# II. The classical examples

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March 27, 2008

- X complex manifold, compact and simply-connected.
- X hyperkähler = Kähler, with holonomy Sp(r).
- $\Leftrightarrow H^0(X, \Omega^2_X) = \mathbb{C}\sigma, \ \sigma$  everywhere non-degenerate.

## Examples?

• K3 surface S:  $\omega \in H^0(S, \Omega^2_S)$ . Note that Sp(2) = SU(2).

• Idea: S<sup>r</sup> has (too) many symplectic structures:

$$\sigma = \lambda_1 \, \rho_1^* \omega + \ldots + \lambda_r \, \rho_r^* \omega \, , \quad \text{with} \ \ \lambda_1, \ldots, \lambda_r \in \mathbb{C} \, .$$

- Try to get unicity by imposing λ<sub>1</sub> = ... = λ<sub>r</sub>, i.e. σ invariant under G<sub>r</sub>, i.e. σ comes from S<sup>(r)</sup> := S<sup>r</sup>/G<sub>r</sub> = {subsets of r points of S, counted with multiplicities}
- $S^{(r)}$  is singular, but admits a natural desingularization

 $S^{[r]} := \{ \text{finite analytic subspaces of } S \text{ of length } r \}$ 

(Hilbert scheme or "Douady space")

## Understanding the Hilbert scheme

• "Hilbert-Chow morphism"  $h: S^{[r]} \to S^{(r)}$ ; induces  $S_0^{[r]} \xrightarrow{\sim} S_0^{(r)}$ ,

where 
$$S_0^{[r]} \cong S_0^{(r)} = \{p_1, ..., p_r\}$$
 distinct.

 $\Delta_r := S^{[r]} - S_0^{[r]}$  is an irreducible divisor,  $\operatorname{codim} h(\Delta_r) = 2$ .

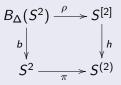
- subscheme of length 2 at  $p \iff$  tangent direction  $v \in \mathbb{P}(T_p(S))$ .
- Exercise: length 3 subscheme at p: ideal of the form (x, y<sup>3</sup>) or (x<sup>2</sup>, xy, y<sup>2</sup>).
- $S^{[r]}$  is smooth (Fogarty, infinitesimal calculation).

#### Theorem

For S K3,  $S^{[r]}$  is a hyperkähler manifold.

Idea of proof: r = 2.

Cartesian diagram



 b<sup>\*</sup>(p<sub>1</sub><sup>\*</sup>ω + p<sub>2</sub><sup>\*</sup>ω) invariant by the involution, hence = ρ<sup>\*</sup>σ, σ holomorphic 2-form on S<sup>[2]</sup>.

$$\begin{array}{l} \textcircled{0} \quad \operatorname{div}(b^*(p_1^*\omega+p_2^*\omega)^2)=E:=b^{-1}(\Delta) \ ; \\ \operatorname{div}(\rho^*\sigma^2)=\rho^*\operatorname{div}(\sigma^2)+E \end{array} \end{array}$$

$$\implies \operatorname{div}(\sigma^2) = \mathsf{0}$$
, hence  $\sigma$  symplectic.

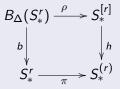
#### Sketch of proof (*r* arbitrary).

•  $S^{[r]} \supset S^{[r]}_* := {$ subschemes with at most one double point $}$ 

then 
$$\operatorname{codim}(S^{[r]} - S^{[r]}_*) = 2 \implies$$

enough to find  $\sigma$  symplectic on  $S_*^{[r]}$  (Hartogs), and to prove  $\pi_1(S_*^{[r]}) = 0$ .

Same cartesian diagram



and same argument works.

The same construction gives a symplectic form on A<sup>[r]</sup>, A complex torus of dimension 2; but π<sub>1</sub>(A<sup>[r]</sup>) ≠ 0.

• Consider 
$$A^{[r+1]} \xrightarrow{h} A^{(r+1)} \xrightarrow{S} A$$
, where  $S =$  addition map  $\left(S(a_1, \ldots, a_{r+1}) = \sum a_i\right)$  Define  $K_r(A) := S^{-1}(0)$ .

#### Theorem

 $K_r(A)$  is a hyperkähler manifold.

