Pic}(\hat{S}) = b^* \operatorname{Pic}(S) \overset{\perp}{\oplus} \mathbb{Z}[E]$, $(b^*C \cdot b^*D) = (C \cdot D)$, $E^2 = -1$.
2. $K_{\hat{S}} = b^* K_S + E$.
3. $b_2(\hat{S}) = b_2(S) + 1$.

Proof : • $\Gamma \subset \hat{S}$, $\Gamma \neq E \Rightarrow \Gamma =$ strict transform of $b(\Gamma) \subset S$ $\Rightarrow \Gamma = b^* b(\Gamma) - mE$.

• $\forall\, C \subset S$, $C \equiv A \not\ni p \Rightarrow (b^*C \cdot E) = 0$, $(b^*C \cdot b^*D) = (C \cdot D)$.

• Take $H \ni p$, $m_p(H) = 1$. Then $(\hat{H} \cdot E) = 1$; $b^*H = \hat{H} + E$, $(b^*H \cdot E) = 0 \Rightarrow E^2 = -1$.

• $b^*K_S = K_{\hat{S}} + kE \Rightarrow K_{\hat{S}} \cdot E + kE^2 = 0$. $K_{\hat{S}} \cdot E = -1$ (genus formula) $\Rightarrow k = -1$.

• The claim on b_2 follows from standard topological arguments. ∎

Rational maps

Corollary

$C \subset S$, strict transform $\hat{C} \subset \hat{S}$. Then $\hat{C}^2 \leqslant C^2$, $K_{\hat{S}} \cdot \hat{C} \geqslant K_S \cdot C$.

Proof : • $\hat{C}^2 = (b^*C - mE)^2 = C^2 - m^2$.

• $K_{\hat{S}} \cdot \hat{C} = (b^*K_S + E) \cdot (b^*C - mE) = K_S \cdot C + m$. ∎

Definition : Rational map $\varphi : S \dashrightarrow T :=$ morphism $S \supset U \to T$.
We'll always take the largest U such that $\varphi_{|U}$ is a morphism.

• φ is **birational** if $\exists\, U \subset S$, $V \subset T$ such that $\varphi : U \xrightarrow{\sim} V$

– then we say that S and T are birational.

Elimination of indeterminacy

Theorem (Elimination of indeterminacy)

1. $\exists\, u, v$ *morphisms,* $u = b_1 \circ \ldots \circ b_n$ *blowups.*

$$\begin{array}{ccc} & \hat{S} & \\ {\scriptstyle u}\swarrow & & \searrow{\scriptstyle v} \\ S & \overset{\varphi}{\dashrightarrow} & T \end{array}$$

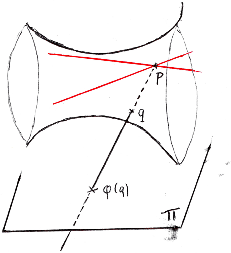
2. *A birational* morphism *is a composition of blowups.*

Remark : (1) holds in higher dimension ("Hironaka's little roof"), but not (2).

Example: stereographic projection

$Q \subset \mathbb{P}^3$ smooth quadric $XT - YZ = 0$. Segre embedding
$s : \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sim} Q \subset \mathbb{P}^3$, $s(U, V; W, S) = (UW, US, VW, VS)$.

For each $p = s(a, b) \in Q$, there are 2 lines $\subset Q$ passing through p:
$s(\mathbb{P}^1 \times b)$ and $s(a \times \mathbb{P}^1)$.



Let $\Pi \subset \mathbb{P}^3$ plane $\not\ni p$.

$\varphi : Q \dashrightarrow \Pi$: $q \neq p \rightsquigarrow \langle p, q \rangle \cap \Pi$.

Extension $f : \hat{Q} \to \Pi$: $\ell \in \mathbb{P}(T_p(Q)) \mapsto \ell \cap \Pi$.

f birational, contracts the 2 lines through p.

$$\begin{array}{ccc} & \hat{Q} & \\ \swarrow^{b} & & \searrow^{f} \\ \mathbb{P}^1 \times \mathbb{P}^1 = Q & \overset{\varphi}{\dashrightarrow} & \Pi = \mathbb{P}^2 \end{array}$$

Some consequences

Corollary

$\varphi : S \dashrightarrow T$ rational. $\exists\, F \subset S$ finite, $\varphi : S \smallsetminus F \to T$ morphism.

Remark : Direct proof easy, see exercises.

Consequences : • Since $\mathrm{Div}(S) \xrightarrow{\sim} \mathrm{Div}(S \smallsetminus F)$ and $\mathrm{Pic}(S) \xrightarrow{\sim} \mathrm{Pic}(S \smallsetminus F)$, $\varphi^* : \mathrm{Div}(T) \to \mathrm{Div}(S)$ and $\mathrm{Pic}(T) \to \mathrm{Pic}(S)$ defined.

• For $C \subset S$, $\varphi(C) := \overline{\varphi(C \smallsetminus F)}$ well-defined.

• $\varphi : S \overset{\sim}{\dashrightarrow} T \;\Rightarrow\; H^0(T, K_T) \xrightarrow{\sim} H^0(S, K_S)$.

(Beware! Not true that $\varphi^* K_T = K_S$, think of blowups)

Proof : $\varphi^* : H^0(T, K_T) \to H^0(S \smallsetminus F, K_S) \xleftarrow{\sim} H^0(S, K_S)$, then $(\varphi^{-1})^* : H^0(T, K_T) \to H^0(S, K_S)$ inverse of φ^*. ■

• $H^0(T, nK_T) \xrightarrow{\sim} H^0(S, nK_S)$ for $n > 0$ (same argument).

• $H^0(T, \Omega^1_T) \xrightarrow{\sim} H^0(S, \Omega^1_S)$ (same argument).

Birational invariants

- The numerical invariants $p_g(S) := h^0(K_S)$ (**geometric genus**), $P_n(S) := h^0(nK_S)$ (**plurigenera**), $q(S) := h^0(\Omega^1_S)$ (**irregularity**) are **birational invariants**.

Definition

A surface is **ruled** if it is birational to $C \times \mathbb{P}^1$.

Proposition

S ruled $\Rightarrow\ P_n(S) = 0\ \forall n \geqslant 1$.

Proof : Suffices to prove it for $S = C \times \mathbb{P}^1$.

$F = \{c\} \times \mathbb{P}^1$ satisfies $F^2 = 0$, hence $K \cdot F = -2$ (genus formula).

