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Classical theta functions and their generalization

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Abstract. We first recall the modern theory of classical theta functions, viewed as sections of line bundles on complex tori. We emphasize the case of theta functions associated to an algebraic curve C: they are sections of a natural line bundle (and of its tensor powers) on the Jacobian of C, a complex torus which parametrizes topologically trivial line bundles on C. Then we explain how replacing the Jacobian by the moduli space of higher rank vector bundles leads to a natural generalization ("non-abelian theta functions"). We present some of the main results and open problems about these new theta functions.

Introduction

Theta functions are holomorphic functions on $\mathbb{C}^g$, quasi-periodic with respect to a lattice. For $g=1$ they have been introduced by Jacobi; in the general case they have been thoroughly studied by Riemann and his followers. From a modern point of view they are sections of line bundles on certain complex tori; in particular, the theta functions associated to an algebraic curve C are viewed as sections of a natural line bundle (and of its tensor powers) on a complex torus associated to C, the *Jacobian*, which parametrizes topologically trivial line bundles on C.

Around 1980, under the impulsion of mathematical physics, the idea emerged gradually that one could replace in this definition line bundles by higher rank vector bundles. The resulting sections are called *generalized* (or *non-abelian*) *theta functions*; they turn out to share some (but not all) of the beautiful properties of classical theta functions.

These notes follow closely my lectures in the Duksan workshop on algebraic curves and Jacobians. I will first develop the modern theory of classical theta functions (complex tori, line bundles, Jacobians), then explain how it can be generalized by considering higher rank vector bundles. A more detailed version can be found in [**B5**].

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1. The cohomology of a torus

1.1. Real tori. Let V be a real vector space, of dimension n. A *lattice* in V is a $\mathbb{Z}$-module $\Gamma \subset V$ such that the induced map $\Gamma \otimes_{\mathbb{Z}} \mathbb{R} \to V$ is an isomorphism; equivalently, any basis of Γ over $\mathbb{Z}$ is a basis of V. In particular $\Gamma \cong \mathbb{Z}^n$.

The quotient $T := V/\Gamma$ is a smooth, compact Lie group, isomorphic to $(\mathbb{S}^1)^n$. The quotient homomorphism $\pi : V \to V/\Gamma$ is the *universal covering* of T. Thus Γ is identified with the fundamental group $\pi_1(T)$.

We want to consider the cohomology algebra $H^*(T,\mathbb{C})$. We think of it as being de Rham cohomology: recall that a smooth p-form ω on T is *closed* if $d\omega = 0$, *exact* if $\omega = d\eta$ for some $(p-1)$-form η. Then

$$H^p(T,\mathbb{C}) = \frac{\{\text{closed } p\text{-forms}\}}{\{\text{exact } p\text{-forms}\}} \ .$$

Let ℓ be a linear form on V. The 1-form $d\ell$ on V is invariant by translation, hence is the pull back by π of a 1-form on T that we will still denote $d\ell$. Let $(x_1,\ldots,x_n)$ be a system of coordinates on V. The forms $(dx_1,\ldots,dx_n)$ form a basis of the cotangent space $T^*_a(T)$ at each point $a \in T$; thus a p-form ω on T can be written in a unique way

$$\omega = \sum_{i_1<\ldots<i_p} \omega_{i_1\ldots i_p}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

where the $\omega_{i_1\ldots i_p}$ are smooth functions on T (with complex values).

An important role in what follows will be played by the translations $t_a : x \mapsto x+a$ of T. We say that a p-form ω is *constant* if it is invariant by translation, that is, $t_a^*\omega = \omega$ for all $a \in T$; in terms of the above expression for ω, it means that the functions $\omega_{i_1\ldots i_p}$ are constant. Such a form is determined by its value at 0, which is a skew-symmetric p-linear form on $V = T_0(T)$. We will denote by $\mathrm{Alt}^p(V,\mathbb{C})$ the space of such forms, and identify it to the space of constant p-forms. A constant form is closed, hence we have a linear map $\delta^p : \mathrm{Alt}^p(V,\mathbb{C}) \to H^p(T,\mathbb{C})$. Note that $\mathrm{Alt}^1(V,\mathbb{C})$ is simply $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C})$, and δ^1 maps a linear form ℓ to $d\ell$.

Proposition 1.1. *The map $\delta^p : \mathrm{Alt}^p(V,\mathbb{C}) \to H^p(T,\mathbb{C})$ is an isomorphism.*

Proof : There are various elementary proofs of this, see for instance [**D**], III.4. To save time we will use the Künneth formula. We choose our coordinates $(x_1,\ldots,x_n)$ so that $V = \mathbb{R}^n$, $\Gamma = \mathbb{Z}^n$. Then $T = T_1 \times \ldots \times T_n$, with $T_i \cong \mathbb{S}^1$ for each i, and dx_i is a 1-form on T_i, which generates $H^1(T_i,\mathbb{C})$. The Künneth formula gives an isomorphism of graded algebras $H^*(T,\mathbb{C}) \xrightarrow{\sim} \bigotimes_i H^*(T_i,\mathbb{C})$. This means that $H^*(T,\mathbb{C})$ is the exterior algebra on the vector space with basis $(dx_1,\ldots,dx_n)$, and this is equivalent to the assertion of the Proposition. ■

What about $H^*(T,\mathbb{Z})$? The Künneth isomorphism shows that it is torsion free, so it can be considered as a subgroup of $H^*(T,\mathbb{C})$. By definition of the de Rham isomorphism the image of $H^p(T,\mathbb{Z})$ in $H^p(T,\mathbb{C})$ is spanned by the closed p-forms ω such that $\int_\sigma \omega \in \mathbb{Z}$ for each p-cycle σ in $H_p(T,\mathbb{Z})$. Write again

$T = \mathbb{R}^n/\mathbb{Z}^n$; the closed paths $\gamma_i : t \mapsto te_i$, for $t \in [0,1]$, form a basis of $H_1(T,\mathbb{Z})$, and we have $\int_{\gamma_i} d\ell = \ell(e_i)$. Thus $H^1(T,\mathbb{Z})$ is identified with the subgroup of $H^1(T,\mathbb{C}) = \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C})$ consisting of linear forms $V \to \mathbb{C}$ which take integral values on Γ; it is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z})$. Applying again the Künneth formula gives:

Proposition 1.2. *For each p, the image of $H^p(T,\mathbb{Z})$ in $H^p(T,\mathbb{C}) \cong \mathrm{Alt}^p(V,\mathbb{C})$ is the subgroup of forms which take integral values on Γ; it is isomorphic to $\mathrm{Alt}^p(\Gamma,\mathbb{Z})$.* ∎

1.2. Complex tori. From now on we assume that V has a complex structure, that is, V is a complex vector space, of dimension g. Thus $V \cong \mathbb{C}^g$ and $\Gamma \cong \mathbb{Z}^{2g}$. Then $T := V/\Gamma$ is a complex manifold, of dimension g , in fact a complex Lie group; the covering map $\pi : V \to V/\Gamma$ is holomorphic. We say that T is a *complex torus.* **Beware :** while all real tori of dimension n are diffeomorphic to $(\mathbb{S}^1)^n$, there are many non-isomorphic complex tori of dimension g – more about that in section 3.3 below.

The complex structure of V provides a natural decomposition

$$\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) = V^* \oplus \overline{V}^* ,$$

where $V^* := \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ and $\overline{V}^* = \mathrm{Hom}_{\mathbb{C}}(\overline{V},\mathbb{C})$ are the subspaces of $\mathbb{C}$-linear and $\mathbb{C}$-antilinear forms respectively. We write the corresponding decomposition of $H^1(T,\mathbb{C})$

$$H^1(T,\mathbb{C}) = H^{1,0}(T) \oplus H^{0,1}(T) .$$

If $(z_1,\ldots,z_g)$ is a coordinate system on V , $H^{1,0}(T)$ is the subspace spanned by the classes of $dz_1,\ldots,dz_g$, while $H^{1,0}(T)$ is spanned by the classes of $d\bar z_1,\ldots,d\bar z_g$.

The decomposition $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) = V^* \oplus \overline{V}^*$ gives rise to a decomposition

$$\mathrm{Alt}^p(V,\mathbb{C}) \cong \wedge^p V^* \oplus (\wedge^{p-1}V^* \otimes \overline{V}^*) \oplus \ldots \oplus \wedge^p\overline{V}^*$$

which we write

$$H^p(T,\mathbb{C}) = H^{p,0}(T) \oplus \ldots \oplus H^{0,p}(T) .$$

The forms in $\mathrm{Alt}^p(V,\mathbb{C})$ which belong to $H^{p,0}(T)$ (resp. $H^{0,p}(T)$) are those which are $\mathbb{C}$-linear (resp. $\mathbb{C}$-antilinear) in each variable. It is not immediate to characterize those which belong to $H^{q,r}(T)$ for $q,r > 0$; for $p = 2$ we have:

Proposition 1.3. *Via the identification $H^2(T,\mathbb{C}) = \mathrm{Alt}^2(V,\mathbb{C})$, $H^{2,0}$ is the space of $\mathbb{C}$-bilinear forms, $H^{0,2}$ the space of $\mathbb{C}$-biantilinear forms, and $H^{1,1}$ is the space of $\mathbb{R}$-bilinear forms E such that $E(ix,iy) = E(x,y)$.*

Proof : We have only to prove the last assertion. For $\varepsilon \in \{\pm 1\}$, let $\mathcal{E}_\varepsilon$ be the space of forms $E \in \mathrm{Alt}^2(V,\mathbb{C})$ satisfying $E(ix,iy) = \varepsilon E(x,y)$. We have $\mathrm{Alt}^2(V,\mathbb{C}) = \mathcal{E}_1 \oplus \mathcal{E}_{-1}$, and $H^{2,0}$ and $H^{0,2}$ are contained in $\mathcal{E}_{-1}$.

