Non-abelian theta functions and the theta map

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line bundles trivial on $\mathbb{C}^g$ $\Rightarrow$ theta functions lift to functions on $\mathbb{C}^g$, quasi-periodic w.r.t. Γ.

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Gives explicit description of J as submanifold of $\mathbb{P}^N$; much is known about its equations, geometry etc.

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G-theta functions of level k := elements of $H^0(\mathcal{M}_G, \mathcal{L}^k)$

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In other words, $H^0(\mathcal{M}_G, \mathcal{L}^k)$ carries a (projective) representation of the modular group $\Gamma_g = \pi_1(\mathcal{M}_g)$.

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Aim of the talk : understand $\mathcal{L}$ and $H^0(\mathcal{M}_G, \mathcal{L})$, in particular, the **theta map** $\varphi_{\mathcal{L}} : \mathcal{M}_G \dashrightarrow |\mathcal{L}|^*$.

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Thus get map $\theta : \mathcal{M}_{SL(r)} \dashrightarrow |r\Theta|$, $\theta(E) = \Theta_E$.

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In particular $\qquad H^0(\mathcal{M}_{SL(r)}, \mathcal{L}) \xrightarrow{\sim} H^0(J, \mathcal{O}_J(r\Theta))^*.$

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$\mathcal{S}^* \subset |3\Theta|^*$ is the Coble cubic, the unique cubic hypersurface in $|3\Theta|^*$ singular along the image of J.

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Map $\mathcal{M}_{SO(r)} \twoheadrightarrow \mathcal{M}^{\mathcal{O}}_{O(r)} := \{(E,q) \in \mathcal{M}_{O(r)} \mid \wedge^r E = \mathcal{O}_C\}$.

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Technical point: $O(r)$ versus $GL(r)$

Theorem (Serman)

The map $\mathcal{M}_{O(r)} \to \mathcal{M}_{GL(r)}$ *is an embedding.*

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   Thus $\Theta_E \in |r\Theta|^+$ or $|r\Theta|^-$, the eigenspaces of i^* in $|r\Theta|$.

$H^0(\mathcal{M}^{\pm}_{SO(r)}, \mathcal{L})$

Theorem

$$\begin{array}{ccc} \mathcal{M}^{\pm}_{SO(r)} & \overset{\theta^{\pm}}{\dashrightarrow} & |r\Theta|^{\pm} \\ \downarrow & & \downarrow \\ \mathcal{M}_{SL(r)} & \overset{\theta}{\dashrightarrow} & |r\Theta| \end{array}$$

$\theta^{\pm} =$ *theta map for* $\mathcal{M}^{\pm}_{SO(r)}$.

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(Essential ingredient: Verlinde formula for $SO(r)$.)

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|3\Theta|^{-}_{(\cong \mathbb{P}^3)} & \hookrightarrow & |3\Theta|_{\ (\cong \mathbb{P}^8)} & \hookleftarrow & |3\Theta^{+}|_{\ (\cong \mathbb{P}^4)} & \hookleftarrow & \mathcal{Q} \\
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$\mathcal{S} \cap |3\Theta^+| = \mathcal{Q} + 2H$, $\mathcal{Q} =$ Igusa quartic, $H = \Theta + |2\Theta| \subset |3\Theta|^+$.

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$\Big($That is, $\big(H^0(J, \mathcal{O}_J(2r\Theta))^*\big)^+ \longrightarrow H^0(\mathcal{M}_{Sp(2r)}, \mathcal{L})$ not bijective$\Big)$

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Replace J by $\mathcal{N} := \{F \in \mathcal{M}_{GL(2)} \mid \det F = K_C\} \quad (\cong \mathcal{M}_{SL(2)})$

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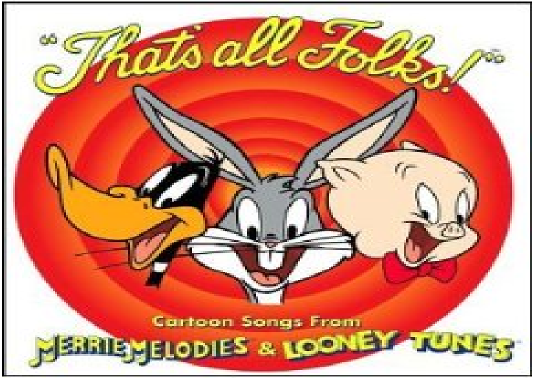
(Proof relies on the rank-level duality $SL(2) - GL(r)$ proved by Marian-Oprea and Belkale.)

The end


"That's all Folks!"
Cartoon Songs From
MERRIE MELODIES & LOONEY TUNES