If $nK \equiv D \geqslant 0$, D must contain $\{c\} \times \mathbb{P}^1$ for all $c \in C$, impossible. ∎

Irregularity of ruled surfaces

The converse is true, but difficult:

Theorem (Enriques)

$P_n(S) = 0\ \forall n \ \Rightarrow\ S$ ruled.

In fact Enriques proved a more precise result: $P_{12} = 0 \ \Rightarrow\ S$ ruled.

Proposition

S birational to $C \times \mathbb{P}^1 \ \Rightarrow\ q(S) = g(C)$.

Proof: $S = C \times \mathbb{P}^1 \xrightarrow{p} C$. **Claim:** $p^* : H^0(C, K_C) \xrightarrow{\sim} H^0(S, \Omega^1_S)$.
$\omega \in H^0(\Omega^1_S)$, $s : C \hookrightarrow C \times \mathbb{P}^1$, $s(c) = (c, 0)$. Suffices: $\omega = p^* s^* \omega$.
Local coordinates z on C, t on $\mathbb{P}^1 \rightsquigarrow\ \omega = a(z,t)dz + b(z,t)dt$.
$\omega_{\{c\}\times\mathbb{P}^1} = 0 \ \Rightarrow\ b(c,t) \equiv 0\ \forall c \ \Rightarrow\ b = 0$.
$d\omega \in H^0(K_S) = 0 \ \Rightarrow\ \dfrac{\partial}{\partial t} a(z,t) = 0 \ \Rightarrow\ a(z,t) = a(z,0)$,
$\omega = a(z,0)dz = p^* s^* \omega$. ∎

Minimal surfaces

Definition

S **minimal** if any birational morphism $S \to T$ is an isomorphism.

Proposition

Every S admits a birational morphism onto a minimal surface.

Proof : If not, $\exists$ an infinite chain $S \to S_1 \to \cdots \to S_n \to \cdots$ of blowups. This is impossible since $b_2(S_n) = b_2(S) - n$. ∎

Theorem (Castelnuovo's criterion)

Let $E \subset S$, $E \cong \mathbb{P}^1$, $E^2 = -1$. There exists a surface T and a blowing up $b : S \to T$ with exceptional curve E.

Corollary

S minimal $\Leftrightarrow$ $S \not\ni E \cong \mathbb{P}^1$ with $E^2 = -1$.

Exercises

1) Let $b : \hat{S} \to S$ the blowup of $p \in S$, $\hat{C}$ the strict transform of $C \subset S$. Using the genus formula, compute $g(\hat{C})$. Deduce that after a finite number of appropriate blowups, the strict transform of C becomes smooth.

2) Let $\sigma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be given by $\sigma(X, Y, Z) = (YZ, ZX, XY)$ ("*standard quadratic transformation*"). Let $b : P \to \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Show that there is an automorphism s of P, with $s^2 = \mathrm{Id}_P$ and $b \circ s = s \circ \sigma$.

3) Let $\varphi : S \dashrightarrow \mathbb{P}^n$ be a rational map.
a) Show that there exists rational functions $\varphi_0, \ldots, \varphi_n$ on S such that $\varphi(p) = [\varphi_0(p), \ldots, \varphi_n(p)]$ (observe that there is an open subset $U \subset S$ such that $\varphi_{|U}$ is a morphism into $\mathbb{A}^n \subset \mathbb{P}^n$).

b) Prove that there is a finite subset $F \subset S$ such that φ is well-defined outside F (suppose φ is not defined along a curve C; let $p \in C$, $g \in \mathcal{O}_p$ a local equation for C. We can assume that all φ_i are in $\mathcal{O}_p$, with no common factor. But $\varphi_i = 0$ along $C \;\Rightarrow\; g \mid \varphi_i \,\forall\, i$, contradiction.)

4) Let $u : S \to T$ be a birational morphism of surfaces, $C \subset S$ an irreducible curve such that $u(C)$ is a point. Show that $C \cong \mathbb{P}^1$, and $C^2 < 0$.

5) Let $S \subset \mathbb{P}^3$ be a smooth surface of degree d. Using $K_S \equiv (d-4)H$ and the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^3}(-d) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_S \to 0$, compute $P_n(S)$.

Algebraic surfaces

Lecture III: minimal models

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Geometrically ruled surfaces

Definition

- A surface S is **ruled** if it is birational to $C \times \mathbb{P}^1$.
- If $C = \mathbb{P}^1$, we say that S is **rational**.
- S is **geometrically ruled** if $\exists\, p : S \to C$ smooth, fibers $\cong \mathbb{P}^1$.

The last definition is justified by:

Theorem (Noether-Enriques)

$p : S \to C$ geometrically ruled $\Rightarrow$ S ruled.

Note that this is specific to surfaces: there exist smooth morphisms $X \to S$ (S surface) with all fibers $\cong \mathbb{P}^1$, but X not birational to $S \times \mathbb{P}^1$ (*Severi-Brauer varieties*).

Minimal ruled surfaces

Theorem

S ruled not rational. S minimal $\Leftrightarrow$ S geometrically ruled.

Proof : 1) $p : S \to C$ with fibers $\cong \mathbb{P}^1$, $g(C) \geqslant 1$.

If $E \subset S$, $p(E) = q \in \mathbb{P}^1$ since $g(C) \geqslant 1 \Rightarrow E = p^{-1}(q) \Rightarrow E^2 = 0$.

2) $S \cong C \times \mathbb{P}^1 \rightsquigarrow$ rational map $p : S \dashrightarrow C$, $g(C) \geqslant 1$.

Claim : p is a morphism.

If not,

$$\begin{array}{ccccc} & & S_n & & \\ & \swarrow^{u} & & \searrow^{v} & \\ S & & \overset{p}{\dashrightarrow} & & C \end{array}$$

$u : S_n \to S_{n-1} \to \cdots \to S_0 = S$.

$E_n \subset S_n$ exceptional curve; since $g(C) \geqslant 1$, $v(E_n) = \{\text{pt}\} \Rightarrow$ can replace S_n by S_{n-1}, then ... till $S_0 \Rightarrow$ ■ .

End of the proof

3) $p : S \to C$, general fiber $F \cong \mathbb{P}^1$. Want to prove **all** fibers $\cong \mathbb{P}^1$.
Recall: $F^2 = 0$, $K \cdot F = -2$ (genus formula).