For $\ell \in V^*$, $\ell' \in \overline{V}^*$, $v,w \in V$, we have

$$(\ell \wedge \ell')(iv,iw) = \ell(iv)\ell'(iw) - \ell(iw)\ell'(iv) = (\ell \wedge \ell')(v,w) ,$$

hence $H^{1,1}$ is contained in $\mathcal{E}_1$; it follows that $H^{2,0}\oplus H^{0,2}=\mathcal{E}_{-1}$ and $H^{1,1}=\mathcal{E}_1$. ■

2. Line bundles on complex tori

2.1. The Picard group of a manifold. Our next goal is to describe all holomorphic line bundles on our complex torus T. Recall that line bundles on a complex manifold M form a group, the *Picard group* $\mathrm{Pic}(M)$ (the group structure is given by the tensor product of line bundles). It is canonically isomorphic to the first cohomology group $H^1(M,\mathcal{O}_M^*)$ of the sheaf $\mathcal{O}_M^*$ of invertible holomorphic functions on M. To compute this group a standard tool is the *exponential exact sequence* of sheaves

$$0\to\mathbb{Z}_M\to\mathcal{O}_M\xrightarrow{\mathrm{e}}\mathcal{O}_M^*\to 1$$

where $\mathrm{e}(f):=\exp(2\pi i f)$, and $\mathbb{Z}_M$ denotes the sheaf of locally constant functions on M with integral values. This gives a long exact sequence in cohomology

$$(1)\qquad H^1(M,\mathbb{Z})\longrightarrow H^1(M,\mathcal{O}_M)\longrightarrow \mathrm{Pic}(M)\xrightarrow{c_1}H^2(M,\mathbb{Z})\longrightarrow H^2(M,\mathcal{O}_M)$$

For $L\in\mathrm{Pic}(M)$, the class $c_1(L)\in H^2(M,\mathbb{Z})$ is the *first Chern class* of L. It is a topological invariant, which depends only on L as a topological complex line bundle (this is easily seen by replacing holomorphic functions by continuous ones in the exponential exact sequence).

When M is a projective (or compact Kähler) manifold, Hodge theory provides more information on this exact sequence.[1] The image of c_1 is the kernel of the natural map $H^2(M,\mathbb{Z})\to H^2(M,\mathcal{O}_M)$. This map is the composition of the maps $H^2(M,\mathbb{Z})\to H^2(M,\mathbb{C})\to H^2(M,\mathcal{O}_M)$ deduced from the injections of sheaves $\mathbb{Z}_M\hookrightarrow\mathbb{C}_M\hookrightarrow\mathcal{O}_M$. Now the map $H^2(M,\mathbb{C})\to H^2(M,\mathcal{O}_M)\cong H^{0,2}$ is the projection onto the last summand of the Hodge decomposition

$$H^2(M,\mathbb{C})=H^{2,0}\oplus H^{1,1}\oplus H^{0,2}$$

(for the experts: this can be seen by comparing the de Rham complex with the Dolbeault complex.)

Thus the image of c_1 consists of classes $\alpha\in H^2(M,\mathbb{Z})$ whose image $\alpha_{\mathbb{C}}=\alpha^{0,2}+\alpha^{1,1}+\alpha^{0,2}$ in $H^2(M,\mathbb{C})$ satisfies $\alpha^{0,2}=0$. But since $\alpha_{\mathbb{C}}$ comes from $H^2(M,\mathbb{R})$ we have $\alpha^{2,0}=\overline{\alpha^{0,2}}=0$: *the image of c_1 consists of the classes in $H^2(M,\mathbb{Z})$ whose image in $H^2(M,\mathbb{C})$ belongs to $H^{1,1}$* ("Lefschetz theorem").

The kernel of c_1, denoted $\mathrm{Pic}^{\mathrm{o}}(M)$, is the group of topologically trivial line bundles. The exact sequence (1) shows that it is isomorphic to the quotient of $H^1(M,\mathcal{O}_M)$ by the image of $H^1(M,\mathbb{Z})$. We claim that this image is a lattice in $H^1(M,\mathcal{O}_M)$: this is equivalent to saying that the natural map $H^1(M,\mathbb{R})\to H^1(M,\mathcal{O}_M)$ is bijective. By Hodge theory, this map is identified with the restriction to $H^1(M,\mathbb{R})$ of the projection of $H^1(M,\mathbb{C})=H^{1,0}\oplus H^{0,1}$ onto $H^{0,1}$.

[1] In this section and the following we use standard Hodge theory, as explained in [**G-H**], 0.6. Note that Hodge theory is much easier in the two cases of interest for us, namely complex tori and algebraic curves.

Since $H^1(M,\mathbb{R})$ is the subspace of classes $\alpha+\bar\alpha$ in $H^1(M,\mathbb{C})$, the projection $H^1(M,\mathbb{R})\to H^{0,1}$ is indeed bijective. Thus $\mathrm{Pic}^{\mathrm{o}}(M)$ is naturally identified with the complex torus $H^1(M,\mathcal{O}_M)/H^1(M,\mathbb{Z})$.

2.2. Systems of multipliers. We go back to our complex torus $T=V/\Gamma$.

Lemma 2.1. *Every line bundle on V is trivial.*

Proof : We have $H^2(V,\mathbb{Z})=0$ and $H^1(V,\mathcal{O}_V)=0$ (see [**G-H**], p. 46), hence $\mathrm{Pic}(V)=0$ by the exact sequence (1). ■

Let L be a line bundle on T. We consider the diagram

$$\begin{array}{ccc} \pi^*L & \longrightarrow & L \\ \downarrow & & \downarrow \\ V & \xrightarrow{\pi} & T \end{array}$$

The action of Γ on V lifts to an action on $\pi^*L=V\times_T L$. We know that π^*L is trivial; we choose a trivialization $\pi^*L\xrightarrow{\sim}V\times\mathbb{C}$. We obtain an action of Γ on $V\times\mathbb{C}$, so that L is the quotient of $V\times\mathbb{C}$ by this action. An element γ of Γ acts linearly on the fibers, hence by

$$\gamma\cdot(z,t)=(z+\gamma,e_\gamma(z)\,t)\quad\text{for}\quad z\in V,\ t\in\mathbb{C}$$

where e_γ is a holomorphic invertible function on V. This formula defines a group action of Γ on $V\times\mathbb{C}$ if and only if the functions e_γ satisfy

$$e_{\gamma+\delta}(z)=e_\gamma(z+\delta)\,e_\delta(z)\quad(\text{“cocycle condition”}).$$

A family $(e_\gamma)_{\gamma\in\Gamma}$ of holomorphic invertible functions on V satisfying this condition is called a *system of multipliers.* Every line bundle on T is defined by such a system.

A *theta function* for the system $(e_\gamma)_{\gamma\in\Gamma}$ is a holomorphic function $V\to\mathbb{C}$ satisfying

$$\theta(z+\gamma)=e_\gamma(z)\,\theta(z)\quad\text{for all }\gamma\in\Gamma,z\in V\,.$$

Proposition 2.2. *Let $(e_\gamma)_{\gamma\in\Gamma}$ be a system of multipliers, and L the associated line bundle. The space $H^0(T,L)$ is canonically identified with the space of theta functions for $(e_\gamma)_{\gamma\in\Gamma}$.*

Proof : Any global section s of L lifts to a section $\hat s=\pi^*s$ of $\pi^*L=V\times_T L$ over V, defined by $\hat s(z)=(z,s(\pi z))$; it is Γ-invariant in the sense that $\hat s(z+\gamma)=\gamma\cdot\hat s(z)$. Conversely, a Γ-invariant section of π^*L comes from a section of L. Now a section of $\pi^*L\cong V\times\mathbb{C}$ is of the form $z\mapsto(z,\theta(z))$, where $\theta:V\to\mathbb{C}$ is holomorphic. It is Γ-invariant if and only if θ is a theta function for $(e_\gamma)_{\gamma\in\Gamma}$. ■

Let $(e_\gamma)_{\gamma\in\Gamma}$ and $(e'_\gamma)_{\gamma\in\Gamma}$ be two systems of multipliers, defining line bundles L and L' . The line bundle $L\otimes L'$ is the quotient of the trivial line bundle $V\times(\mathbb{C}\otimes\mathbb{C})$ by the tensor product action $\gamma\cdot(z,t\otimes t')=(z+\gamma,e_\gamma(z)\,t\otimes e'_\gamma(z)\,t')$;

therefore it is defined by the system of multipliers $(e_\gamma e'_\gamma)_{\gamma\in\Gamma}$. In other words, multiplication defines a group structure on the set of systems of multipliers, and we have a surjective group homomorphism

$$\{\text{systems of multipliers}\} \longrightarrow \operatorname{Pic}(T)\ .$$

A system of multipliers $(e_\gamma)_{\gamma\in\Gamma}$ lies in the kernel if and only if the associated line bundle admits a section which is everywhere $\neq 0$; in view of Proposition 2.2, this means that *there exists a holomorphic function* $h : V \rightarrow \mathbb{C}^*$ *such that* $e_\gamma(z) = \frac{h(z+\gamma}{h(z)}$. We will call such systems of multipliers trivial.

Remark 2.3. (only for the readers who know group cohomology) Put $H^* := H^0(V, \mathcal{O}_V^*)$. The system of multipliers are exactly the 1-cocycles of Γ with values in H^*, and the trivial systems are the coboundaries. Thus we get a group isomorphism $H^1(\Gamma, H^*) \xrightarrow{\sim} \operatorname{Pic}(T)$ (see [**M1**], §2 for a more conceptual explanation of this isomorphism).

2.3. Interlude: hermitian forms. There are many holomorphic invertible functions on V, hence many systems of multipliers giving rise to the same line bundle. Our next goal will be to find a subset of such systems such that each line bundle corresponds exactly to one system of multipliers in that subset. This will involve hermitian forms on V, so let us fix our conventions.

A *hermitian form* H on V will be $\mathbb{C}$-linear in the second variable, $\mathbb{C}$-antilinear in the first. We put $S(x,y) = \operatorname{Re} H(x,y)$ and $E(x,y) = \operatorname{Im} H(x,y)$. S and E are $\mathbb{R}$-bilinear forms on V, S is symmetric, E is skew-symmetric; they satisfy:

$$S(x,y) = S(ix,iy) \quad , \quad E(x,y) = E(ix,iy) \quad , \quad S(x,y) = E(x,iy)$$

Using these relations one checks easily that the following data are equivalent:

- The hermitian form H;
- The symmetric $\mathbb{R}$-bilinear form S with $S(x,y) = S(ix,iy)$;
- The skew-symmetric $\mathbb{R}$-bilinear form E with $E(x,y) = E(ix,iy)$.

Moreover,

$$H \text{ non-degenerate} \iff E \text{ non-degenerate} \iff S \text{ non-degenerate.}$$

2.4. Systems of multipliers associated to hermitian forms. We denote by $\mathcal{P}$ the set of pairs (H,α), where H is a hermitian form on V, α a map from Γ to $\mathbb{S}^1$, satisfying:

$$E := \operatorname{Im}(H) \text{ takes integral values on } \Gamma\,; \quad \alpha(\gamma+\delta) = \alpha(\gamma)\alpha(\delta)(-1)^{E(\gamma,\delta)}\ .$$

(We will say that α is a *semi-character* of Γ with respect to E).

