Holomorphic symplectic manifolds

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I. Manifolds with $c_1 = 0$

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- Conjecturally, and **very roughly**, the “building bricks” of algebraic geometry are:
- The Fano manifolds (K^{-1} ample);
- The manifolds with K trivial;
- The manifolds with K ample.

We will consider the case K trivial.

Proposition

Let X compact Kähler with $c_1(X) = 0$ in $H^2(X, \mathbb{C})$.

There exists $T \times Y \to X$ finite étale with

$T =$ complex torus, Y simply-connected with $K_Y = \mathcal{O}_Y$.

Corollary

$\pi_1(X)$ almost abelian; K_X torsion.

Theorem (Decomposition theorem)

X compact Kähler simply-connected with $K_X = \mathcal{O}_X$. Then

$$X = \prod_i Y_i \times \prod_j Z_j$$

- *$Y_i = Y$ simply-connected projective,* $\dim Y \geq 3$, *$H^0(Y, \Omega_Y^*) = \mathbb{C} \oplus \mathbb{C}\omega$, where ω is a generator of K_Y.*

  (these are Calabi-Yau manifolds)

- *$Z_j = Z$ simply-connected, $H^0(Z, \Omega_Z^*) = \mathbb{C}[\sigma]$, where $\sigma \in H^0(Z, \Omega_Z^2)$ is everywhere non-degenerate.*

  (these are the irreducible symplectic manifolds)

Remarks :

- Many examples of Calabi-Yau: hypersurfaces of degree $n+1$ in $\mathbb{P}^n$, etc.
- In contrast, very few examples of symplectic manifolds.

Idea of the proof

The proof uses differential geometry, in particular the concept of holonomy.

- (M,g) Riemannian manifold $\rightsquigarrow$ parallel transport:

$$\rightsquigarrow \quad \varphi_\gamma : T_p(M) \xrightarrow{\sim} T_q(M)$$

with $\varphi_\gamma \circ \varphi_\delta = \varphi_{\delta\gamma}$.

- In particular, $\varphi : \{\text{loops at } p\} \longrightarrow O(T_p(M))$

  Image $= H_p =$ holonomy (sub-)group at p

- independent of p up to conjugacy.

- For simplicity, we will assume M simply-connected and compact.
- $\Rightarrow$ H_p closed, connected Lie subgroup of $SO(T_p(M))$.

Theorem (de Rham)

$T_p(M) = \bigoplus_i V_i$ *stable under* $H_p \Rightarrow M \cong \prod_i M_i$ *and* $H_p \cong \prod_i H_{p_i}$.

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- So we are reduced to irreducible manifolds, i.e. those for which the holonomy representation is irreducible.
- There is one well-known class of Riemannian manifolds that we want to exclude, the symmetric spaces: G/H, where G compact Lie group, $H = \mathrm{Fix}(\sigma)^{\mathrm{o}}$, σ involution of G. Completely understood: $H_p = H$.

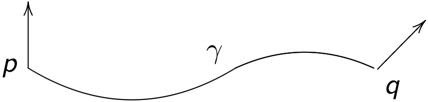
Theorem (Berger)

M irreducible ($\pi_1(M) = 0$), not symmetric. Then $H =$

H	$\dim(M)$	*metric*
$SO(n)$	n	generic
$U(m)$	$2m$	Kähler
$SU(m)$ $(m \geq 3)$	$2m$	Calabi-Yau
$\mathrm{Sp}(r)$	$4r$	hyperkähler
$\mathrm{Sp}(r)\mathrm{Sp}(1)$ $(r \geq 2)$	$4r$	quaternion-Kähler
G_2	7	
$\mathrm{Spin}(7)$	8	

WHAT DOES THE HOLONOMY MEAN?

A vector field (more generally, a tensor field) τ is parallel if



$$\varphi_\gamma(\tau(p)) = \tau(q)$$

for each path γ.

Holonomy principle.

Evaluation at p establishes a one-to-one correspondence between:

– parallel tensor fields;

– tensors on $T_p(M)$ invariant under H_p.

Examples

- $H = SO(n) \iff$ no parallel tensor fields (except g and dv_g)

- $H \subset U(\frac{n}{2}) \iff$ parallel complex structure J

  $\iff$ M Kähler

- $H \subset SU(\frac{n}{2}) \iff$ same + parallel holomorphic m-form

  $\iff$ M Kähler, $\mathrm{Ric}_g = 0$

  $(\mathrm{Ric}_g =$ curvature of $K_M)$

Yau's theorem: M admits a Kähler metric with $\mathrm{Ric}_g = 0 \iff$ M Kähler with trivial canonical bundle.

Proof of the decomposition theorem

1. Yau: X admits a metric with holonomy $\subset SU(m)$.
2. de Rham: $X \cong \prod_i X_i$, X_i irreducible with holonomy $H_i \subset SU(m_i)$.
3. Berger's list: either $H_i = SU(m)$, or $H_i = Sp(r)$, $r = m/2$.
4. $H_i = SU(m) \implies H^0(X_i, \Omega^p) = 0$ for $0 < p < m$;
   if $m \geq 3$, $H^0(X_i, \Omega^2) = 0 \Rightarrow X_i$ projective
   (use Bochner's principle : $\mathrm{Ric}_g = 0 \implies$ any holomorphic tensor field is parallel)
5. $H_i = Sp(r)$: X_i hyperkähler.

End of the proof: hyperkähler manifolds

Hyperkähler manifolds: X compact, Kähler, with holonomy $Sp(r)$.

$Sp(r) := U(r, \mathbb{H})$ = subgroup of $GL(r, \mathbb{H})$ preserving the hermitian form $\psi(x, y) = \sum x_i \bar{y}_i$. 2 interpretations:

1. $\mathbb{H}^r = \mathbb{R}^{4r}$; $Sp(r) = \{$elements of $SO(4r)$ linear w.r.t. $I, J, K\}$

   $\rightsquigarrow$ parallel complex structures I, J, K with $IJ = -JI = K$

   $\rightsquigarrow$ a sphere $\mathbb{S}^2$ of parallel complex structures:

   $$\mathbb{S}^2 = \{aI + bJ + cK,\ a^2 + b^2 + c^2 = 1\}\ .$$

2. $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}J, \quad \mathbb{H}^r = \mathbb{C}^r \oplus \mathbb{C}^r J = \mathbb{C}^{2r}, \quad \psi = h + \varphi J\ ;$

   then h hermitian and φ $\mathbb{C}$-linear alternating. Thus

   $$Sp(r) = U(2r, \mathbb{C}) \cap Sp(2r, \mathbb{C})\ .$$

$\rightsquigarrow$ X Kähler + parallel non-degenerate holomorphic 2-form

End of the proof: hyperkähler manifolds

Conversely, any $\sigma \in H^0(X, \Omega^2_X)$ is parallel for the Ricci-flat metric (Bochner principle).

- **Conclusion**: $H = Sp(r) \iff X$ holomorphic symplectic :

  X Kähler (compact, simply-connected)

  $H^0(X, \Omega^2_X) = \mathbb{C}\sigma$, σ everywhere non-degenerate

  (and automatically closed). ∎

- Consequences:

  – $\dim(X) = 2r$; σ^r generator of K_X.

  – $H^0(X, \Omega^*_X) = \mathbb{C}[\sigma]$.