- F irreducible $\Rightarrow$ $F \cong \mathbb{P}^1$ (genus formula).
- $F = mF'$? Only possibility $m = 2$, $K \cdot F' = -1$, contradicts genus formula.
- $F = \sum n_i C_i$. **Claim :** $\Rightarrow$ $C_i^2 < 0 \ \forall i$.

Because: $n_i C_i^2 = C_i \cdot (F - \sum_{j \neq i} n_j C_j)$, $C_i \cdot F = 0$, $C_i \cdot C_j \geqslant 0$, and $C_i \cdot C_j > 0$ for some j since F is connected.

- Then $K \cdot C_i = 2g(C_i) - 2 - C_i^2 \geqslant -1$, $= -1 \Leftrightarrow C_i$ exceptional.

So if S minimal, $(K \cdot C_i) \geqslant 0 \ \forall i \ \Rightarrow \ (K \cdot F) \geqslant 0$, contradiction. ∎

Projective bundles

E rank 2 vector bundle on C $\rightsquigarrow$ projective bundle $p : \mathbb{P}_C(E) \to C$, $p^{-1}(x) = \mathbb{P}(E_x)$, so $\mathbb{P}_C(E)$ is a geometrically ruled surface.

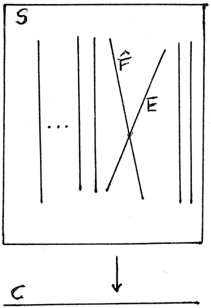
The following can be deduced from the Noether-Enriques theorem:

Proposition

Every geometrically ruled surface is a projective bundle.

There is a highly developed theory of vector bundles on curves, particularly in rank 2; therefore the classification of minimal ruled surfaces is well understood.

Elementary transformation



$f : S \to C$ geometrically ruled. Choose $p \in C$, $q \in F := f^{-1}(p)$. Blow up q.

$\hat{f} : \hat{S} \xrightarrow{b} S \xrightarrow{f} C$. Fiber above $p = E \cup \hat{F}$.

$0 = (\hat{f}^* p)^2 = (E + \hat{F})^2 = E^2 + \hat{F}^2 + 2 \ \Rightarrow$ $\hat{F}^2 = -1$, hence $\hat{F}$ is an exceptional curve (Castelnuovo). Contraction $c : \hat{S} \to S'$:

$$\Longrightarrow \quad \begin{array}{ccccc} & & \hat{S} & & \\ & \overset{b}{\swarrow} & \downarrow \hat{f} & \overset{c}{\searrow} & \\ S & & & & S' \\ & \underset{f}{\searrow} & & \underset{g}{\swarrow} & \\ & & C & & \end{array}$$

$\hat{f}$ induces $g : S' \to C$ geometrically ruled.

Elementary transformation with section

$$S \xleftarrow{\;b\;} \hat{S} \xrightarrow{\;c\;} S'$$

Let $\Sigma \subset S$ be a section of f passing through q.
Then Σ and F are transverse, so $\hat{\Sigma} \cap \hat{F} = \varnothing$ in $\hat{S}$,
and c maps $\hat{\Sigma}$ isomorphically to Σ' section of g.

Then $$\Sigma'^2 = \hat{\Sigma}^2 = (b^*\Sigma - E)^2 = \Sigma^2 - 1\,.$$

Lemma

Suppose $\mathrm{Pic}(S) = \mathbb{Z}[F] \oplus \mathbb{Z}[\Sigma]$. *Then* $\mathrm{Pic}(S') = \mathbb{Z}[F'] \oplus \mathbb{Z}[\Sigma']$.

Proof : It suffices to prove that $(c^*F', c^*\Sigma', \hat{F})$ basis of $\mathrm{Pic}(\hat{S})$.

But $$c^*F' = b^*F,\ c^*\Sigma' = \hat{\Sigma} = b^*\Sigma - E,\ \hat{F} = b^*F - E$$

and $(b^*F, b^*\Sigma, E)$ basis of $\mathrm{Pic}(\hat{S})$. ∎

The surfaces $\mathbb{F}_n$

Proposition

- *For $n \geqslant 0$, $\exists$ a geometrically ruled rational surface $\mathbb{F}_n \to \mathbb{P}^1$, with a section Σ of square $-n$, and $\mathrm{Pic}(\mathbb{F}_n) = \mathbb{Z}[F] \oplus \mathbb{Z}[\Sigma]$.*
- *For $n > 0$, the curve Σ is the only curve of square < 0 on $\mathbb{F}_n$.*

Proof : We start with $\mathbb{F}_0 := \mathbb{P}^1 \times \mathbb{P}^1$, with $f = \mathrm{pr}_1$ and $\Sigma = \mathbb{P}^1 \times \{0\}$. Once $(\mathbb{F}_n, \Sigma)$ is constructed, we choose $q \in \Sigma$: elementary transformation $\rightsquigarrow$ $\mathbb{F}_{n+1} = S'$ with $\Sigma'^2 = -n-1$.

- By the Lemma, $\mathrm{Pic}(\mathbb{F}_n) = \mathbb{Z}[F] \oplus \mathbb{Z}[\Sigma]$.
- Let $C \neq \Sigma$ irreducible curve on $\mathbb{F}_n$. $C \equiv a\Sigma + bF$.

$(C \cdot F) \geqslant 0 \;\Rightarrow\; a \geqslant 0$; $\qquad (C \cdot \Sigma) = -an + b \geqslant 0$

$\Rightarrow\; C^2 = -na^2 + 2ab = a(2b - an) \geqslant an^2 \geqslant 0$. ∎

Minimal rational surfaces

Corollary

$\mathbb{F}_n$ is minimal for $n \neq 1$.

$\mathbb{F}_1$ is obtained by blowing up a point q in $\mathbb{P}^1 \times \mathbb{P}^1$ and contracting one of the lines through q; by stereographic projection, $\mathbb{F}_1 \cong \hat{\mathbb{P}}^2$.

Theorem

The minimal rational surfaces are $\mathbb{P}^2$ and $\mathbb{F}_n$ for $n \neq 2$.

Remark : Being geometrically ruled, the surfaces $\mathbb{F}_n$ are of the form $\mathbb{P}_{\mathbb{P}^1}(E)$. It is not difficult to show that all vector bundles on $\mathbb{P}^1$ are direct sums of line bundles; in fact, it was observed by Hirzebruch that $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$.