The law $(H,\alpha)\cdot(H',\alpha') = (H+H', \alpha\,\alpha')$ defines a group structure on $\mathcal{P}$. For $(H,\alpha)\in\mathcal{P}$, we put

$$e_\gamma(z) = \alpha(\gamma)\, e^{\pi[H(\gamma,z)+\frac{1}{2}H(\gamma,\gamma)]}\ .$$

We leave as an (easy) exercise to check that this defines a system of multipliers. The corresponding line bundle will be denoted $L(H,\alpha)$. The map $(H,\alpha)\mapsto L(H,\alpha)$ from $\mathcal{P}$ onto $\mathrm{Pic}(T)$ is a group homomorphism; we want to prove that it is an isomorphism.

Theorem 2.4. *The map $(H,\alpha)\mapsto L(H,\alpha)$ defines a group isomorphism $\mathcal{P}\xrightarrow{\sim}\mathrm{Pic}(T)$.*

Sketch of proof : One proves first that the first Chern class $c_1(L(H,\alpha))$ is equal to $E\in\mathrm{Alt}^2(\Gamma,\mathbb{Z})\cong H^2(T,\mathbb{Z})$. This can be done by using the differential-geometric definition of the Chern classes ([**G-H**], p. 141), or in terms of group cohomology ([**M1**], §2).

Let $\mathcal{Q}$ be the group of hermitian forms H on V such that $\mathrm{Im}(H)$ is integral on Γ. The projection $p:\mathcal{P}\to\mathcal{Q}$ is surjective, because a semi-character is determined by its values on the elements of a basis of Γ, and these values can be chosen arbitrarily. The kernel of p is the group of characters $\mathrm{Hom}(\Gamma,\mathbb{S}^1)$. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\Gamma,\mathbb{S}^1) \arrow[r] \arrow[d, "L^{\mathrm{o}}"] & \mathcal{P} \arrow[r, "p"] \arrow[d, "L"] & \mathcal{Q} \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & \mathrm{Pic}^{\mathrm{o}}(T) \arrow[r] & \mathrm{Pic}(T) \arrow[r, "c_1"] & H^2(T,\mathbb{Z}) &
\end{tikzcd}$$

where $L^{\mathrm{o}}(\alpha):=L(0,\alpha)$, and $\iota(H)=\mathrm{Im}(H)\in\mathrm{Alt}^2(\Gamma,\mathbb{Z})\cong H^2(T,\mathbb{Z})$. The equality $c_1(L(H,\alpha))=E=\iota(H)$ implies that the diagram is commutative.

Now we claim that ι is bijective onto $\mathrm{Im}(c_1)$: indeed we have seen in section 2.1 that a form $E\in\mathrm{Alt}^2(\Gamma,\mathbb{Z})\cong H^2(T,\mathbb{Z})$ belongs to $\mathrm{Im}(c_1)$ if and only if it belongs to $H^{1,1}$, that is satisfies $E(ix,iy)=E(x,y)$ (Proposition 1.3). By section 2.3 this is equivalent to $E=\mathrm{Im}(H)$ for a hermitian form $H\in\mathcal{Q}$; moreover H is uniquely determined by E, hence our assertion.

Finally one proves using Hodge theory that any line bundle M in $\mathrm{Pic}^{\mathrm{o}}(T)$ admits a unique flat unitary structure, that is, $M\cong L(0,\alpha)$ for a unique character α of Γ. In other words L^{o} is bijective, hence L is bijective. ■

2.5. The theorem of the square. This section is devoted to an important result, Theorem 2.6 below, which is actually an easy consequence of our description of line bundles on T (we encourage the reader to have a look at the much more elaborate proof in [**M1**], §6, valid over any algebraically closed field).

Lemma 2.5. *Let $a\in V$. We have $t^*_{\pi(a)}L(H,\alpha)=L(H,\alpha')$ with $\alpha'(\gamma)=\alpha(\gamma)\,\mathbb{e}(E(\gamma,a)$.*

Proof : In general, let L be a line bundle on T defined by a system of multipliers $(e_\gamma)_{\gamma\in\Gamma}$. Then $(e_\gamma(z+a))_{\gamma\in\Gamma}$ is a system of multipliers, defining a line bundle L'; the self-map $(z,t)\mapsto(z+a,t)$ of $V\times\mathbb{C}$ is equivariant w.r.t. the actions of Γ

defined by $(e_\gamma(z+a))$ on the source and $(e_\gamma(z))$ on the target, so it induces an isomorphism $L' \xrightarrow{\sim} t^*_{\pi(a)}L$.

We apply this to the multiplier $e_\gamma(z) = a(\gamma)e^{\pi[H(\gamma,z)+\frac{1}{2}H(\gamma,\gamma)]}$; we find $e_\gamma(z+a) = e_\gamma(z)\, e^{\pi H(\gamma,a)}$. Recall that we are free to multiply $e_\gamma(z)$ by $\frac{h(z+\gamma)}{h(z)}$ for some holomorphic invertible function h; taking $h(z) = e^{-\pi H(a,z)}$, our multiplier becomes $e_\gamma\, e^{\pi[H(\gamma,a)-H(a,\gamma)]} = e_\gamma\, e^{2\pi i E(\gamma,a)}$. ■

Theorem 2.6. *Let L be a line bundle on T.*

1) (Theorem of the square) *The map*

$$\lambda_L : T \to \mathrm{Pic}^{\mathrm{o}}(T) \quad , \quad \lambda_L(a) = t^*_a L \otimes L^{-1}$$

is a group homomorphism.

2) *Let $E \in \mathrm{Alt}^2(\Gamma,\mathbb{Z})$ be the first Chern class of L. We have*

$$\mathrm{Ker}\,\lambda_L = \Gamma^\perp/\Gamma \ , \quad \textit{with} \ \ \Gamma^\perp := \{z \in V \mid E(z,\gamma) \in \mathbb{Z} \ \ \textit{for all}\ \gamma \in \Gamma\}\,.$$

3) *If E is non-degenerate, λ_L is surjective and has finite kernel.*

4) *If E is unimodular, λ_L is a group isomorphism.*

Proof : By the Lemma, λ_L is the composition

$$T \xrightarrow{\ \varepsilon\ } \mathrm{Hom}(\Gamma,\mathbb{S}^1) \xrightarrow{\ L^{\mathrm{o}}\ } \mathrm{Pic}^{\mathrm{o}}(T)\ ,$$

where $\varepsilon(a)$, for $a = \pi(\tilde{a}) \in T$, is the map $\gamma \mapsto \mathbb{e}(E(\gamma,\tilde{a})$, and L^{o} is the isomorphism $\alpha \mapsto L(0,\alpha)$ (Theorem 2.4). Therefore we can replace λ_L by ε in the proof. Then 1) and 2) become obvious.

Assume that E is non-degenerate. Let $\chi \in \mathrm{Hom}(\Gamma,\mathbb{S}^1)$. Since Γ is a free $\mathbb{Z}$-module, we can find a homomorphism $u : \Gamma \to \mathbb{R}$ such that $\chi(\gamma) = \mathbb{e}(u(\gamma))$ for each $\gamma \in \Gamma$. Extend u to a $\mathbb{R}$-linear form $V \to \mathbb{R}$; since E is non-degenerate, there exists $a \in V$ such that $u(z) = E(z,a)$, hence $\varepsilon(\pi(a)) = \chi$. Thus ε is surjective.

Let us denote by $e : V \to \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$ the $\mathbb{R}$-linear isomorphism associated to E. The dual $\Gamma^* := \mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z})$ embeds naturally in $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$, and $\Gamma^\perp$ is by definition $e^{-1}(\Gamma^*)$; then e identifies $\Gamma^\perp$ with Γ^*, so that the inclusion $\Gamma \subset \Gamma^\perp$ corresponds to the map $\Gamma \to \Gamma^*$ associated to $E_{|\Gamma}$. This map has finite cokernel, and it is bijective if E is unimodular; this achieves the proof. ■

Remark 2.7. We have seen in section 2.1 that $\mathrm{Pic}^{\mathrm{o}}(T)$ has a natural structure of complex torus; it is not difficult to prove that the map λ_L is holomorphic. In particular, when E is unimodular, λ_L is an isomorphism of complex tori.

Corollary 2.8. *Assume that $c_1(L)$ is non-degenerate. Any line bundle L' with $c_1(L') = c_1(L)$ is isomorphic to $t^*_a L$ for some a in T.*

Proof : $L' \otimes L^{-1}$ belongs to $\mathrm{Pic}^{\mathrm{o}}(T)$, hence is isomorphic to $t^*_a L \otimes L^{-1}$ for some a in T by 3). ■

The following immediate consequence of 1) will be very useful:

Corollary 2.9. *Let* $a_1, \ldots, a_r$ *in* T *with* $\sum a_i = 0$. *Then* $t_{a_1}^* L \otimes \ldots \otimes t_{a_r}^* L \cong L^{\otimes r}$. ■

3. Polarizations

In this section we will consider a line bundle $L = L(H, \alpha)$ on our complex torus T such that the hermitian form H is *positive definite.* We will first look for a concrete expression of the situation using an appropriate basis.

3.1. Frobenius lemma. The following easy result goes back to Frobenius:

Proposition 3.1. *Let* Γ *be a free finitely generated* $\mathbb{Z}$*-module, and* $E : \Gamma \times \Gamma \to \mathbb{Z}$ *a skew-symmetric, non-degenerate form. There exists positive integers* $d_1, \ldots, d_g$ *with* $d_1 \,|\, d_2 \,|\, \ldots \,|\, d_g$ *and a basis* $(\gamma_1, \ldots, \gamma_g; \delta_1, \ldots, \delta_g)$ *of* Γ *such that the matrix of* E *in this basis is* $\begin{pmatrix} 0 & \mathbf{d} \\ -\mathbf{d} & 0 \end{pmatrix}$, *where* $\mathbf{d}$ *is the diagonal matrix with entries* $(d_1, \ldots, d_g)$.

As a consequence we see that the determinant of E is the square of the integer $d_1 \ldots d_g$, called the *Pfaffian* of E and denoted $\mathrm{Pf}(E)$. The most important case for us will be when $d_1 = \cdots = d_g = 1$, or equivalently $\det(E) = 1$; in that case one says that E is *unimodular*, and that $(\gamma_1, \ldots, \gamma_g; \delta_1, \ldots, \delta_g)$ is a *symplectic basis* of Γ.

