III. The period map

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The local period map

X hyperkähler with symplectic form σ;

$\mathcal{X} \to (B, \mathrm{o})$ universal deformation of X: $\mathcal{X}_{\mathrm{o}} \xrightarrow{\sim} X$.

Hodge decomposition $H^2(X,\mathbb{C}) = \mathbb{C}[\sigma] \oplus H^{1,1} \oplus \mathbb{C}[\bar{\sigma}]$, like for K3.

Topologically, locally on B, exists diffeomorphism u

$$\begin{array}{ccc} X \times B & \xrightarrow{\quad u \quad} & \mathcal{X} \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

Choose σ_b symplectic form on $\mathcal{X}_b$ for each b; period map

$$\wp : B \to \mathbb{P}(H^2(X,\mathbb{C}))\ ;\ \wp(b) := u_b^*[\sigma_b]\ .$$

Proposition

$\wp$ local isomorphism of B into a quadric in $\mathbb{P}(H^2(X,\mathbb{C}))$.

Proof, I

- Let $\alpha = u_b^*\sigma_b$. Then $\sigma_b^{r+1} = 0 \;\Rightarrow\; \alpha^{r+1} = 0$.

- Write $\alpha = a\sigma + \omega + b\bar{\sigma}$, with $\omega \in H^{1,1}(X)$. Then

$$0 = (a\sigma + \omega + b\bar{\sigma})^{r+1} = (r+1)a^r b\sigma^r\bar{\sigma} + \binom{r+1}{2} a^{r-1}\sigma^{r-1}\omega^2$$
$$(\in H^{2r,2}) + \text{terms in } H^{p,q},\ q \geq 3.$$

- Multiply by $\bar{\sigma}^{r-1}$, and integrate $\rightsquigarrow$ $0 = (r+1)a^{r-1}q(\alpha)$, with

$$q(\alpha) := ab\int_X (\sigma\bar{\sigma})^r + \frac{r}{2}\int_X \omega^2(\sigma\bar{\sigma})^{r-1}$$

Thus $\wp(B) \subset Q \subset \mathbb{P}(H^2(X,\mathbb{C}))$, with Q defined by $q = 0$.

- $q(\sigma,\bar{\sigma}) = \frac{1}{2}\int_X (\sigma\bar{\sigma})^r > 0 \;\Rightarrow\; Q$ smooth at $[\sigma]$.

Proof, II

- Differential of the period map $\wp : B \to \mathbb{P}(H^2(X,\mathbb{C}))$:

$$T_{\mathrm{o}}(\wp) : H^1(X, T_X) \longrightarrow \mathrm{Hom}(H^{2,0}, H^{2,0} \oplus H^{1,1})$$

deduced from cup-product

$$\cup : H^1(X, T_X) \otimes H^0(X, \Omega^2_X) \longrightarrow H^1(X, \Omega^1_X) \ .$$

- Here $H^{2,0} = \mathbb{C}\sigma$, $\cup$ isomorphism $\Rightarrow$ $T_{\mathrm{o}}(\wp)$ isomorphism onto hyperplane of $T_{\wp(\mathrm{o})}(\mathbb{P})$, necessarily $= T_{\wp(\mathrm{o})}(Q)$. ■

The quadratic form

Theorem

1. *Q is defined by an integral quadratic form $q : H^2(X,\mathbb{Z}) \to \mathbb{Z}$, non-degenerate, of signature $(3, b_2 - 3)$.*
2. *$\wp$ is a local isomorphism $B \to \Omega$,*

   where $\Omega =$ open subset of Q defined by $q(\sigma, \bar{\sigma}) > 0$.
3. *$\exists\ f_X \in \mathbb{N}$ (the Fujiki constant) such that $\int_X \alpha^{2r} = f_X\, q(\alpha)^r$.*

Corollary

For $\alpha \in H^2(X,\mathbb{C})$, $q(\alpha) = 0 \Leftrightarrow \alpha^{r+1} = 0 \Leftrightarrow \alpha^{2r} = 0$.

Proof of the corollary.

$\alpha^{r+1} = 0$ for α in an open subset of $q = 0$, and therefore for all α with $q(\alpha) = 0$. ∎

Remarks

Explicit formulas for $q(\alpha)$:

- for any $\omega \in H^2(X, \mathbb{C})$ with $\int_X \omega^{2r} = 1$:

$$q(\alpha) = c\left[(2r-1)\int_X \omega^{2r-2}\alpha^2 - (2r-2)\Big(\int_X \omega^{2r-1}\alpha\Big)^2\right].$$

- For any polynomial $F(c_1, c_2, \ldots)$ in $H^*(X, \mathbb{Z})$,

$$c' q(\alpha) = \int_X F(c_1, c_2, \ldots)\,\alpha^2 \qquad \text{(Looijenga-Lunts)}$$

- $c' \neq 0$ for $F = \sqrt{\mathrm{Todd}(X)}$ (Hitchin-Sawon).

A typical application

Corollary

For b in a dense subset of B, $\mathcal{X}_b$ is projective.

Proof.

1. $H^{2,0} \oplus H^{0,2} = L_{\mathbb{C}}$, $L \subset H^2(X, \mathbb{R})$ positive 2-plane.

   (L spanned by $\sigma + \bar{\sigma}$ and $i(\sigma - \bar{\sigma})$)

2. The line $\mathbb{P}(L_{\mathbb{C}}) \subset \mathbb{P}(H^2(X, \mathbb{C}))$ meet Q at σ, $\bar{\sigma}$.

3. Choose L' close to L defined over $\mathbb{Q}$. Then $\mathbb{P}(L'_{\mathbb{C}})$ meet Q at σ', $\bar{\sigma}'$, with σ' close to σ.

4. $\sigma' = \wp(b)$; in $H^2(\mathcal{X}_b, \mathbb{C})$, $H^{2,0} \oplus H^{0,2}$ is defined over $\mathbb{Q}$, hence also its orthogonal $H^{1,1}$.

5. Any rational class in the Kähler cone is ample (Kodaira). ∎

The global period map

Fix a lattice L.

Marked hyperkähler manifold: (X, τ) where $\tau : H^2(X, \mathbb{Z}) \xrightarrow{\sim} L$

$\mathcal{M}_L$:={iso. classes of (X, τ), $\dim X = 2r$} – complex manifold (non-Hausdorff, see next lecture)

Period domain $\Omega_L = \{[v] \in \mathbb{P}(L_{\mathbb{C}}) \mid v^2 = 0\,,\ v.\bar{v} > 0\}$

Period map $\wp : \mathcal{M}_L \to \Omega_L :\ \ \wp(X, \tau) = \tau_{\mathbb{C}}(H^{2,0}) \subset L_{\mathbb{C}}$.

Theorem (Huybrechts)

$\wp$ *is surjective.*

Torelli?

For K3, Torelli theorem: $\wp(X,\tau) = \wp(X',\tau') \;\Rightarrow\; X \cong X'$.

Does not extend to hyperkähler:

EXAMPLE (Namikawa): A abelian surface, $A \ncong \hat{A}$. Then $K_2(A)$ and $K_2(\hat{A})$ have the same period (for suitable markings)

but they are are not birational.