Non-ruled surfaces

Theorem

Two birational minimal surfaces **not** *ruled are isomorphic.*

Thus a non-ruled surface admits a **unique** minimal model (up to isomorphism); the birational classification of these surfaces is reduced to the classification (up to isomorphism) of the minimal ones. In contrast, ruled surfaces have a simple birational model ($C \times \mathbb{P}^1$), but the determination of the minimal ones is subtle.

The theorem follows easily from an important Lemma (admitted):

Key lemma

If S is minimal not ruled, $(K \cdot C) \geqslant 0$ for all curves C.

We say that K is **nef**. This is the crucial notion to extend the definition of minimal surface in higher dimension.

Proof of the Theorem

Let $\varphi : S \overset{\sim}{\dashrightarrow} T$, with S, T minimal not ruled. We want to prove that φ is an isomorphism.

We choose a diagram:

$$\begin{array}{ccc} & S_n & \\ {}^{u}\swarrow & & \searrow^{v} \\ S & \overset{\varphi}{\dashrightarrow} & T \end{array}$$

v birational, $u : S_n \to S_{n-1} \to \cdots \to S_0 = S$, with $n \geqslant 1$ minimal $\Rightarrow$ v maps E_n to a curve C.

Since v is a composition of blowups,

$(K_T \cdot C) \leqslant (K_{S_n} \cdot E_n) = -1$, contradicting the key lemma.

Thus φ birational morphism; S minimal $\Rightarrow$ φ isomorphism. ■

Exercises

1) Let C be a curve of genus g. Show that the sections Σ of the fibration $C \times \mathbb{P}^1 \to C$ are in bijective correspondence with the maps $f : C \to \mathbb{P}^1$. Using the genus formula, compute Σ^2 in terms of the degree of f. Show that Σ^2 is even, nonnegative, and $\neq 2$ if $g > 0$.

2) a) Show that the canonical divisor of $\mathbb{F}_n$ is $-2\Sigma + (n-2)F$ and that $K^2 = 8$.

b) We say that a divisor D (or the corresponding line bundle) on a surface S is *nef* if $D \cdot C \geqslant 0$ for all curves C on S. Show that the anticanonical divisor $-K$ on $\mathbb{F}_n$ is nef if and only if $n \leqslant 2$.

c) We say that D is *ample* if $D \cdot C > 0$ for all curves C, and $D^2 > 0$. Show that $-K_{\mathbb{F}_n}$ is ample if and only if $n \leqslant 1$.

Exercises

d) Let S be a surface with $-K_S$ ample. Show that S is obtained from $\mathbb{P}^2$ by blowing up $\leqslant 8$ points (observe that if $-K_T$ is not ample for a surface T, any blowup of T has the same property).

3) We consider the divisor class $H_k := \Sigma + kF$ on the surface $\mathbb{F}_n$.

a) For $k < n$, show that the effective divisors $\equiv H_k$ are sum of Σ and k fibers.

b) Compute $\chi(H_k)$ by Riemann-Roch; deduce that $H^1(H_{n-1}) = 0$.

c) Using the exact sequences

$0 \to \mathcal{O}(H_k) \to \mathcal{O}(H_{k+1}) \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0$, show that $H^1(H_k) = 0$ for $k \geqslant n-1$, and $h^0(H_k) = 2k + 2 - n$.

Algebraic surfaces

Lecture IV: Rational surfaces

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Linear systems and rational maps

$L = \mathcal{O}_S(D) \in \mathrm{Pic}(S)$. (Complete) **linear system :**

$$|L| = |D| := \{E \geqslant 0 \mid E \equiv D\} = \mathbb{P}(H^0(L)).$$

$B_L =$ **Base locus** of $L := \bigcap_{E \in |L|} E = Z \bigcup \{p_1, \ldots, p_s\}$

$Z = \bigcup C_i =$ **fixed part**, p_i **base points**.

Rational map defined by L:

$\varphi_L : S \smallsetminus B_L \to |L|^\vee$, $\varphi_L(p) = \{E \mid p \in E\} =$ hyperplane in $|L|$.

If $Z =$ fixed part of $|L|$, $\varphi_L = \varphi_{L(-Z)}$: can assume L has no fixed part, i.e. B_L finite.

$E \in |L| \rightsquigarrow$ hyperplane $H_E \subset |L|^\vee$;

$\varphi_L^* H_E = \{p \in S \smallsetminus B_L \mid E \in \varphi_L(p) \Leftrightarrow p \in E\} = E \smallsetminus B_L$: $\varphi_L^* H_E = E$.

Properties of φ_L

Properties of φ_L

- φ_L morphism $\Leftrightarrow |L|$ base point free (i.e. $B_L = \varnothing$).
- φ_L injective $\Leftrightarrow \forall\, p \neq q,\ \exists\, E \in |L|,\ p \in E,\ q \notin E$. If this holds:
- φ_L embedding $\Leftrightarrow \forall\, p, v \neq 0 \in T_p(S),\ \exists\, p \in E \in |L|,\ v \notin T_p(E)$.

  If this is the case, we say that L is **very ample**.
- φ_L embedding $\Rightarrow \deg(\varphi_L(S)) = L^2$.

Remark : If D is very ample and $|E|$ is base point free, $D + E$ is very ample.

Examples : • Let H be a line in $\mathbb{P}^2$. The linear system $|nH|$ of curves of degree n ($n \geqslant 1$) is very ample. In particular, φ_{2H} is an isomorphism of $\mathbb{P}^2$ onto a surface $V \subset \mathbb{P}^5$, the **Veronese surface**. We have $\deg(V) = (2H)^2 = 4$; the hyperplane sections of V are conics.

Examples

• On $\mathbb{P}^1 \times \mathbb{P}^1$, let $A = \mathbb{P}^1 \times \{0\}$ and $B = \{0\} \times \mathbb{P}^1$. The linear systems $|A|$ and $|B|$ are base point free, and φ_{A+B} is the Segre embedding in $\mathbb{P}^3$. Hence $aA + bB$ is very ample for $a, b \geqslant 1$. In particular, $|2A + B|$ gives an isomorphism onto a surface of degree 4 in $\mathbb{P}^5$ ("quartic scroll"). Since $A \cdot (2A + B) = 1$, the curves in $|A|$ are mapped to lines in $\mathbb{P}^5$.