Proof : Let d_1 be the minimum of the numbers $E(\alpha, \beta)$ for $\alpha, \beta \in \Gamma$, $E(\alpha, \beta) > 0$; choose γ, δ such that $E(\gamma, \delta) = d_1$. For any $\varepsilon \in \Gamma$, $E(\gamma, \varepsilon)$ is divisible by d_1 – otherwise using Euclidean division we would find ε with $0 < E(\gamma, \varepsilon) < d_1$. Likewise $E(\delta, \varepsilon)$ is divisible by d_1. Put $U = \mathbb{Z}\gamma \oplus \mathbb{Z}\delta$; we claim that $\Gamma = U \oplus U^{\perp}$. Indeed, for $x \in \Gamma$, we have

$$x = \frac{E(x, \delta)}{d_1}\gamma + \frac{E(\gamma, x)}{d_1}\delta + (x - \frac{E(x, \delta)}{d_1}\gamma - \frac{E(\gamma, x)}{d_1}\delta) .$$

Reasoning by induction on the rank of Γ, we find integers $d_2 \,|\, d_3 \,|\, \ldots \,|\, d_g$ and a basis $(\gamma, \gamma_2, \ldots, \gamma_g;\ \delta, \delta_2, \ldots, \delta_g)$ of Γ, such that the matrix of E is $\begin{pmatrix} 0 & \mathbf{d} \\ -\mathbf{d} & 0 \end{pmatrix}$. It remains to prove that d_1 divides d_2; otherwise, using Euclidean division again, we can find $k \in \mathbb{Z}$ such that $0 < E(\gamma + \gamma_2, k\delta + \delta_2) < d_1$, a contradiction. ■

3.2. Polarizations and the period matrix. Going back to our complex torus $T = V/\Gamma$, we assume given a *positive definite* hermitian form H on V, such that $E := \mathrm{Im}(H)$ takes integral values on Γ. Such a form is called a *polarization* of T; if E is unimodular, we say that H is a *principal polarization.* A complex torus which admits a polarization is classically called a (polarized) *abelian variety*; we will see below that it is actually a projective manifold. It is common to use the abbreviation *p.p.a.v.* for “principally polarized abelian variety”.

In what follows we will treat only the case of principal polarizations, which is sufficient for our purpose. The general case is completely analogous but requires some more complicated notations, see e.g. [**D**], ch. VI.

We choose a symplectic basis $(\gamma_1, \ldots, \gamma_g; \delta_1, \ldots, \delta_g)$ as in Proposition 3.1.

Lemma 3.2. *$(\gamma_1, \ldots, \gamma_g)$ is a basis of V over $\mathbb{C}$.*

Proof : Let $W = \mathbb{R}\gamma_1 \oplus \ldots \oplus \mathbb{R}\gamma_g$. Our statement is equivalent to $V = W \oplus iW$. But if $x \in W \cap iW$, we have $H(x,x) = E(x,ix) = 0$ since $E_{|W} = 0$, hence $x = 0$. ■

Expressing the δ_j's in this basis gives a matrix $\tau \in M_g(\mathbb{C})$ with $\delta_j = \sum_i \tau_{ij}\gamma_i$. In the corresponding coordinates, we have

$$\Gamma = \mathbb{Z}^g \oplus \tau\mathbb{Z}^g \ ;$$

in other words, the elements of Γ are the column vectors $p + \tau q$ with $p, q \in \mathbb{Z}^g$. The matrix τ is often called the *period matrix.*

Proposition 3.3. *The matrix τ is symmetric, and $\mathrm{Im}(\tau)$ is positive definite.*

Proof : Put $\tau = A + iB$, with $A, B \in M_g(\mathbb{R})$. We will compare the bases $(\gamma_1, \ldots, \gamma_g; \delta_1, \ldots, \delta_g)$ and $(\gamma_1, \ldots, \gamma_g; i\gamma_1, \ldots, i\gamma_g)$ of V over $\mathbb{R}$. The change of basis matrix (expressing the vectors of the first basis in the second one) is $P = \begin{pmatrix} \mathbf{d} & A \\ 0 & B \end{pmatrix}$. Therefore the matrix of E in the second basis is

$$ {}^tP^{-1} \begin{pmatrix} 0 & \mathbf{d} \\ -\mathbf{d} & 0 \end{pmatrix} P^{-1} = \begin{pmatrix} 0 & B^{-1} \\ -{}^tB^{-1} & {}^tB^{-1}(A - {}^tA)B^{-1} \end{pmatrix} $$

(exercise!). Now the condition $E(ix, iy) = E(x,y)$, expressed in the basis $(\gamma_1, \ldots, \gamma_g; i\gamma_1, \ldots, i\gamma_g)$, is equivalent to $A = {}^tA$ and $B = {}^tB$; we have $H(\gamma'_j, \gamma'_k) = E(\gamma'_j, i\gamma'_k)$, so the matrix of H in the basis $(\gamma'_1, \ldots, \gamma'_g)$ (over $\mathbb{C}$) is B^{-1}, and the positivity of H is equivalent to that of B. ■

3.3. The moduli space of p.p.a.v. We have seen that the choice of a symplectic basis determines the matrix τ, which in turn completely determines T and H: we have

$$V = \mathbb{C}^g \quad \text{and} \quad \Gamma = \Gamma_\tau := \mathbb{Z}^g \oplus \tau\mathbb{Z}^g \ ;$$

the hermitian form H is given by the matrix $\mathrm{Im}(\tau)^{-1}$, and its imaginary part E by $E(p + \tau q, p' + \tau q') = {}^tpq' - {}^tqp'$.

The space of symmetric matrices $\tau \in M_g(\mathbb{C})$ with $\mathrm{Im}(\tau)$ positive definite is denoted $\mathbb{H}_g$, and called the *Siegel upper half space.* It is an open subset of the vector space of complex symmetric matrices. From what we have seen it follows that $\mathbb{H}_g$ parametrizes p.p.a.v. $(V/\Gamma, H)$ endowed with a symplectic basis of the lattice Γ.

Now we want to get rid of the choice of the symplectic basis. We have associated to a symplectic basis an isomorphism $V \xrightarrow{\sim} \mathbb{C}^g$ which maps Γ to the lattice Γ_τ. A change of the basis amounts to a linear automorphism M of $\mathbb{C}^g$, inducing a symplectic isomorphism $\Gamma_\tau \xrightarrow{\sim} \Gamma_{\tau'}$. Such an isomorphism is given by $\begin{pmatrix} p' \\ q' \end{pmatrix} = P \begin{pmatrix} p \\ q \end{pmatrix}$, where P belongs to the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$, that is, $P \in M_{2g}(\mathbb{Z})$ and ${}^tPJP = J$, with $J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$.

We have $M(p+\tau q) = p' + \tau' q'$, hence

$$\begin{pmatrix} \mathbf{1} & \tau' \end{pmatrix} = M \begin{pmatrix} \mathbf{1} & \tau \end{pmatrix} P^{-1} \quad \text{or equivalently} \quad \begin{pmatrix} \mathbf{1} \\ \tau' \end{pmatrix} = {}^tP^{-1} \begin{pmatrix} \mathbf{1} \\ \tau \end{pmatrix} {}^tM \ .$$

If $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $a,b,c,d \in M_g(\mathbb{Z})$, we have ${}^tP^{-1} = -JPJ = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$. We find

$$\mathbf{1} = (d - c\tau)^t M \ , \quad \tau' = (-b + a\tau)^t M \ , \quad \text{hence} \quad \tau' = (a\tau - b)(-c\tau + d)^{-1} \ .$$

Thus the group $\mathrm{Sp}(2g,\mathbb{Z})$ acts on $\mathbb{H}_g$ by $(P,\tau) \mapsto (a\tau - b)(-c\tau + d)^{-1}$, and two matrices τ, τ' correspond to the same p.p.a.v. with different symplectic bases iff they are conjugate under this action. To get a nicer formula, we observe that

$$\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = tPt \ , \ \text{with} \ \ t = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \ ;$$

since $tJt = -J$, the map $P \mapsto tPt$ is an automorphism of $\mathrm{Sp}(2g,\mathbb{Z})$. Composing our action with this automorphism, we obtain:

Proposition 3.4. *The group* $\mathrm{Sp}(2g,\mathbb{Z})$ *acts on* $\mathbb{H}_g$ *by* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = (a\tau + b)(c\tau + d)^{-1}$. *The quotient* $\mathcal{A}_g := \mathbb{H}_g/\mathrm{Sp}(2g,\mathbb{Z})$ *parametrizes isomorphism classes of* g*-dimensional p.p.a.v.*

It is not difficult to show that the action of $\mathrm{Sp}(2g,\mathbb{Z})$ on $\mathbb{H}_g$ is nice ("properly discontinuous"), so that $\mathcal{A}_g$ is an analytic space ([**D**], VII.1). A much more subtle result is that $\mathcal{A}_g$ is Zariski open in a projective variety, the *Satake compactification* $\overline{\mathcal{A}_g}$.

We have not made precise in which sense $\mathcal{A}_g$ parametrizes p.p.a.v. It is actually what is called a moduli space; we will give a precise definition in the case of vector bundles (see section 4.2 below), which can be adapted without difficulty to this case.

3.4. Theta functions. Let H be a polarization on T; we keep the notation of the previous sections. Let $\alpha : \Gamma \to \mathbb{S}^1$ be any semi-character of Γ w.r.t. E.

Theorem 3.5. $\dim H^0(T, L(H,\alpha)) = d_1 \dots d_g = \mathrm{Pf}(E)$.

Proof : We first treat the case $d_1 = \ldots = d_g = 1$. According to Prop. 2.2, we are looking for theta functions satisfying

$$\theta(z+\gamma) = \alpha(\gamma) e^{\pi[H(\gamma,z)+\frac{1}{2}H(\gamma,\gamma)]}\theta(z) \ .$$

Recall that we are free to multiply $e_\gamma(z)$ by $\frac{h(z+\gamma)}{h(z)}$ for some $h\in H^0(V,\mathcal{O}_V^*)$ (this amounts to multiply θ by h). We will use this to make θ *periodic* with respect to the basis elements $\gamma_1,\ldots,\gamma_g$ of Γ.

