V. Further developments

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Cohomology

Proposition (Bogomolov-Verbitsky)

X hyperkähler, A sub-algebra of $H^*(X,\mathbb{Q})$ spanned by $H^2(X,\mathbb{Q})$.

Then A satisfies Poincaré duality; $H^*(X,\mathbb{Q}) = A \oplus A^\perp$;

$$A = S^*H^2(X,\mathbb{Q})/J \ \text{ with } \ J = \langle x^{r+1} \mid x \in H^2(X,\mathbb{Q}) \, , \; q(x) = 0 \, \rangle$$

Corollary

$S^pH^2(X,\mathbb{Q}) \to H^{2p}(X,\mathbb{Q})$ injective for $p \le r$.

Proof.

1. Geometric input:
   - $q(\alpha) = 0 \;\Rightarrow \alpha^{r+1} = 0$;
   - $\exists\, \omega \in H^2(X, \mathbb{Q}),\ \omega^{2r} \neq 0$.

2. Put $H = H^2(X, \mathbb{Q})$, $B = S^*H/J$. Then $S^*H \to H^*(X, \mathbb{Q})$ maps J to 0, hence factors as $\lambda : B \twoheadrightarrow A$, with $\lambda(B_{2r}) \neq 0$.

3. Representation theory of $O(H, q) \;\Rightarrow B$ Gorenstein, i.e. $B_p \times B_{2r-p} \to B_{2r} = \mathbb{Q}$ perfect $\forall p$.

4. If $\mathrm{Ker}\,\lambda \neq 0$, contains B_{2r}, contradiction. ∎

REMARK : A depends only on (H, q) and r.

Lagrangian fibrations

X hyperkähler, $\dim X = 2r$. Lagrangian fibration :

$f : X \to B$ with connected fibres, B Kähler of dimension r,

smooth fibres Lagrangian (i.e. $\sigma_{|X_b} = 0$).

Proposition (Arnold-Liouville)

The smooth fibres of f are complex tori.

Proof.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{X/B} & \longrightarrow & T_X & \longrightarrow & f^*T_B & \longrightarrow & 0 \\
 & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\
0 & \longrightarrow & f^*\Omega^1_B & \longrightarrow & \Omega^1_X & \longrightarrow & \Omega^1_{X/B} & \longrightarrow & 0
\end{array}$$

$\Rightarrow\ \Omega^1_{X_b} \cong \mathcal{O}^r_{X_b}\ \Rightarrow\ X_b$ complex torus. ∎

REMARK : Lagrangian fibrations correspond to completely integrable hamiltonian system in symplectic geometry.

Theorem (Matsushita + Hwang)

X hyperkähler, B Kähler with $0 < \dim B < 2r$, $f : X \twoheadrightarrow B$ with connected fibers. Then:

1. *f is a Lagrangian fibration;*
2. *B Fano with $b_2 = 1$ (and $\dim B = r$);*
3. *If X projective, $B \cong \mathbb{P}^r$.*

Proof.

1. For $\alpha \in H^2(B,\mathbb{C})$,

$$\alpha^{2r} = 0 \;\Rightarrow\; (f^*\alpha)^{2r} = 0 \;\Rightarrow\; (f^*\alpha)^{r+1} = 0 \;\Rightarrow\; \alpha^{r+1} = 0$$

$\Rightarrow\;$ $\dim B \leq r$ (take α Kähler).

2. $\alpha \neq 0 \;\Rightarrow\; f^*\alpha \neq 0 \;\Rightarrow\; (f^*\alpha)^r \neq 0 \;\Rightarrow\; \alpha^r \neq 0 \;\Rightarrow\; \dim B \geq r$.
3. $f^* : H^2(B,\mathbb{C}) \to H^2(X,\mathbb{C})$ injective $\;\Rightarrow\;$ $H^{2,0}(B) = 0$.
4. $f^*(H^2(B,\mathbb{C})) \subset H^{1,1}(X)$ totally isotropic for q;

   signature $q_{|H^{1,1}} = (1, h^{1,1} - 1) \;\Rightarrow\;$ $\dim H^2(B,\mathbb{C}) \leq 1$.
5. $\mathrm{Pic}(B) = \mathbb{Z}\cdot[L]$, $K_B = L^{\otimes n}$. *Idea* : $H^{r,0}(B) = 0$ (as above)