• Let $p_1, \ldots, p_s \in S$. Let $|D|$ be a linear system on S, and $P \subset |D|$ the subspace of divisors passing through $p_1, \ldots, p_s$. Assume that at each p_i the curves of P have different tangent directions. Let $b : \hat{S} \to S$ be the blowing up of $p_1, \ldots, p_s$, E_i the exceptional curve above p_i. The system $\hat{D} := b^*D - \sum E_i$ is base point free and defines a morphism $\varphi_{\hat{D}} : \hat{S} \to |\hat{D}|^\vee$ to which we can apply the previous remarks.

Examples (continued)

- Let $p \in \mathbb{P}^2$; consider the system of conics passing through p. It is easy to check that $|2b^*H - E|$ on $\hat{\mathbb{P}}^2_p$ is very ample. It gives an isomorphism onto a surface $S \subset \mathbb{P}^4$, with $\deg(S) = (4H^2 + E^2) = 3$. The strict transforms of the lines through p in $\mathbb{P}^2$ form the linear system $b^*H - E$; since $(b^*H - E)\cdot(2b^*H - E) = 1$, they are mapped to lines in $\mathbb{P}^4$. S is the **cubic scroll**.

- Now let us pass to linear systems of cubic curves.

Proposition

For $s \leqslant 6$, let $p_1, \ldots, p_s \in S = \mathbb{P}^2$, such that no 3 of them lie on a line and no 6 on a conic. The linear system $|-K|$ on $\hat{S}$ is very ample, and defines an isomorphism of $\hat{S}$ onto a surface Σ_d of degree $d := 9 - s$ in $\mathbb{P}^d$, called a **del Pezzo surface***.*

In prticular, Σ_3 is a (smooth) cubic surface in $\mathbb{P}^3$.

Sketch of proof

Sketch of proof : The proof is a long exercise, with no essential difficulty; I will just give an idea. We have $-K_{\hat{S}} = 3b^*H - \sum E_i$, corresponding to the system P of cubics passing through the p_i.

Let us show that φ_{-K} is injective in the most difficult case $s = 6$.

- Let $p \neq q \in \mathbb{P}^2 \smallsetminus \{p_i\}$. Can assume p_1 is not on the line $\langle p, q \rangle$.
- $\exists\,!$ conic Q_{ij} passing through p and the p_k for $k \neq i, j$.
- $Q_{1i} \cap Q_{1j} = \{p\} \cup 3$ other $p_k \Rightarrow q \in$ at most one Q_{1i}, say Q_{12}.
- q is at most on one $\langle p_1, p_i \rangle$, say $\langle p_1, p_3 \rangle$.
- Then $Q_{14} \cup \langle p_1, p_4 \rangle \in P$, $\ni p$, $\not\ni q \;\Rightarrow\; \varphi_{-K}(p) \neq \varphi_{-K}(q)$.
- Then: $\deg(\Sigma_d) = (3b^*H - \sum E_i)^2 = 9 - s = d$; one has $h^0(3H) = 10$, and one checks that $p_1, \ldots, p_s$ impose s independent conditions. ∎

Example : Σ_3 is a smooth cubic surface in $\mathbb{P}^3$; we will see that one obtains all smooth cubic surfaces in that way.

Lines on del Pezzo surfaces

Proposition

lines $\subset \Sigma_d =$ *exceptional curves* $=$ *the* E_i*, the strict transforms of the lines* $\langle p_i, p_j\rangle$ *and of the conics passing through* 5 *of the* p_i *(for* $s = 5$ *or* 6*). Their number is* $s + \binom{s}{2} + \binom{s}{5}$.

Proof : $E \subset \hat{S} \rightsquigarrow$ line in $\Sigma \Leftrightarrow K_{\hat{S}} \cdot E$=-1, i.e. E exceptional.
$E \neq E_i \Rightarrow E \equiv mb^*H - \sum a_i E_i$ in $\mathrm{Pic}(\hat{S})$; $a_i = E \cdot E_i = 0$ or 1.
$(-K) \cdot E = 3m - \sum a_i = 1 \Rightarrow \sum a_i = 2$ and $m = 1$, or $\sum a_i = 5$ and $m = 2$. ∎

Remark : We know more than the number of lines, namely their classes in $\mathrm{Pic}(\Sigma_d)$, their incidence properties, etc. The configuration of lines has been intensively studied in the 19th and 20th century. Let us just mention that the lattice $K^\perp \subset \mathrm{Pic}(\Sigma_d)$ is a *root system*, of type E_6, D_5, A_4, $A_2 \times A_1$ for $s = 6, 5, 4, 3$.

The cubic surface

Proposition

Any smooth cubic surface $S \subset \mathbb{P}^3$ is a del Pezzo surface Σ_3.
In particular, S contains 27 lines.

Strategy of the proof : show that S contains a line, then 2 skew lines; then deduce from that a map $S \to \mathbb{P}^2$ composite of blowups. There are many details to check, left to the reader.

① $\mathbb{G} := \{\text{lines} \subset \mathbb{P}^3\}$, $\dim \mathbb{G} = 4$.

$\mathcal{C} := |\mathcal{O}_{\mathbb{P}^3}(3)| = \{\text{cubic surfaces} \subset \mathbb{P}^3\} \cong \mathbb{P}^c$ $(c = 19)$.

Incidence correspondence: $Z \subset \mathbb{G} \times \mathcal{C} = \{(\ell, S) \mid \ell \subset S\}$.

$$\begin{array}{ccc} & Z & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ \mathbb{G} & & \mathcal{C} \end{array}$$

Fibers of p $\cong \mathbb{P}^{c-4}$ $(S : F = 0$ contains $Z = T = 0 \Leftrightarrow F$ has no $X^3, X^2Y, XY^2, Y^3)$.

Thus $\dim Z = \dim \mathcal{C}$. We want q surjective.

Cubic surface (continued)

If $q : Z \to \mathcal{C}$ not surjective, $\dim q(Z) \leqslant c - 1 \;\Rightarrow\; \dim q^{-1}(S) \geqslant 1$ for $S \in q(Z)$. But $q^{-1}(\Sigma_3)$ finite $\Rightarrow$ impossible.

② $S \supset \ell$. The planes $\Pi \supset \ell$ cut S along a conic.

Claim : 5 of these conics are degenerate, i.e. of the form $\ell_1 \cup \ell_2$.