As before we put $W = \mathbb{R}\gamma_1\oplus\ldots\oplus\mathbb{R}\gamma_g$. Since $E_{|W} = 0$, the form $H_{|W}$ is a real symmetric form; since $V = W\oplus iW$ (lemma 3.2), it extends as a $\mathbb{C}$-*bilinear* symmetric form B on V. We put $h(z) = e^{-\frac{\pi}{2}B(z,z)}$: this amounts to replace H in $e_\gamma(z)$ by $H' := H - B$. We have

Lemma 3.6. $H'(p+\tau q, z) = -2i\,{}^tqz$.

Proof : Let $w\in W$. We have $H'(w,y) = 0$ for $y\in W$, hence also for any $y\in V$ because H' is $\mathbb{C}$-linear in y. On the other hand for $z\in V$ we have $H'(z,w) = (H-B)(z,w) = (\bar H - B)(w,z) = (\bar H - H)(w,z) = 2iE(z,w)$. Thus for $z = \sum z_i\gamma_i\in V$ we have $H'(\gamma_j,z) = 0$ and $H'(\delta_j,z) = \sum_k z_kH'(\delta_j,\gamma_k) = -2iz_j$, hence the lemma. ■

Put $L = L(H,\alpha)$. By Corollary 2.8, changing α amounts to replace L by t_a^*L for some $a\in T$. Since the pull back map $t_a^* : H^0(T,L)\to H^0(T,t_a^*L)$ is an isomorphism, it suffices to prove the theorem for a particular value of α; we choose $\alpha(p+\tau q) = (-1)^{{}^tpq}$. Indeed we have mod. 2, for $p,q,p',q'\in\mathbb{Z}^g$:

$${}^t(p+p')(q+q') \equiv {}^tpq + {}^tp'q' + ({}^tpq' - {}^tp'q) = {}^tpq + {}^tp'q' + E(p+\tau q, p'+\tau q') \ .$$

Thus our theta functions must satisfy the quasi-periodicity condition

$$\theta(z+p+\tau q) = \theta(z)\,\mathbb{e}(-{}^tqz - \frac{1}{2}{}^tq\tau q) \qquad \text{for } \ z\in\mathbb{C}^g,\ p,q\in\mathbb{Z}^g \ .$$

In particular, they are periodic with respect to the subgroup $\mathbb{Z}^g\subset\mathbb{C}^g$. This implies that they admit a Fourier expansion of the form $\theta(z) = \sum_{m\in\mathbb{Z}^g} c(m)\mathbb{e}({}^tmz)$. Now let us express the quasi-periodicity condition; we have:

$$\theta(z+p+\tau q) = \sum_{m\in\mathbb{Z}^g} c(m)\mathbb{e}({}^tm\tau q)\mathbb{e}({}^tmz)$$

and

$$\theta(z)\mathbb{e}(-{}^tqz - \frac{1}{2}{}^tq\tau q) = \sum_{m\in\mathbb{Z}^g} c(m)\mathbb{e}({}^t(m-q)z - \frac{1}{2}{}^tq\tau q) = \sum_{m\in\mathbb{Z}^g} c(m+q)\mathbb{e}(-\frac{1}{2}{}^tq\tau q)\mathbb{e}({}^tmz) \ .$$

Comparing we find $c(m+q) = c(m)\mathbb{e}({}^t(m+\frac{q}{2})\tau q)$. Taking $m = 0$ gives $c(q) = c(0)\,\mathbb{e}(\frac{1}{2}{}^tq\tau q)$. Thus our theta functions, if they exist, are all proportional to

$$\theta(z) = \sum_{m\in\mathbb{Z}^g} \mathbb{e}({}^tmz + \frac{1}{2}{}^tm\tau m) \ .$$

It remains to prove that this function indeed exists, that is that the series converges. But the coefficients $c(m)$ of the Fourier series satisfy $|c(m)| = e^{-q(m)}$, where q is a positive definite quadratic form, and therefore they decrease very fast as $m\to\infty$.

Now we treat the case $d_1 = \ldots = d_g = d$. In this case the form $\frac{1}{d}H$ is a principal polarization, so we can take $L = L_0^{\otimes d}$, where L_0 is the line bundle considered above. The corresponding theta functions satisfy

$$\theta(z+p+\tau q) = \theta(z)\mathrm{e}(-d\,{}^tqz - \frac{d}{2}{}^tq\tau q)) \qquad \text{for } \; z\in\mathbb{C}^g,\; p,q\in\mathbb{Z}^g$$

("theta functions of order d"). We write again $\theta(z) = \sum_{m\in\mathbb{Z}^g} c(m)\mathrm{e}({}^tmz)$; the quasi-periodicity condition gives

$$c(m+dq) = c(m)\mathrm{e}({}^t(m+\frac{d}{2})\tau q) = c(m)\mathrm{e}(\frac{-1}{d}{}^tm\tau m)\mathrm{e}(\frac{1}{2d}{}^t(m+dq)\tau(m+dq))$$

This determines up to a constant all coefficients $c(m)$ for m in a given coset ε of $\mathbb{Z}^g$ modulo $d\mathbb{Z}^g$; the corresponding theta function is

$$\theta[\varepsilon](z) = \sum_{m\in\varepsilon} \mathrm{e}({}^tmz + \frac{1}{2d}{}^tm\tau m)\ . \tag{2}$$

By what we have seen the functions $\theta[\varepsilon]$, where ε runs through $\mathbb{Z}^g/d\mathbb{Z}^g$, form a basis of the space of theta functions of order d; in particular, the dimension of this space is d^g.

The proof of the general case is completely analogous but requires more complicated notations. We will not need it in these lectures, so we leave it as an exercise for the reader. ∎

3.5. Comments. The proof of the theorem gives much more than the dimension of the space of theta functions, namely an explicit basis $(\theta[\varepsilon])_{\varepsilon\in\mathbb{Z}^g/d\mathbb{Z}^g}$ of this space given by formula (2). In particular, when the polarization H is principal, the line bundles $L(H,\alpha)$ have a unique non-zero section (up to a scalar); the divisor of this section is called a *theta divisor* of the p.p.a.v. (T,H). By Cor. 2.8 it is well-defined up to translation, so one speaks sometimes of "the" theta divisor. The choice of a symplectic basis gives a particular theta divisor Θ_τ, defined by the celebrated Riemann theta function

$$\theta(z) = \sum_{m\in\mathbb{Z}^g} \mathrm{e}({}^tmz + \frac{1}{2}{}^tm\tau m)\ .$$

It is quite remarkable that starting from a linear algebra data (a lattice Γ in V and a polarization), we get a hypersurface $\Theta\subset T = V/\Gamma$. When the p.p.a.v. comes from a geometric construction (Jacobians, Prym varieties, intermediate Jacobians), this divisor has a rich geometry, which reflects the objects we started with. In particular it is often possible to recover these objects from the data (T,Θ) ("Torelli theorem"), or to characterize the p.p.a.v. obtained in this way ("Schottky problem").

3.6. Reminder: line bundles and maps into projective space. Let M be a projective variety, and L a line bundle on M. The *linear system* $|L|$ is by definition $\mathbb{P}(H^0(M,L))$. Sending a nonzero section to its divisors identifies $|L|$ with the set of effective divisors E on M such that $\mathcal{O}_M(E)\cong L$.

The *base locus* $B(L)$ of L is the intersection of the divisors in $|L|$. Assume that L has no base point, that is, $B(L)=\varnothing$. Then the divisors of $|L|$ passing through a point $m\in M$ form a hyperplane in $|L|$, corresponding to a point $\varphi_L(m)$ in the dual projective space $|L|^*$. This defines a morphism $\varphi_L: M\to |L|^*$. Choosing a basis $(s_0,\ldots,s_n)$ of $H^0(M,L)$ identifies $|L|$ and its dual $|L|^*$ to $\mathbb{P}^n$; then $\varphi_L(m)=(s_0(m),\ldots,s_n(m))$, where we have fixed an isomorphism $L_m \xrightarrow{\sim} \mathbb{C}$ to evaluate the s_i at m.

If $E\in|L|$, we also denote the linear system $|L|$ by $|E|$, and the map φ_L by φ_E. Thus $|E|$ is the set of effective divisors linearly equivalent to E.

3.7. The Lefschetz theorem.

Theorem 3.7 (Lefschetz)**.** *Let L be a line bundle on T.*

1) *Assume $H^0(T,L)\neq 0$. For $k\geq 2$, the linear system $|L^{\otimes k}|$ has no base points.*

2) *Assume that the hermitian form associated to L is positive definite. For $k\geq 3$, the map $\varphi_{L^{\otimes k}}: T\to |L^{\otimes k}|^*$ is an embedding.*

Proof : We will only prove 1) – the proof of 2) uses the same idea but is technically more involved, see [**M1**], §17. Let $D\in|L|$. A simple but crucial observation is that

$$x\in t_a^*D \iff a\in t_x^*D\ .$$

By Cor. 2.9 we have $(k-1)t_a^*D+t_{-(k-1)a}^*D\in|L^{\otimes k}|$ for all a in T. Given $x\in T$, we choose a such that a and $-(k-1)a$ do not belong to t_x^*D; then $x\notin(k-1)t_a^*D+t_{-(k-1)a}^*D$, which proves 1). ■

Remark 3.8. A line bundle L such that $\varphi_{L^{\otimes k}}$ is an embedding for k large enough is said to be *ample.* The celebrated (and difficult) Kodaira embedding theorem states that this is the case if and only if the class $c_1(L)$ can be represented by a $(1,1)$-form which is everywhere positive definite (see [**G-H**], section I.4, for a precise statement and a proof). The Lefschetz theorem gives a much more elementary version for complex tori. It is also more precise, since it says that $k\geq 3$ is enough for $L^{\otimes k}$ to give an embedding.

In particular, for a theta divisor Θ on a p.p.a.v., the linear system $3|\Theta|$ already gives an embedding, while $|\Theta|$ is just a point. What about $|2\Theta|$? From formula (2) above one sees easily that theta functions of order 2 are *even.* It follows that $\varphi_{2\Theta}$ factors through the quotient of T by the involution $i_T: z\mapsto -z$. This quotient $K:=T/i_T$ is called the *Kummer variety* of T; it has 2^{2g} singular points, which are the images of the points of order 2 in T. Using again the theorem of the square one can prove (see [**L-N**]):

Proposition 3.9. *Let* Θ *be an* irreducible *symmetric theta divisor on* T. *The map* $\varphi_{2\Theta} : T \to |2\Theta|^*$ *factors through* i_T *and embeds* $K = T/i_T$ *into* $|2\Theta|^*$.