   $\Rightarrow\;$ $n \neq 0$, more work $\rightsquigarrow$ $n < 0$.
6. Proof that X_b Lagrangian: Skip proof

Proof that the fibres are Lagrangian

Lemma

$\alpha, \beta, \gamma \in H^2(X, \mathbb{C})$ with $q(\alpha) = q(\alpha, \beta) = 0$. Then

$$\int_X \alpha^p \beta^q \gamma^m = 0 \quad \textit{for } p > m\ .$$

Proof of the lemma.

- $\forall \gamma \in H^2(X, \mathbb{C})$, $q(t\alpha + \beta + s\gamma) = c\,st + P(s)$
- $\Rightarrow \int_X (t\alpha + \beta + s\gamma)^{2r} = f_X\,(c\,st + P(s))^r = \sum_{m \geq p} a_{p,m} t^p s^m$
- $\Rightarrow \int_X \alpha^p \beta^q \gamma^m = 0$ for $p > m$. ∎

Proof that the fibres are Lagrangian.

- APPLY WITH : $\alpha = f^*\alpha_0$ with $\int_B \alpha_0^r = m \neq 0$, $\beta = \sigma + \bar{\sigma}$, $\gamma =$ Kähler class on X.
- $i : X_b \hookrightarrow X$. Then $\int_X \alpha^r \omega = m \int_{X_b} i^*\omega$. Thus:
- $0 = \int_X \alpha^r \beta^2 \gamma^{r-2} = m \int_{X_b} i^*(\beta^2\gamma^{r-2}) =$
$$2m \int_{X_b} (i^*\sigma)(i^*\bar{\sigma})(i^*\gamma)^{r-2} \ .$$
- $i^*\gamma$ Kähler $\Rightarrow$ hermitian form $(\alpha,\beta) \mapsto \int_X \alpha\bar{\beta}(i^*\gamma)^{r-2} > 0$ on $H^{2,0}(X_b) \Rightarrow i^*\sigma = 0$. ∎

Some open questions

If $f : X \to B$ Lagrangian and M ample on B, f^*M nef and $q(f^*M) = 0$.

1. $L \in \mathrm{Pic}(X)$ nef, $q(L) = 0 \;\Rightarrow\; \exists\, f : X \to B$ s.t. $L = f^*M$?
2. Variant: $L \in \mathrm{Pic}(X)$, $q(L) = 0 \;\Rightarrow\; \exists\, f : X \dashrightarrow B$?

   EXAMPLE: S K3 with $\mathrm{Pic}(S) = \mathbb{Z}[L]$. Recall:

   $\mathrm{Pic}(X) = \mathbb{Z}[L^{[r]}] \overset{\perp}{\oplus} \mathbb{Z}[\delta_r]$, $\;q(L^{[r]}) = L^2$, $q(\delta_r) = -2(r-1)$.

   Assume $L^2 = 2(r-1)n^2$, then $M = L^{[r]}(-n\delta_r)$ has $q(M) = 0$.

   THEOREM (Sawon, Markushevich): $\exists\, f : S^{[r]} \to \mathbb{P}^r$ with $f^*\mathcal{O}_{\mathbb{P}^r}(1) = M$.

3. Recall $H^*(X, \mathbb{Q}) = A \oplus A^\perp$. What about $A^\perp$?

   Known: $4 \mid b_{2i+1}$ (Wakakuwa).

Some open questions, II

1. Can we say more for hyperkähler 4-folds?

   THEOREM (Guan): either $b_2 = 23$, or $3 \leq b_2 \leq 8$. Improve?

2. Do they have only finitely many deformation types?

3. Is there a correct formulation of a Torelli-type property?

4. Most important: Find more examples!

THE END