Proof : $\ell : \; Z = T = 0 \;\Rightarrow$

$F = AX^2 + 2BXY + CY^2 + 2DX + 2EY + G$, with $A, \ldots, G$ homogeneous polynomials in Z, T. The conic is degenerate

$$\Leftrightarrow \det \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & G \end{vmatrix} = 0,$$ degree 5 in Z, T. $\geqslant 2$ distinct roots $\Rightarrow$

$S \supset 2$ triangles: $\ell \cup \ell_1 \cup \ell_1'$, $\ell \cup \ell_2 \cup \ell_2'$. Then $\ell_1 \cap \ell_2 = \varnothing$.

Cubic surface (continued)

③ $\ell \subset S$, given by $X = Y = 0$. Projection from ℓ: $S \xrightarrow{(X,Y)} \mathbb{P}^1$.

Well-defined: $S : XB - YA = 0$, $(X, Y) = (A, B)$ on S,

$X = Y = A = B = 0 \;\Rightarrow\; S$ singular.

$\varphi_i : S \to \mathbb{P}^1$ projection from $\ell_i \;\rightsquigarrow\; \varphi = (\varphi_1, \varphi_2) : S \to \mathbb{P}^1 \times \mathbb{P}^1$.

Geometrically, $\varphi_i(p) =$ plane $\langle \ell_i, p\rangle$ through ℓ_i.

Birational: for $(\pi_1, \pi_2) \in \mathbb{P}^1 \times \mathbb{P}^1$, $\pi_1 \cap \pi_2 =$ line meeting ℓ_1 and ℓ_2, intersects S along a unique third point p.

$\Rightarrow \;\varphi =$ composition of blowups. Blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at 1 point $=$ blowup of $\mathbb{P}^2$ at 2 points $\Rightarrow \;\varphi' : S \to \mathbb{P}^2$ composition of blowups.

λ line contracted by $\varphi \iff \pi_1(\lambda) = \{p\}, \pi_2(\lambda) =$ pts

$\iff \lambda$ meets ℓ_1 and ℓ_2.

For each of the 5 triangles $\ell_1, \ell_1', \ell_1''$, ℓ_2 meets one of $\ell_1', \ell_1'' \;\Rightarrow$

5 lines contracted $\Rightarrow \;S \cong \mathbb{P}^2$ with 6 points blown up.

Exercises

1) Show that the linear system $|\Sigma + nF|$ on $\mathbb{F}_n$ defines a morphism $\mathbb{F}_n \to \mathbb{P}^{n+1}$, which is an embedding outside Σ and contracts Σ to a point p. Show that the image of $\mathbb{F}_n$ is a cone with vertex p, and that the hyperplane sections not passing through p are rational normal curves of degree n in $\mathbb{P}^n$ (use exercise 3 of Lecture II).

2) Show that the linear system $|\Sigma + kF|$ on $\mathbb{F}_n$ for $k > n$ defines an isomorphism of $\mathbb{F}_n$ onto a surface of degree $2k - n$ in $\mathbb{P}^{2k-n+1}$. The images of the fibers are disjoint lines, and that of Σ is a rational normal curve of degree $n + k$.

3) Let $\mathcal{S}$ be the vector space of symmetric 3×3 matrices. Show that the locus of rank 1 matrices in $\mathbb{P}(\mathcal{S}) \cong \mathbb{P}^5$ is a Veronese surface V. Deduce that all secants to V (i.e. the lines $\langle p, q\rangle$, $p \neq q \in V$) are contained in a cubic hypersurface.

[Note: the secant lines depend on $2+2$ parameters, so one would expect that their union fills $\mathbb{P}^5$. It is a classical theorem of Severi that the Veronese surface is the only smooth surface in $\mathbb{P}^5$ (not contained in a hyperplane) with this property.]

4) a) Let C be a smooth rational curve of degree e on a del Pezzo surface Σ_d. Show that $C^2 = e - 2$. Prove that the linear system $|C|$ has dimension $e - 1$ (use the exact sequence $0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_S(C)_{|C} \to 0$).

b) Describe in terms of $\mathbb{P}^2$ with $9 - d$ points blown up the pencils (= linear systems of dimension 1) of conics on Σ_d. Find their number.

Exercises

c) We fix $e = 3$. Show that the linear system $|C|$ is base point free, and defines a birational morphism to $\mathbb{P}^2$ (use the exact sequence of a). Conversely, any birational morphism $\Sigma_d \to \mathbb{P}^2$ is defined by a net (= linear systems of dimension 2) of twisted cubics.

d) Describe the nets of twisted cubics on Σ_3. Show that there are 72 such nets.

5) A *double-six* in $\mathbb{P}^3$ consists of 2 sets of disjoint lines $\ell_1, \ldots, \ell_6$ and $\ell'_1, \ldots, \ell'_6$, such that $\ell_i \cap \ell'_j \neq \varnothing$ for $i \neq j$ and $\ell_i \cap \ell'_i = \varnothing$.

a) Show that in a cubic surface Σ_3, the images of $E_1, \ldots, E_6$ and of the conics passing through 5 of the p_i form a double-six.

b) Conversely, given a double-six (ℓ_i, ℓ'_j) on Σ_3, there is a birational morphism $\mathbb{S}_3 \to \mathbb{P}^2$ contracting the ℓ_i to points p_i and mapping the ℓ'_j to conics through 5 of the p_i.

c) Conclude that there are 36 double-six on Σ_3.

Algebraic surfaces

Lecture V: The Kodaira dimension

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Kodaira dimension

The key ingredient to distinguish different projective varieties is the behaviour of the canonical bundle.

Definition

The **Kodaira dimension** of a surface S is

$$\kappa(S) := \max_n \dim \varphi_{nK}(S)$$

with the convention $\dim \varnothing = -\infty$.

Using the plurigenera $P_n = h^0(nK)$, this translates as

- $\kappa(S) = -\infty \iff P_n = 0 \ \forall n \iff S$ ruled (Enriques theorem).
- $\kappa(S) = 0 \iff P_n = 0$ or $1 \ \forall n$, and $= 1$ for some n.
- $\kappa(S) = 1 \iff P_n \geqslant 2$ for some n, and $\dim \varphi_{mK}(S) \leqslant 1 \ \forall m$;
- $\kappa(S) = 2 \iff \dim \varphi_{nK}(S) = 2$ for some n.