Remark 3.10. What if Θ is reducible? It is not difficult to show that T must be a product of lower-dimensional p.p.a.v.; that is, $T = T_1 \times \ldots \times T_p$ and $\Theta = \Theta_1 \times T_2 \times \ldots \times T_p + \ldots + T_1 \times \ldots \times T_{p-1} \times \Theta_p$. In that case the geometry of (T, Θ) is determined by that of the (T_i, Θ_i).

EXAMPLE. Suppose $g = 2$. Then $\varphi_{2\Theta}$ embeds $K = T/i_T$ in $\mathbb{P}^3$. It is easy to see that K has degree 4 (hint: use $K_T = \mathcal{O}_T = \varphi_{2\Theta}^* \mathcal{O}_{\mathbb{P}^3}(\deg(K) - 4)$); it has 16 double points corresponding to the 16 points of order 2 in T. This is the celebrated Kummer quartic surface, found by Kummer in 1864.

4. Curves and their Jacobians

In this section we denote by C a smooth projective curve (= compact Riemann surface) of genus g.

4.1. Hodge theory for curves. We first recall briefly Hodge theory for curves, which is much easier than in the general case. We start from the exact sequence of sheaves

$$0 \to \mathbb{C}_C \longrightarrow \mathcal{O}_C \xrightarrow{d} K_C \to 0 ,$$

where $\mathbb{C}_C$ is the sheaf of locally constant complex functions, and K_C (also denoted Ω^1_C or ω_C) is the sheaf of holomorphic 1-forms. Taking into account $H^0(C, \mathcal{O}_C) = \mathbb{C}$ and $H^1(C, K_C) \cong \mathbb{C}$ (Serre duality), we obtain an exact sequence

$$0 \to H^0(C, K_C) \xrightarrow{\partial} H^1(C, \mathbb{C}) \xrightarrow{p} H^1(C, \mathcal{O}_C) \to 0 .$$

By definition $g = \dim H^0(C, K_C)$; by Serre duality we have also $\dim H^1(C, \mathcal{O}_C) = g$, hence $\dim H^1(C, \mathbb{C}) = 2g$.

We put $H^{1,0} := \operatorname{Im} \partial$ and $H^{0,1} := \overline{H^{1,0}}$; $H^{1,0}$ is the subspace of classes in $H^1(C, \mathbb{C})$ which can be represented by holomorphic forms, and $H^{0,1}$ by antiholomorphic forms.

Lemma 4.1. *Let* $\alpha \neq 0$ *in* $H^0(C, K_C)$*; then* $i \int_C \alpha \wedge \bar{\alpha} > 0$.

Proof : Let $z = x + iy$ be a local coordinate in an open subset U of C. We can write $\alpha = f(z) dz$ in U, so that

$$i \int_U \alpha \wedge \bar{\alpha} = \int_U |f(z)|^2 \, i \, dz \wedge d\bar{z} = \int_U |f(z)|^2 \, 2 \, dx \wedge dy > 0 . \quad \blacksquare$$

Proposition 4.2. $H^1(C, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$*; the map* p *induces an isomorphism* $H^{0,1} \to H^1(C, \mathcal{O}_C)$.

Proof : The second assertion follows from the first and from the above exact sequence. For dimension reasons it suffices to prove that $H^{1,0} \cap H^{0,1} = (0)$. Let $x \in H^{1,0} \cap H^{0,1}$. There exists $\alpha, \beta \in H^0(C, K_C)$ such that $x = [\alpha] = [\bar{\beta}]$, hence $\alpha - \bar{\beta} = df$ for some C^∞ function f on C. Then $\beta \wedge \bar{\beta} = df \wedge \beta = d(f\beta)$, hence $\int_C \beta \wedge \bar{\beta} = 0$ by Stokes theorem. By the Lemma this implies $\beta = 0$ hence $x = 0$. ■

Proposition 4.3. *$p(H^1(C,\mathbb{Z}))$ is a lattice in $H^{0,1}$; the hermitian form H on $H^{0,1}$ defined by $H(\alpha,\beta) := 2i \int_C \bar{\alpha} \wedge \beta$ induces a principal polarization on the complex torus $H^{0,1}/p(H^1(C,\mathbb{Z}))$.*

Proof : The first assertion has already been proved (section 2.1). Lemma 4.1 shows that the form H is positive definite on $H^{0,1} = \overline{H^{1,0}}$. Let $a, b \in H^1(C,\mathbb{Z})$; we have

$$a = \bar{\alpha} + \alpha \ , \quad b = \bar{\beta} + \beta \quad \text{with} \quad \alpha = p(a) \ , \ \beta = p(b) \ .$$

Their cup-product in $H^2(C,\mathbb{Z}) = \mathbb{Z}$ is given by

$$a \cdot b = \int_C (\bar{\alpha} + \alpha) \wedge (\bar{\beta} + \beta) = \frac{1}{2i}\big(H(\alpha,\beta) - H(\beta,\alpha)\big) = \operatorname{Im}(H)(\alpha,\beta) \ ;$$

thus $\operatorname{Im}(H)$ induces on $H^1(C,\mathbb{Z})$ the cup-product, which is unimodular by Poincaré duality. ■

The g-dimensional abelian variety $JC := H^{0,1}/p(H^1(C,\mathbb{Z}))$ with the principal polarization H is called the *Jacobian* of C; it plays an essential role in the study of the curve.

4.2. Line bundles on C. To study line bundles on C we use again the exact sequence (1):

$$0 \to H^1(C,\mathbb{Z}) \xrightarrow{i} H^1(C,\mathcal{O}_C) \longrightarrow \operatorname{Pic}(C) \xrightarrow{c_1} H^2(C,\mathbb{Z}) \cong \mathbb{Z} \to 0 \ .$$

Here for a line bundle L on C, $c_1(L)$ is simply the degree $\deg(L)$ (through the canonical isomorphism $H^2(C,\mathbb{Z}) \cong \mathbb{Z}$): $\deg(L) = \deg(D)$ for any divisor D such that $\mathcal{O}_C(D) \cong L$.

Note that i is the composition of the maps $H^1(C,\mathbb{Z}) \to H^1(C,\mathbb{C}) \xrightarrow{p} H^1(C,\mathcal{O}_C)$ deduced from the inclusions of sheaves $\mathbb{Z}_C \subset \mathbb{C}_C \subset \mathcal{O}_C$. Hence:

Proposition 4.4. *We have an exact sequence* $0 \to JC \longrightarrow \operatorname{Pic}(C) \xrightarrow{\deg} \mathbb{Z} \to 0$. ■

Thus JC is identified with $\operatorname{Pic}^{\mathrm{o}}(C)$, the group of isomorphism classes of degree 0 line bundles on C – or the group of degree 0 divisors modulo linear equivalence. More precisely, one can show that JC is a *moduli space* for degree 0 line bundles on C. This means the following. Let S be a complex manifold (or analytic space), and let $\mathcal{L}$ be a line bundle on $C \times S$. For $s \in S$, put $\mathcal{L}_s := \mathcal{L}_{C\times\{s\}}$. We say that $(\mathcal{L}_s)_{s\in S}$ is a *holomorphic family* of line bundles on C parametrized by S. If the line bundles $\mathcal{L}_s$ have degree 0, we get a map $S \to JC$; we want this map to be holomorphic.

4.3. The Theta divisor. Jacobians give the first (and fundamental) examples of the situation we discussed in 3.5: while the the principal polarization is defined in a linear algebra way, the theta divisor admits a very simple geometric description. Let us define indeed, for any line bundle L on C of degree $g-1$:

$$\Theta_L := \{M \in JC \mid H^0(M \otimes L) \neq 0 \ .\}$$

Theorem 4.5 (Riemann). Θ_L *is a theta divisor of* JC.

We have to refer to [**ACGH**], p. 23 for the proof. Let us only observe that the fact that Θ_L is a divisor is easy: it is the image of C^{g-1} by the map $(p_1, \ldots, p_{g-1}) \mapsto L^{-1}(p_1 + \ldots + p_{g-1})$; this map is generically finite because for $p_1, \ldots, p_{g-1}$ general enough the linear system $|p_1 + \ldots + p_{g-1}|$ consists of one point. ■

Remark 4.6. 1) Replacing L by another line bundle L' (of the same degree) amounts to translate Θ_L by the element $L' \otimes L^{-1}$ of JC. Thus the ambiguity in the choice of L corresponds to the fact that a theta divisor is defined only up to translation.

Still there is a way to define a canonical theta divisor, which lives on a variety isomorphic to JC. For any $d \in \mathbb{Z}$, let J^d denote the set of isomorphism classes of line bundles of degree d on C. Choosing a line bundle L of degree d defines a bijection $t_L : JC \to J^d$ (by $M \mapsto M \otimes L$). This provides a structure of projective variety on J^d which does not depend on the choice of L. By construction J^d is isomorphic to JC, but there is no canonical isomorphism.

Now we have a canonical divisor $\Theta \subset J^{g-1}$, the locus of line bundles L with $H^0(L) \neq 0$; for each $L \in J^{g-1}$ we have $\Theta_L = t_L^* \Theta$.

2) A consequence of the Riemann theorem is that the theta divisor is *irreducible*, so a Jacobian cannot be a product of non-trivial p.p.a.v. (Remark 3.10).

5. Vector bundles on curves

As explained in the introduction, generalized theta functions appear when we replace JC, the moduli space of degree 0 line bundles on C, by the analogous moduli spaces for higher rank vector bundles. We will now explain what this means.

5.1. Elementary properties. Let E be a vector bundle on C, of rank r. The maximum wedge power $\wedge^r E$ is a line bundle on C, denoted $\det(E)$. Its degree is denoted by $\deg(E)$. It has the following properties:

- In an exact sequence $0 \to F \to E \to G \to 0$ we have $\det(E) \cong \det(F) \otimes \det(G)$;
- For any line bundle L on C, we have $\det(E \otimes L) = \det(E) \otimes L^{\otimes r}$.
- (Riemann-Roch) $h^0(E) - h^1(E) = \deg(E) + r(1-g)$.

It will be convenient to introduce the *slope* $\mu(E) := \frac{\deg(E)}{r} \in \mathbb{Q}$. Thus Riemann-Roch can be written $h^0(E) - h^1(E) = r(\mu(E) + 1 - g)$.

