Riemannian holonomy and algebraic geometry

Arnaud Beauville

Université de Nice

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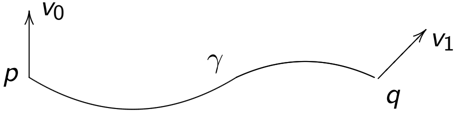
Parallel transport

(M, g) Riemannian manifold $\rightsquigarrow$ parallel transport:



$\rightsquigarrow \ \varphi_\gamma : T_p(M) \xrightarrow{\sim} T_q(M)$

$$\text{with} \quad \varphi_\gamma \circ \varphi_\delta = \varphi_{\delta\gamma} \ .$$

Idea : $\gamma : [0,1] \to M$, we look for $t \mapsto v(t) \in T_{\gamma(t)}(M)$

- If $M = \mathbb{R}^n$ (euclidean), one imposes $\dot{v}(t) = 0$;
- If $M \subset \mathbb{R}^n$, one imposes $\dot{v}(t) \perp T_{\gamma(t)}(M)$;

  linear first order ODE, unique solution with $v(0) = v_0$.

Holonomy

In particular, $\varphi : \{\text{loops at } p\} \longrightarrow O(T_p(M))$

$$\text{Image} = H_p = \text{holonomy (sub-)group at } p$$

- independent of p up to conjugacy (M connected).

For simplicity, we assume M simply connected and compact.

$\Rightarrow$ H_p compact, connected Lie subgroup of $SO(T_p(M))$

(Borel-Lichnerowicz)

de Rham's theorem

Theorem (de Rham)

$T_p(M) = \bigoplus_i V_i$ *stable under* $H_p \Rightarrow M \cong \prod_i M_i$ *et* $H_p \cong \prod_i H_{p_i}$.

We are reduced to irreducible manifolds, i.e. with irreducible holonomy representation.

We first exclude a well-known class of manifolds, the symmetric spaces: G/H, with G compact Lie group, $H = \mathrm{Fix}(\sigma)^{\mathrm{o}}$, σ involution of G. Complete list (E. Cartan), $H_p = H$.

Berger's theorem

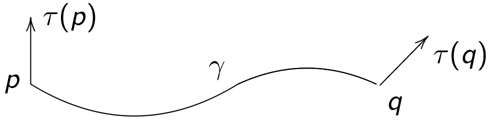
Theorem (Berger)

M irreducible ($\pi_1(M) = 0$), *non symmetric. Then* $H =$

H	$\dim(M)$	*metric*
$SO(n)$	n	generic
$U(m)$	$2m$	Kähler
$SU(m)$ $(m\geq 3)$	$2m$	Calabi-Yau
$\mathrm{Sp}(r)$	$4r$	hyperkähler
$\mathrm{Sp}(r)\mathrm{Sp}(1)$ $(r\geq 2)$	$4r$	quaternion-Kähler
G_2	7	
$\mathrm{Spin}(7)$	8	

What is holonomy good for?

A vector field (more generally, a tensor field) τ is parallel if



$$\varphi_\gamma(\tau(p)) = \tau(q)$$

for every path γ from p to q.

Holonomy principle

Evaluation at p gives a bijective correspondence between:

- parallel tensor fields;
- tensors on $T_p(M)$ invariant under H_p.

Examples: SO, U

Hence : fixing $H \iff$ imposing certain parallel tensor fields.

(small holonomy $\Rightarrow$ special manifold)

$$SU(m) \subset U(m) \subset SO(2m)\,,\ Sp(r) \subset SU(2r)\,,\ Sp(r) \subset Sp(1)Sp(r)$$

Examples

- $H = SO(n) \iff$ no parallel tensor fields

  (except g and dv_g): generic metric.

- $H \subset U(m) \subset SO(2m)$

  $\iff$ H commutes with endomorphism $v \mapsto iv$ of $\mathbb{R}^{2m} \cong \mathbb{C}^m$

  $\iff$ parallel endomorphism J of $T(M)$, $J^2 = -I$

  $\iff$ M has a Kähler complex structure J.

$SU(m)$

$H \subset SU(m) \iff H \subset U(m)$ and H preserves the $\mathbb{C}$-multi-linear alternating m-form $\det : \mathbb{C}^m \to \mathbb{C}$

$\iff M$ Kähler + holomorphic parallel m-form $\omega \neq 0$

(locally, $\omega = f(z)\, dz_1 \wedge \ldots \wedge dz_m$)

Theorem (Yau)

M complex manifold, $\dim_{\mathbb{C}}(M) = m$.

$\Updownarrow$

M admits a Kähler metric with holonomy $\subset SU(m)$;

M kählerian, $\exists$ holomorphic m-form everywhere $\neq 0$.

$\Rightarrow$ many examples: hypersurfaces of degree $n+1$ in $\mathbb{P}^n$, etc.

$Sp(r)$ – hyperkähler point of view

$Sp(r) := U(r, \mathbb{H}) =$ subgroup of $GL(r, \mathbb{H})$ preserving the hermitian form $\psi(x, y) = \sum x_i \bar{y}_i$.

2 ways of looking at quaternions:

- "Hamilton": $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$, $\mathbb{H}^r \cong \mathbb{R}^{4r}$.

  $Sp(r) =$ subgroup of $O(\mathbb{R}^{4r})$ commuting with i, j, k .

  $H \subset Sp(r) \iff$ parallel complex structures I, J, K,

  actually a sphere $\mathbb{S}^2$:

$$\mathbb{S}^2 = \{aI + bJ + cK,\ a^2 + b^2 + c^2 = 1\} \ .$$

We say that M est hyperkähler.