Examples

- Let B, C be two curves of genus b, c. Then:
  - $\kappa(B \times C) = -\infty \iff bc = 0$;
  - $\kappa(B \times C) = 0 \iff b = c = 1$;
  - $\kappa(B \times C) = 1 \iff b$ or $c = 1$, $bc > 1$;
  - $\kappa(B \times C) = 2 \iff b$ and $c \geqslant 2$.
- Let $S_d \subset \mathbb{P}^3$ of degree d; then S_d is rational for $d \leqslant 3$, $\kappa(S_4) = 0$, $\kappa(S_d) = 2$ for $d \geqslant 5$.

These examples show a general pattern: most surfaces have $\kappa = 2$ (they are called **of general type**), some have $\kappa = 1$, and the cases $\kappa = 0$ and $\kappa = -\infty$ are completely classified.

Remark : S minimal, $\kappa(S) \geqslant 0 \Rightarrow K_S^2 \geqslant 0$.
Indeed $|nK_S| \ni E$ for some $n \geqslant 1$, and $K \cdot E \geqslant 0$ by the key lemma.

$\kappa = 2$

Proposition

Let S be a minimal surface. The following are equivalent:

1. $\kappa(S) = 2$;
2. $K^2 > 0$ *and S not ruled;*
3. φ_{nK} *birational onto its image for* $n \gg 0$.

Proof : ③ $\Rightarrow$ ① clear.

② $\Rightarrow$ ③: let H be a very ample divisor on S. Riemann-Roch $\rightsquigarrow$
$\chi(nK - H) \sim \frac{1}{2}n^2K^2 > 0$ for $n \gg 0$, hence
$h^0(nK - H) + h^0((1-n)K + H) > 0$.
But $\big((1-n)K + H\big) \cdot K < 0$ for $n \gg 0$, hence $h^0 = 0$ by key Lemma
$\Rightarrow h^0(nK - H) > 0$, hence $nK \equiv H + E$, $E \geqslant 0 \Rightarrow \varphi_{nK}$ birational.

$\kappa = 2$ (continued)

① $\Rightarrow$ ②: $\kappa(S) = 2 \Rightarrow S$ not ruled and $K^2 \geqslant 0$. But $K^2 > 0$ by:

Lemma

S minimal, $K^2 = 0$, $|nK| = Z + M$ with Z fixed part. Then M is base-point free, and $\varphi_M = \varphi_{nK} : S \to C \subset |nK|^\vee$.

.

Proof : Key lemma $\Rightarrow$ $(K \cdot Z)$ and $(K \cdot M) \geqslant 0$, hence $= 0$.

$0 = M \cdot (Z + M) \Rightarrow M^2 = 0 \Rightarrow |M|$ base-point free, hence $\varphi_M : S \to C \subset |nK|^\vee$. $M^2 = 0 \Rightarrow C$ curve. ∎

Remark: $\exists$ much more precise results for ③ (Kodaira, Bombieri): φ_{nK} morphism for $n \geqslant 4$, birational for $n \geqslant 5$.

Example: For $S = B \times C$ as above,

$K^2_{B\times C} = (p^*K_B \cdot q^*K_C) = (2b-2)(2c-2)$: $K^2_X > 0 \Leftrightarrow b, c \geqslant 2$.

Surfaces with $\kappa = 1$

Proposition

S minimal, $\kappa(S) = 1 \;\Rightarrow\; K^2 = 0$, and $\exists\, p : S \to B$ with general fiber elliptic curve.

(We say that S is an **elliptic surface**.)

Proof : Choose n such that $h^0(nK) \geqslant 2$, $|nK| = Z + |M|$. By the Lemma, $\varphi_M : S \to C \subset |nK|^\vee$.

Stein factorization: $\varphi_M : S \xrightarrow{p} B \to C$, with fibers of p connected.

F smooth fiber. $F \leqslant M \;\Rightarrow\; K \cdot F = 0,\, F^2 = 0 \;\Rightarrow\; g(F) = 1$ (genus formula). ∎

Remark : An elliptic surface can be rational, ruled, or have $\kappa = 0$.

Surfaces with $\kappa = 0$

Theorem

S minimal with $\kappa = 0$.

1. *$q = 0$, $K \equiv 0$: S is a* **K3 surface**;
2. *$q = 0$, $2K \equiv 0$, $K \not\equiv 0$: S is an* **Enriques surface** *– quotient of a K3 by a fixed-point free involution.*
3. *$q = 1$: S is a* **bielliptic surface**, *quotient of a product $E \times F$ of elliptic curves by a finite group acting freely (7 cases).*
4. *$q = 2$: S is an* **abelian surface** *(projective complex torus).*

We will treat only the cases with $q = 0$ (the other cases require the theory of the Albanese variety). If $K \equiv 0$, we are in case ①.

We want to prove that $\quad q = 0,\ K \not\equiv 0 \ \Rightarrow\ 2K \equiv 0.$

S minimal, $q = 0$, $K \not\equiv 0$

Proof : We have $h^0(nK) = 0$ or 1 $\forall n \geqslant 1$, and $K^2 = 0$ by the case $\kappa = 2$. We first prove $p_g = h^0(K) = 0$.

If $h^0(K) = 1$ Riemann-Roch gives

$$h^0(-K) + h^0(2K) \geqslant \chi(\mathcal{O}_S) = 1 - q + p_g = 2\,,$$

hence $h^0(-K) \geqslant 1$. Thus $\exists\, A \in |K|, B \in |-K| \;\Rightarrow\; A + B \equiv 0$ $\Rightarrow\; A = B + 0$, $K \equiv 0$, excluded. Hence $h^0(K) = 0$.

Then: $\qquad h^0(-K) + h^0(2K) \geqslant \chi(\mathcal{O}_S) = 1.$

If $h^0(-K) > 0$, $|-K| \ni D \geqslant 0$, $|nK| \ni E \geqslant 0$, $nD + E \equiv 0 \;\Rightarrow$ $D \equiv 0$, contradiction. Hence $h^0(2K) > 0$.

Riemann-Roch: $h^0(3K) + h^0(-2K) \geqslant 1$. Suppose $h^0(3K) \geqslant 1$.

$D \in |2K|$, $E \in |3K|$; $3D, 2E \in |6K| \;\Rightarrow\; 3D = 2E \Rightarrow$ $D = 2F, E = 3F$ with $F \geqslant 0$. But $F \equiv E - D \equiv K$, contradiction.