5.2. Moduli spaces. We have seen that the Jacobian of C parametrizes line bundles of degree 0, in the sense that for any holomorphic family $(\mathcal{L}_s)_{s\in S}$ the corresponding map $S \to JC$ is holomorphic. Unfortunately such a nice moduli space does not exist in higher rank. Indeed, let L be a non-trivial line bundle on C with no base point (section 3.6). One can construct a holomorphic family of vector bundles $(\mathcal{E}_t)_{t\in\mathbb{C}}$ on C such that:

$$\mathcal{E}_t \cong \mathcal{O}_C \oplus \mathcal{O}_C \quad \text{for} \quad t \neq 0 \qquad \mathcal{E}_0 \cong L \oplus L^{-1}$$

(The construction is quite elementary, see [**B5**], lemma 5.1). This implies that there is no reasonable moduli space $\mathcal{M}$ containing both $\mathcal{O}_C^{\oplus 2}$ and $L \oplus L^{-1}$: the above family would give rise to a holomorphic map $\mathbb{C} \to \mathcal{M}$ mapping $\mathbb{C} \smallsetminus \{0\}$ to a point, and 0 to a different point. There are two ways to deal with this problem. The sophisticated one, which we will not discuss here, replaces moduli spaces by a more elaborate notion called *moduli stacks.* The reader interested by this point of view may look at [**G**].

Instead we will follow the classical (by now) approach, which eliminates certain vector bundles, for instance those of the form $L \oplus L^{-1}$ which appear in the Lemma; this is done as follows:

Definition 5.1. A vector bundle E on C is *stable* if $\mu(F) < \mu(E)$ for every sub-bundle $0 \subsetneq F \subsetneq E$. It is *polystable* if it is a direct sum of stable sub-bundles of slope $\mu(E)$.

Theorem 5.2. *There exists a moduli space $\mathcal{M}^s(r,d)$ for stable vector bundles of rank r and degree d. It is a smooth connected quasi-projective manifold; it admits a projective compactification $\mathcal{M}(r,d)$ whose points correspond to isomorphism classes of polystable bundles.*

We refer to [**LP**] for a precise statement as well as the proof.

5.3. The moduli space $\mathcal{M}(r)$. As for line bundles, degree 0 vector bundles (those which are topologically trivial) are particularly important. We will actually focus on the subspace $\mathcal{M}(r)$ of $\mathcal{M}(r,0)$ parametrizing vector bundles with trivial determinant; it is simpler than $\mathcal{M}(r,0)$, but we will see that it carries enough information to recover the latter moduli space. Let J_r be the subgroup of line bundles $\alpha \in JC$ with $\alpha^{\otimes r} \cong \mathcal{O}_C$. It is isomorphic to $(\mathbb{Z}/r\mathbb{Z})^{2g}$ (as a group, JC is isomorphic to $(\mathbb{S}^1)^{2g}$).

Proposition 5.3. *The map $\mathcal{M}(r) \times JC \to \mathcal{M}(r,0)$ given by $(E,\lambda) \mapsto E \otimes \lambda$ identifies $\mathcal{M}(r,0)$ with the quotient of $\mathcal{M}(r) \times JC$ by J_r acting by $\alpha \cdot (E,\lambda) = (E \otimes \alpha, \lambda \otimes \alpha^{-1})$.*

Proof : Let E in $\mathcal{M}(r,0)$. The pairs (F,λ) with $F\in\mathcal{M}(r)$, $\lambda\in JC$ and $E\cong F\otimes\lambda$ are obtained by taking $\lambda\in JC$ with $\lambda^{\otimes r}=\det(E)\otimes L^{-1}$ and $F=E\otimes\lambda^{-1}$. We can always find such a λ, hence a pair (F,λ), and two such pairs differ by the action of J_r. ■

Thus, up to an étale covering (more precisely, the quotient by a finite abelian group acting freely), $\mathcal{M}(r,0)$ is the product of $\mathcal{M}(r)$ and JC. We will therefore focus on $\mathcal{M}(r)$. This is also the moduli space of principal $\mathrm{SL}(r)$-bundles, so its study fits into the more general theory of principal G-bundles for a semisimple group G.

Let us summarize in the next Proposition some elementary properties of $\mathcal{M}(r)$, which follow from its construction (see [**LP**]). From now on we will assume that the genus g of C is ≥ 2 (for $g\leq 1$ there are no stable bundles of degree 0 and rank >1).

Proposition 5.4. $\mathcal{M}(r)$ *is a projective normal irreducible variety, of dimension* $(r^2-1)(g-1)$, *with mild singularities (so-called* rational *singularities). Except when* $r=g=2$, *its singular locus is the locus of non-stable bundles.*

As an algebraic variety, $\mathcal{M}(r)$ is very different from a complex torus:

Proposition 5.5. *The moduli space* $\mathcal{M}(r)$ *is* unirational*; that is, there exists a rational dominant map*[2] $\mathbb{P}^N \dashrightarrow \mathcal{M}(r,L)$.

I refer to [**B5**], Prop. 5.6 for a proof.

Corollary 5.6. *Any rational map from* $\mathcal{M}(r)$ *to a complex torus is constant.*

Proof : Let $T=V/\Gamma$ be a complex torus. In view of the Proposition, it suffices to show that any rational map $\varphi:\mathbb{P}^N\dashrightarrow T$ is constant. Let p,q be two general points of $\mathbb{P}^N$. The restriction of φ to the line $\langle p,q\rangle$ defines a map $\mathbb{P}^1\to T$, which factors through V since $\mathbb{P}^1$ is simply connected, hence is constant. Thus $\varphi(p)=\varphi(q)$. ■

Corollary 5.7. $\mathcal{M}(r)$ *is simply connected.*

Indeed a unirational variety is simply connected [**S**].

5.4. Rationality. The *Lüroth problem* asks whether a unirational variety X is necessarily rational. The answer is positive when X is a curve (Lüroth, 1876) or a surface (Castelnuovo, 1895), but not in higher dimension (see for instance [De]). The rationality of $\mathcal{M}(r)$ is an open problem, already for $r=2$ and $g=3$.

[2] In the rest of this section we assume some familiarity with the notion of rational maps – see e.g. [**G-H**], p. 490.

6. Generalized theta functions

6.1. The theta divisor. Since $\mathcal{M}(r)$ is simply connected, there is no hope to describe its line bundles by systems of multipliers as for complex tori. However we may try to mimic the definition of the theta divisor: for $L \in J^{g-1}$, we put

$$\Delta_L := \{E \in \mathcal{M}(r) \mid H^0(E \otimes L) \neq 0\}.$$

Theorem 6.1 ([**D-N**])**.** 1) *Δ_L is a Cartier divisor on $\mathcal{M}(r)$.*

2) *The line bundle $\mathcal{L} = \mathcal{O}(\Delta_L)$ is independent of L, and* $\mathrm{Pic}(\mathcal{M}(r)) = \mathbb{Z}[\mathcal{L}]$.

Recall that an effective Cartier divisor is a subvariety locally defined by an equation – or, globally, as the zero locus of a section of a line bundle. On a singular variety (as is $\mathcal{M}(r)$) this is stronger than having codimension 1.

Proof : We will only show why Δ_L is a divisor on the stable locus $\mathcal{M}^s(r)$, referring to [**D-N**] for the rest of the proof. It is a consequence of the following lemma:

Lemma 6.2. *Let S be a complex variety, $(\mathcal{E}_s)_{s\in S}$ a family of vector bundles on C, with $\mu(\mathcal{E}_s) = g-1$ for all $s \in S$. Then the locus*

$$\{s \in S \mid H^0(C, \mathcal{E}_s) \neq 0\}$$

is defined locally by one equation (possibly trivial).

Proof : We need to know how the cohomology of $\mathcal{E}_s$ varies when s runs over S. This is described by the following (quite non-trivial) result (see [**M1**], §5) : locally on S there exist vector bundles F, G and a homomorphism $u : F \to G$ such that we have for each s in S an exact sequence

$$0 \to H^0(C, \mathcal{E}_s) \longrightarrow F(s) \xrightarrow{u(s)} G(s) \longrightarrow H^1(C, \mathcal{E}_s) \to 0\ .$$

By Riemann-Roch we have $h^0(\mathcal{E}_s) = h^1(\mathcal{E}_s)$, hence F and G have the same rank. We see that $H^0(C, \mathcal{E}_s) \neq 0$ if and only if $\det(u(s)) = 0$, that is, the section $\det(u)$ of $\det(G) \otimes \det(F)^{-1}$ vanishes at s, hence the lemma. ■

Coming back to $\mathcal{M}(r)$, the construction of the moduli space implies that locally for the complex topology, there is a "Poincaré bundle", that is a rank r vector bundle $\mathcal{E}$ on $C \times V$ such that $\mathcal{E}_{|C\times\{E\}} \cong E$ for E in V. Applying the lemma to $\mathcal{E} \otimes L$ shows that Δ_L is a divisor on $\mathcal{M}^s(r)$, unless $\Delta_L = \mathcal{M}(r)$. But this cannot hold: if α is a general element of JC, we have $H^0(L \otimes \alpha) = 0$, hence $\alpha^{\oplus r} \notin \Delta_L$. ■

6.2. Generalized theta functions. By analogy with the case of Jacobians, the sections of $H^0(\mathcal{M}(r), \mathcal{L}^{\otimes k})$ are called *generalized* (or *non-abelian*) *theta functions* of order k. They are associated to the group $\mathrm{SL}(r)$ (there are more general theta functions associated to each complex reductive group, but we will not discuss them in these notes).

Like for complex tori, the first question we can ask about these generalized theta functions is the dimension of the space $H^0(\mathcal{M}(r),\mathcal{L}^{\otimes k})$. The answer, much more intricate than Theorem 3.5 for complex tori, is known as the *Verlinde formula*; it has been first found by E. Verlinde using physics arguments, then proved mathematically in many different ways – see e.g. [**So**]. The formula is as follows:

$$\dim H^0(\mathcal{M}(r),\mathcal{L}^{\otimes k}) = \Big(\frac{r}{r+k}\Big)^g \sum_{\substack{S\amalg T=[1,r+k]\\ |S|=r}} \prod_{\substack{s\in S\\ t\in T}} \big|2\sin\pi\frac{s-t}{r+k}\big|^{g-1} \tag{3}$$

For $r=2$ it reduces (after some trigonometric manipulations) to:

$$\dim H^0(\mathcal{M}(2),\mathcal{L}^{\otimes k}) = (\frac{k}{2}+1)^{g-1}\sum_{i=1}^{k+1}\frac{1}{(\sin\frac{i\pi}{k+2})^{2g-2}}\ .$$

Even in rank 2, it is not at all obvious that the right hand side is an integer!