$Sp(r)$ – holomorphic symplectic point of view

- "Cayley": $\mathbb{C} = \mathbb{R} + \mathbb{R}i$, $\mathbb{H} = \mathbb{C}(j)$ with $jz = \bar{z}j$; $\mathbb{H}^r \cong \mathbb{C}^{2r}$.

  $\psi = h + \varphi j$ with h $\mathbb{C}$-hermitian and φ $\mathbb{C}$-bilinear alternating.

  Thus $$Sp(r) = U(2r,\mathbb{C}) \cap Sp(2r,\mathbb{C}) \ .$$

$$H = Sp(r) \Longleftrightarrow \begin{cases} \text{complex Kähler structure } + \\ \text{parallel holomorphic symplectic 2-form } \varphi, \\ \text{unique up to a scalar} \end{cases}$$

Theorem

M kählerian with holomorphic symplectic 2-form φ $\Rightarrow$

M admits a hyperkähler metric.

Proof: φ^r $(2r)$-form $\neq 0$ $\Rightarrow$ M admits a Kähler metric with holonomy $\subset SU(m)$ (Yau); for such a metric, every holomorphic tensor field is parallel (Bochner).

Examples of hyperkähler manifolds

Examples

1. $r = 1$: $Sp(1) = SU(2)$, $M =$ complex surface (compact) with holomorphic 2-form everywhere $\neq 0 \stackrel{\text{def}}{=}$ K3 surface.

2. $r > 1$? Idea: S^r admits symplectic forms, in fact too many:

$$\sigma = \lambda_1 p_1^*\varphi + \ldots + \lambda_r p_r^*\varphi \ , \quad \text{with} \ \ \lambda_1, \ldots, \lambda_r \in \mathbb{C}^* \ .$$

   To get unicity, try to impose $\lambda_1 = \ldots = \lambda_r$, i.e.:

   σ comes from $S^{(r)} := S^r/\mathfrak{S}_r$.

   $S^{(r)}$ is singular, but admits a resolution $S^{[r]}$, the Hilbert scheme (or Douady space).

   σ symplectic form on $S^{[r]} \ \Rightarrow \ S^{[r]}$ hyperkähler.

Examples of hyperkähler manifolds 2

1. a
2. a

Examples (continued)

3. Analogous construction starting from a 2-dim'l complex torus

   $\rightsquigarrow$ generalized Kummer varieties K_r.

4. 2 isolated examples (O'Grady), of dimension 6 and 10.

No other example known! (up to deformation)

$Sp(1)Sp(r)$

$Sp(r) = U(r, \mathbb{H})$ commutes with homotheties, in particular with

$$\mathbb{H}_1^\times = \{\text{quaternions with norm } 1\} \cong Sp(1) \ ,$$

hence a group $Sp(1)Sp(r) \subset SO(4r)$, $\not\subset U(2r)$.

It preserves the sphere

$$\mathbb{S}^2 = \{aI + bJ + cK,\ a^2 + b^2 + c^2 = 1\} \subset \mathrm{End}(\mathbb{R}^{4r}) \ .$$

For M with holonomy $Sp(1)Sp(r)$ ("quaternion-Kähler"), get sphere $\mathbb{S}^2_p \subset \mathrm{End}(T_p(M))$ at each $p \in M$.

The union of these spheres is the twistor space $t : Z \to M$.

Theorem (Salamon)

Z has a natural complex structure, such that $t^{-1}(m) \cong \mathbb{P}^1 \quad \forall m$, and a holomorphic contact structure.

Contact and complex structures on Z

contact structure = odd-dim'l analogue of symplectic structure

= sub-bundle of hyperplanes $H \subset T(Z)$,

defined locally by 1-form η such that $d\eta_{|H}$ symplectic.

Idea of the construction

$$\text{For } (p,J) \in Z\,,\ \ T_{(p,J)}(Z) = T_p(M) \oplus T_J(\mathbb{S}^2)$$

- complex structure J on $T_p(M)$, standard on $T_J(\mathbb{S}^2)$
- contact structure: $H_{(p,J)} = T_p(M) \subset T_{(p,J)}(Z)$. ∎

Two cases, according to the sign of the scalar curvature.

Negative case: Z not Kähler, no example known.

Contact projective manifolds

Positive case : Z is a projective manifold, even Fano.

(i.e.: K_Z^{-N} has many sections for $N \gg 0$).

Examples of contact projective manifolds

1. $\mathbb{P}T^*(X)$ for every projective manifold X;
2. $\mathfrak{g}$ simple Lie algebra; $\mathcal{O}_{min} \subset \mathbb{P}(\mathfrak{g})$ unique closed adjoint orbit.

   (example: rank 1 matrices in $\mathbb{P}(\mathfrak{sl}_r)$.)

Conjectures

1. These are the only contact projective manifolds.
2. Every quaternion-Kähler positive manifold is symmetric.

① ⇒ ② : Z Fano $\Rightarrow$ $Z = \mathcal{O}_{min}$ $\Rightarrow$ M symmetric (Wolf space).

Partial results

Every contact projective manifold is $\mathbb{P}T^*(X)$ or $\mathcal{O}_{min}$?

Partial results

Z contact projective manifold, $L := T(Z)/H$

1. If Z is not Fano, $Z \cong \mathbb{P}T^*(X)$

   (Kebekus, Peternell, Sommese, Wiśniewski + Demailly)

2. Z Fano and L has "enough sections" $\Rightarrow$ $Z \cong \mathcal{O}_{min} \subset \mathbb{P}(\mathfrak{g})$

   (AB; note: Z Fano $\Rightarrow$ L^N has many sections for $N \gg 0$)

THE END