Therefore $h^0(-2K) > 0$, and $2K \equiv 0$. ∎

The double cover of an Enriques surface

Let S be an Enriques surface. View $\mathcal{K}_S$ as a line bundle $p : \mathcal{K} \to S$; we have a non-vanishing section ω of $H^0(2K)$. Let

$$X = \{x \in \mathcal{K} \mid x^2 = \omega(px)\}$$

It is a closed subvariety of $\mathcal{K}$; for each $y \in S$ there are 2 points in X above y, exchanged by the involution $\sigma : x \mapsto -x$. This involution acts freely, and p_X identifies S with X/σ.

The morphism $p_X : X \to S$ is étale, hence $p_X^*\mathcal{K}_S \cong \mathcal{K}_X$.

Consider the pull back diagram:

$$\begin{array}{ccc} \mathcal{K} \times_S \mathcal{K} & \longrightarrow & \mathcal{K} \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{K} & \xrightarrow{\ p\ } & S \end{array}$$

p' has a canonical section $x \mapsto (x, x)$; this section does not vanish outside the zero section of $\mathcal{K}$. Therefore $p^*\mathcal{K}_{|S} = \mathcal{K}_X$ is trivial.

We will admit $q = 0$, so X is a K3 surface. ■

Examples

- $S_4 \subset \mathbb{P}^3$ (smooth) is a K3 surface.

Indeed $K_{S_d} \equiv (d-4)H$, so $\equiv 0$ for $d = 4$. To prove $q = 0$ we admit a classical result:

Lemma

$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = 0$ for all k and $0 < i < n$.

Then from the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_S \to 0$ we get $H^1(\mathcal{O}_S) = 0$. ■

- More generally, for each $g \geqslant 3$, there is a family of K3 surfaces of degree $2g-2$ in $\mathbb{P}^g$: in $\mathbb{P}^4$ we get the intersection of a quadric and a cubic, in $\mathbb{P}^5$ the intersection of 3 quadrics, etc. These surfaces have a rich geometry and have been, and still are, extensively studied.

In $\mathbb{P}^5$, with homogeneous coordinates $X_0, X_1, X_2, X'_0, X'_1, X'_2$, consider the surface S defined by

$$P(X) + P'(X') = Q(X) + Q'(X') = R(X) + R'(X') = 0\,,$$

where $P, Q, R; P', Q', R'$ are general quadratic forms in 3 variables. The involution $\sigma : (X_i, X'_j) \mapsto (-X_i, X'_j)$ preserves S; its fixed points are the 2-planes $X_i = 0$ and $X'_j = 0$, which are not on S since the quadratic forms are general. The surface quotient S/σ is an Enriques surface.

THE END

Exercises

1)Let S be a K3 surface, $C \subset S$ a curve of genus g.

a) Show that $C^2 = 2g - 2$ and $h^0(C) = g + 1$ (deduce from the exact sequence $0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$ that $H^1(\mathcal{O}_S(-C)) = 0$).

b) Show that the restriction of $\mathcal{O}_S(C)$ to C has degree $2g - 2$ and $h^0 = g$, hence is $\cong \mathcal{K}_C$.

c) Deduce from b) that $|C|$ is base point free. If C is not hyperelliptic, show the morphism φ_C is birational onto its image.

2) a) Let C, C' two cubic curves in $\mathbb{P}^2$, which intersect transversally at 9 points $p_1, \ldots, p_9$. Let $\hat{P}$ be the bowup of $\mathbb{P}^2$ at these points. Show that the anticanonical system $|-K_{\hat{P}}|$ is base point free, and defines a morphism $\hat{P} \to \mathbb{P}^1$ whose general fiber is a plane cubic, hence an elliptic curve.

b) Let S be a smooth quartic surface in $\mathbb{P}^3$ containing a line ℓ, defined by $X = Y = 0$. Show that (X, Y) define a morphism $S \to \mathbb{P}^1$ whose general fiber is a plane cubic.

3) Let S be a K3 surface, D an effective divisor on S with $D^2 = 0$ and $D \cdot C \geqslant 0$ for every curve C on S. Show that $D \equiv mE$, where $m \geqslant 1$ and E is a smooth elliptic curve.

(Let Z be the fixed part of $|D|$, so that $D \equiv Z + M$; prove $D \cdot Z = 0$, then $Z^2 = 0$, which implies $Z = 0$ by Riemann-Roch. Then use the same argument as in the Lemma.)

Exercises

4) Let S be an Enriques surface, E an elliptic curve on S. Show that either $|E|$ or $|2E|$ is a base point free pencil of elliptic cuves. (Use the exact sequence $0 \to \mathcal{O}_S \to \mathcal{O}_S(E) \to \mathcal{O}_S(E)_{|E} \to 0$. If $\mathcal{O}_S(E)_{|E} = \mathcal{O}_E$, $|E|$ is a base point free pencil. If not, observe that $|K + E|$ contains a divisor E' by Riemann-Roch; then $|2E|$ contains $2E$ and $2E'$, and the above exact sequence tensored by $\mathcal{O}_S(E)$ shows that $h^0(2E) = 2$.)

5) Let S be a surface, $p : S \to B$ a morphism onto a curve with connected fibers. Suppose a fiber F is reducible, i.e. $F = \sum n_i C_i$. Let $D = \sum r_i C_i$, with $r_i \in \mathbb{Z}$. Show that $D^2 \leqslant 0$, and $D^2 = 0$ if and only if $D \equiv kF$ for some $k \in \mathbb{Q}$.
(Write $G_i = n_i C_i$ and $s_i = \dfrac{r_i}{n_i} \in \mathbb{Q}$, so that $D = \sum s_i G_i$; using $G_i^2 = G_i \cdot (F - \sum_{j \neq i} G_j)$, prove that $D^2 = \sum_{i \neq j} (s_i - s_j)^2 G_i \cdot G_j$.)

Exercises

6) Let S be a minimal surface with a morphism $p : S \to B$ onto a curve, whose general fiber is an elliptic curve. By a theorem of Zariski all fibers of p are connected.

a) Suppose a fiber is reducible, hence $= \sum n_i C_i$. Using exercise 5, show that $C_i^2 < 0$ for all i. Deduce that C_i is smooth rational and $C_i^2 = -2$.

b) Suppose $\kappa(S) \geqslant 0$. Show that there exists an integer d such that $dK \equiv p^*D$ for some $D \geqslant 0$ on B (let $D \in |rK|$; since $D \cdot F = 0$, D is contained in some fibers. Apply exercise 5.)