• ("generalized Kummer varieties": r = 1 gives usual Kummer)

X compact manifold. *Deformation* of X over pointed space (B, o):

 $f:\mathcal{X}
ightarrow B$  proper smooth, with  $\mathcal{X}_{\mathrm{o}}\stackrel{\sim}{\longrightarrow} X$ .

If  $H^0(X, T_X) = 0$ , there exists a universal local deformation, parametrized by  $B \subset H^1(X, T_X)$ , with  $T_0(B) = H^1(X, T_X)$ .

If B smooth at 0 (  $\Leftrightarrow B = H^1(X, T_X)$  locally around 0), we say that X is unobstructed.

Theorem (Bogomolov, Tian, Todorov, Ran, Deligne, Kawamata...) If  $K_X = \mathcal{O}_X$ , X is unobstructed.

Back to hyperkähler manifolds:

#### Theorem

Any Kähler deformation of a hyperkähler manifold is hyperkähler.

### Proof.

- $f: \mathcal{X} \to B$  smooth, proper,  $\mathcal{X}_b$  Kähler  $\forall b, \mathcal{X}_o$  hyperkähler. Then
- $c_1(\mathcal{X}_b) = 0$  in  $H^2(\mathcal{X}_b, \mathbb{Z})$  and  $\pi_1(\mathcal{X}_b) = 0$  for all b;
- $h^{p,q}(\mathcal{X}_b)$  independent of *b*, thus  $h^{2p,0} = 1$  for all  $p \leq r$ .
- Decomposition theorem  $\Rightarrow \mathcal{X}_b \cong \prod_{i=1}^s Y_i \times \prod_{j=1}^t Z_j$ ,

with  $Y_i$  hyperkähler and  $Z_j$  Calabi-Yau.

• Define 
$$P_X(t) := \sum_p h^{p,0}(X)t^p$$
; then  $P_{\mathcal{X}_b} = \prod_i P_{Y_i} \prod_j P_{Z_j}$ .  
Exercise :  $\Rightarrow s = 1, t = 0.$ 

## For X hyperkähler, $H^1(X, T_X) \cong H^{1,1}$ . Compute $H^2(S^{[r]}, \mathbb{C})$ ?

#### Proposition

Canonical isomorphism  $H^2(S^{[r]}, \mathbb{C}) \cong H^2(S, \mathbb{C}) \oplus \mathbb{C}[\Delta_r]$ , compatible with Hodge structures.

### Sketch of proof.

4

- $\pi: S^r \to S^{(r)} \text{ induces } \pi^*: H^2(S^{(r)}) \xrightarrow{\sim} H^2(S^r)^{\mathfrak{S}_r} \cong H^2(S)$ (can replace  $S^{(r)}$  by  $S_0^{(r)}$ ).
- A : S<sup>[r]</sup> → S<sup>(r)</sup> induces h<sup>\*</sup> : H<sup>2</sup>(S<sup>(r)</sup>) → H<sup>2</sup>(S<sup>[r]</sup>) injective,
   hence i : H<sup>2</sup>(S) → H<sup>2</sup>(S<sup>[r]</sup>) injective.

• Gysin: 
$$0 \to H^0(\Delta_r) \longrightarrow H^2(S^{[r]}) \longrightarrow H^2(S^{[r]}_{o})$$
 and  
 $H^2(S^{[r]}_{o}) \cong H^2(S^{(r)}_{o}) \cong H^2(S^{(r)}) \cong H^2(S)$ 

• Thus  $H^1(S^{[r]}, T_{S^{[r]}}) \cong H^1(S, T_S) \oplus \mathbb{C} \cong \mathbb{C}^{21}$  :

deformations of  $S^{[r]}$  obtained by deforming S form a hypersurface in the space of all deformations  $(r \ge 2)$ .

• Same result for  $K_r(A)$ :  $H^1(K_r(A), T_{K_r(A)}) \cong H^1(A, T_A) \oplus \mathbb{C}$ .

- Examples:  $S^{[r]}$  and  $K_r(A)$  (1983)
- Mukai (1984): The moduli of stable sheaves on *S* or *A* have a symplectic structure, hence hyperkähler when compact. But
- They are deformations of  $S^{[r]}$  or  $K_r(A)$  (O'Grady, Yoshioka).
- Two new examples by O'Grady, dim. 6 and 10.

That's all what is known!