6.3. Linear systems and rational maps in $\mathbb{P}^n$. This section is the logical continuation of section 3.6; we again assume some familiarity with the notion of rational map ([**G-H**], p. 490). We keep our projective variety M and a line bundle L on M; we do not assume $B(L)=\varnothing$. We still have a map $M\smallsetminus B(L)\to|L|^*$, which we see as a rational map $\varphi_L : M \dashrightarrow |L|^*$.

Conversely, suppose given a rational map φ of M to a projective space $\mathbb{P}(V)$. We assume that M is *normal*; then the indeterminacy locus B of φ has codimension ≥ 2. We assume moreover that the line bundle $\varphi^*\mathcal{O}_{\mathbb{P}^n(1)}$ on $M\smallsetminus B$ extends to a line bundle L on M. By Hartogs theorem the restriction map $H^0(M,L)\to H^0(M\smallsetminus B,L)$ is bijective, so we get a pull back homomorphism $\varphi^* : V^*\to H^0(M,L)$. We have a commutative diagram

$$\begin{tikzcd} & {|L|^*} \arrow[dd, dashed, "\mathbb{P}({}^t\varphi^*)"] \\ M \arrow[ur, dashed, "\varphi_L"] \arrow[dr, dashed, "\varphi"'] & \\ & \mathbb{P}(V) \end{tikzcd}$$

Indeed for m general in M, $\varphi_L(m)$ is the hyperplane of $|L|$ formed by the divisors passing through m; its image under $\mathbb{P}({}^t\varphi^*)$ is the hyperplane of $\mathbb{P}(V)^*$ formed by the hyperplanes of $\mathbb{P}(V)$ passing through $\varphi(m)$, and this corresponds by duality to the point $\varphi(m)\in\mathbb{P}(V)$.

6.4. The theta map. The next step is to ask for the map defined by the linear systems $|\mathcal{L}^{\otimes k}|$. In fact we will concentrate on the simplest one, namely $\varphi_{\mathcal{L}}$. Our next task will be to give a geometric description of this map. In order to do this we associate to each vector bundle $E\in\mathcal{M}(r)$ the locus

$$\theta(E) := \{L\in J^{g-1} \mid H^0(E\otimes L)\neq 0\}$$

Proposition 6.3. *$\theta(E)$ is either equal to J^{g-1}, or is a divisor in J^{g-1}, belonging to the linear system $|r\Theta|$.*

Proof : The fact that $\theta(E)$ is a divisor in J^{g-1} or J^{g-1} itself follows from Lemma 6.2, applied to the family of vector bundles $(E\otimes L)_{L\in J^{g-1}}$. To prove that $\theta(E)$ belongs to $|r\Theta|$ when it is a divisor, we first observe that this holds when $E=L_1\oplus\ldots\oplus L_r$, with $L_1,\ldots,L_r$ distinct in JC; indeed we have

$$\theta(L_1\oplus\ldots\oplus L_r)=t^*_{L_1}\Theta+\ldots+t^*_{L_r}\Theta$$

where t_L denotes the translation $M\mapsto M\otimes L$ of J^{g-1}; and this divisor belongs to $|r\Theta|$ by Cor. 2.9.

Let $\mathcal{M}(r)^{\mathrm{o}}$ be the subset of $E\in\mathcal{M}(r)$ with $\theta(E)\neq J^{g-1}$; it is easy to prove that it is a Zariski open subset of $\mathcal{M}(r)$. When E runs through $\mathcal{M}(r)^{\mathrm{o}}$, the Chern class $c_1(\theta(E))\in H^2(J^{g-1},\mathbb{Z})$ is constant. So if we fix $E_0\in\mathcal{M}(r)^{\mathrm{o}}$, we have a rational map $\mathcal{M}(r)\dashrightarrow \mathrm{Pic}^{\mathrm{o}}(J^{g-1})$ mapping $E\in\mathcal{M}(r)^{\mathrm{o}}$ to $\mathcal{O}_J(\theta(E)-\theta(E_0))$. By Corollary 5.6 this map is constant, hence $\mathcal{O}_J(\theta(E))$ is independent of E. ■

Thus we have a rational map $\theta:\mathcal{M}(r)\dashrightarrow|r\Theta|$.

Theorem 6.4 ([**BNR**])**.** *There is a natural isomorphism*

$$H^0(\mathcal{M}(r),\mathcal{L})\xrightarrow{\ \sim\ }H^0(J^{g-1},\mathcal{O}(r\Theta))^*$$

making the following diagram commutative:

$$\begin{tikzcd} & {|\mathcal{L}|^*} \arrow[dd, "\wr"] \\ \mathcal{M}(r) \arrow[ur, dashed, "\varphi_{\mathcal{L}}"] \arrow[dr, dashed, "\theta"'] & \\ & {|r\Theta|} \end{tikzcd}$$

Sketch of proof : For $L\in J^{g-1}$, let H_L be the hyperplane in $|r\Theta|$ consisting of the divisors passing through L. By definition the pull back of H_L under θ is the divisor Δ_L. Thus, as explained in section 6.3, we get a commutative diagram

$$\begin{tikzcd} & {|\mathcal{L}|^*} \arrow[dd, dashed, "\lambda"] \\ \mathcal{M}(r) \arrow[ur, dashed, "\varphi_{\mathcal{L}}"] \arrow[dr, dashed, "\theta"'] & \\ & {|r\Theta|} \end{tikzcd}$$

with $\lambda:=\mathbb{P}({}^t\theta^*)$. It remains to prove that λ is bijective. Surjectivity is not difficult: if λ is not surjective, the image of θ is contained in a hyperplane of $|r\Theta|$. But this image contains all the divisors $t_{L_1}^*\Theta + \ldots + t_{L_r}^*\Theta$, and it is not difficult to prove that these divisors span the linear system $|r\Theta|$.

We have $\dim|r\Theta| = r^g - 1$ by Theorem 3.5, so the crucial point is to prove the same equality for $\dim|\mathcal{L}|$. Of course this follows (in a non-trivial way) from the Verlinde formula (3); in [**BNR**], since the Verlinde formula was not yet available, we constructed a rational dominant map from a certain abelian variety to the moduli space, and applied Theorem 3.5 to get the result. ■

Corollary 6.5. *The base locus of the linear system $|\mathcal{L}|$ on $\mathcal{M}(r)$ is the set of vector bundles $E \in \mathcal{M}(r)$ such that $\theta(E) = J^{g-1}$.* ■

Thus the rather mysterious map $\varphi_{\mathcal{L}}$ is identified with the more concrete map θ; one usually refers to θ, or $\varphi_{\mathcal{L}}$, as the *theta map*. We will now see that this explicit description allows a good understanding of the theta map in the rank 2 case.

6.5. Rank 2. In rank 2 the theta map is by now fairly well understood. We summarize what is known in one theorem:

Theorem 6.6. 1) *The theta map $\theta : \mathcal{M}(2) \to |2\Theta|$ is a morphism.*

2) *If C is not hyperelliptic or $g = 2$, θ is an embedding.*

3) *If C is hyperelliptic of genus ≥ 3, θ is 2-to-1 onto its image in $|2\Theta|$, and this image admits an explicit description.*

This is the conjunction of various results. Part 1) is due to Raynaud [**R**], part 3) to Bhosle-Ramanan [**D-R**]. In case 2), the fact that θ is generically injective was proved in [**B1**]; from this Brivio and Verra deduced that θ embeds $\mathcal{M}^s(2)$, and this was extended to $\mathcal{M}(2)$ in [**vG-I**].

Recall that $\mathcal{M}(2)$ has dimension $3g-3$. In particular:

Corollary 6.7 ([**N-R1**])**.** *For $g = 2$, $\theta : \mathcal{M}(2) \to |2\Theta| \cong \mathbb{P}^3$ is an isomorphism.* ■

For $g \geq 3$ the singular locus of $\mathcal{M}(2)$ is the locus of vector bundles of the form $L \oplus L^{-1}$ (Proposition 5.4): this is the quotient of JC by the involution $L \mapsto L^{-1}$, that is, the *Kummer variety* K of JC. The embedding $\theta : K \hookrightarrow |2\Theta|$ turns out to be isomorphic to the embedding $K \hookrightarrow |2\Theta|^*$ deduced from $\varphi_{2\Theta}$ (Proposition 3.9), via a natural isomorphism $|2\Theta| \xrightarrow{\sim} |2\Theta|^*$ ("Wirtinger duality", see [**M2**], p. 335).) Thus when C is not hyperelliptic, we can identify $\mathcal{M}(2)$ with a variety in $|2\Theta|$ which is singular along the Kummer variety.

For $g = 3$ and C not hyperelliptic, a very nice application appears in [**N-R2**]. In that case $\dim \mathcal{M}(2) = 6$, so θ embeds $\mathcal{M}(2)$ as a hypersurface in $|2\Theta| \cong \mathbb{P}^7$. It is not difficult to prove that it has degree 4 (for instance by computing its canonical bundle). Now Coble had found long ago that there is a *unique* quartic hypersurface in $|2\Theta|$ which is singular along the Kummer, for which he had written down an explicit equation (see [**B2**] for a modern account). Therefore this hypersurface is $\mathcal{M}(2)$!

6.6. Higher rank. In contrast with the rank 2 case, not much is known in higher rank. It is known since [**R**] that there exist stable bundles E with $\theta(E) = J^{g-1}$ – that is, base points for the linear system $|\mathcal{L}|$; in fact, they exist as soon as $r \geq g+2$, and even $r \geq 4$ if C is hyperelliptic [**P1**]. On the other hand, in rank 3 there are no base points for $g = 2$ [**R**], $g = 3$ [**B3**], or if C is general enough [**R**].

The situation is somewhat particular when $g=2$, since $\dim \mathcal{M}(r) = \dim |r\Theta| = r^2 - 1$.

Proposition 6.8. *Let* $g=2$.

1) $\theta : \mathcal{M}(r) \dashrightarrow |r\Theta|$ *is generically finite.*

2) *Its degree is* 1 *for* $r=2$, 2 *for* $r=3$, 30 *for* $r=4$.

Part 1) is proved in [**B3**]. The rank 2 case has been discussed in Cor. 6.7. In rank three $\theta : \mathcal{M}(3) \to |3\Theta| \cong \mathbb{P}^8$ is a double covering, branched along a sextic hypersurface which can be explicitly described [**O**]. The case $r=4$ is due to Pauly [**P2**].

Let us conclude with a

Conjecture. *For* $g \geq 3$, *the theta map* $\theta : \mathcal{M}(r) \dashrightarrow |r\Theta|$ *is generically* 2*-to-*1 *onto its image if* C *is hyperelliptic, and generically injective otherwise.*

This is unknown even for $r = g = 3$.

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