## Appendix: Lines on Pfaffian Hypersurfaces
by A. Beauville

The aim of this appendix is to prove that a general pfaffian hypersurface of degree $r > 2n-3$ in $\mathbb{P}^n$ contains no lines (Proposition 1). By a simple dimension count (see Corollary 4 below), it suffices to show that the variety of lines contained in the universal pfaffian hypersurface (that is, the hypersurface of degenerate forms in the space of all skew-symmetric forms on a given vector space) has the expected dimension. We will deduce this from an explicit description of the pencils of degenerate skew-symmetric forms, which is the content of the proposition below.

We work over an algebraically closed field $k$. We will need an elementary lemma:

**Lemma 4.** *Given a pencil of skew-symmetric forms on a $n$-dimensional vector space, there exists a subspace of dimension $\left[\frac{n+1}{2}\right]$ which is isotropic for all forms of the pencil.*

*Proof.* By induction on $n$, the cases $n = 0$ and $n = 1$ being trivial. Let $\varphi + t\psi$ be our pencil; we can assume that $\varphi$ is degenerate. Let $D$ be a line contained in the kernel of $\varphi$, and let $D^\perp$ be its orthogonal with respect to $\psi$. Then $\varphi$ and $\psi$ induce skew-symmetric forms $\bar\varphi$ and $\bar\psi$ on $D^\perp/D$; by the induction hypothesis there exists a subspace of dimension $\left[\frac{n-1}{2}\right]$ in $D^\perp/D$ which is isotropic for $\bar\varphi$ and $\bar\psi$. The pull-back of this subspace in $D^\perp$ has dimension $\left[\frac{n+1}{2}\right]$ and is isotropic for $\varphi$ and $\psi$. □

The following result must be well known, but I have not been able to find a reference:

Proposition 5. *Let $V$ be a vector space of dimension $2r$, and $(\varphi_t)_{t\in\mathbb{P}^1}$ a pencil of* degenerate *skew-symmetric forms on $V$. There exists a subspace $L \subset V$ of dimension $r+1$ which is isotropic for $\varphi_t$ for all $t \in \mathbb{P}^1$.*

*Proof.* Again we prove the proposition by induction on $r$, the case $r = 1$ being trivial. The associated maps $\Phi_t : V \to V^*$ form a pencil of singular linear maps. By a classical result in linear algebra (see [G, Chap. XII, Thm. 4]), there exist subspaces $K \in V$ and $L' \in V^*$, with $\dim K = \dim L' + 1$, such that $\Phi_t(K) \subset L'$ for all $t$; equivalently, there exist subspaces $K$ and $L$ of $V$, with $\dim K + \dim L = 2r+1$, which are orthogonal for each $\varphi_t$. Replacing $(K, L)$ by $(K \cap L, K + L)$ we may assume $K \subset L$; the pencil $(\varphi_t)$ restricted to $L$ is singular on $K$, hence induces a pencil $(\bar\varphi_t)$ on $L/K$. Put $\dim K = p$, so that $\dim(L/K) = 2r + 1 - 2p$. By the above lemma there is a subspace of $L/K$, of dimension $r+1-p$, which is isotropic for each $\bar{\varphi}_t$. Its pull-back in $L$ has dimension $r+1$ and is isotropic for each $\varphi_t$. □

Let us give a few consequences of Proposition 5. We keep our vector space $V$ of dimension $2r$; we denote by $\mathcal{S}_r$ the space of skew-symmetric forms on $V$, and by $\mathcal{X}_r$ the hypersurface of degenerate forms in $\mathbb{P}(\mathcal{S}_r)$.

**Corollary 3.** *The variety of lines contained in $\mathcal{X}_r$ is irreducible, of codimension $r+1$ in the Grassmannian of lines of $\mathbb{P}(\mathcal{S}_r)$.*

*Proof.* The $(r+1)$-planes of $V$ are parametrized by a Grassmannian $\mathcal{G}$ of dimension $r^2-1$. For such a plane $L$ the space $\mathcal{S}_{r,L}$ of forms $\varphi \in \mathcal{S}_r$ vanishing on $L$ has dimension

$$\dim \mathcal{S}_{r,L} = \dim \textstyle\bigwedge^2 V^* - \dim \bigwedge^2 L^* = r(2r-1) - \dfrac{r(r+1)}{2} = \dfrac{3r(r-1)}{2}\,.$$

Let $\mathcal{P}$ be the Grassmannian of lines in $\mathbb{P}(\mathcal{S}_r)$ (that is, the variety of pencils of skew-symmetric forms). Consider the locus $Z \in \mathcal{P} \times \mathcal{G}$ of pairs $(\ell, L)$ with $\ell \in \mathcal{S}_{r,L}$. The projection $Z \to \mathcal{G}$ is a smooth fibration; its fibre above a point $L \in \mathcal{G}$ is the Grassmannian of lines in $\mathbb{P}(\mathcal{S}_{r,L})$, which has dimension $2\dim \mathcal{S}_{r,L} - 4$. Thus $Z$ is smooth, irreducible, of dimension $r^2 - 1 + 2\dim \mathcal{S}_L - 4 = 4r^2 - 3r - 5$.

Let $\mathcal{P}_{\text{sing}}$ be the subvariety of $\mathcal{P}$ consisting of lines contained in $\mathcal{X}_r$ (that is, the subvariety of singular pencils). The content of Proposition 5 is that $\mathcal{P}_{\text{sing}}$ is the image of $Z$ under the projection to $\mathcal{P}$. Thus $\mathcal{P}_{\text{sing}}$ is irreducible, of dimension $\leq 4r^2 - 3r - 5$, or equivalently, since $\dim \mathcal{P} = 2\dim \mathcal{S}_r - 4 = 4r^2 - 2r - 4$, of codimension $\geq r+1$. On the other hand, $\mathcal{P}_{\text{sing}}$ is defined locally by $(r+1)$ equations in $\mathcal{P}$, given by the coefficients of the polynomial $\mathrm{Pf}(\varphi_t)$ of degree $r$. The corollary follows. □

Observe that $r+1$ is the number of conditions that the requirement to contain a given line imposes on a hypersurface of degree $r$ in projective space. In other words, Corollary 3 says that the hypersurface $\mathcal{X}_r$ behaves like a general hypersurface of degree $r$ as far as the dimension of its variety of lines is concerned.

Let $L$ be a vector space, of dimension $n+1$, and $\ell = (\ell_{ij})$ a $(2r \times 2r)$-skew-symmetric matrix of linear forms on $L$. The hypersurface $X_\ell$ in $\mathbb{P}(L)$ $(= \mathbb{P}^n)$ defined by $\mathrm{Pf}(\ell_{ij}) = 0$ is called a *pfaffian hypersurface*. It is defined by the equation $\mathrm{Pf}(\ell_{ij}) = 0$, of degree $r$.

**Corollary 4.** *If $r > 2n - 3$ and the forms $\ell_{ij}$ are general enough, $X_\ell$ contains no lines.*

*Proof.* The matrix $(\ell_{ij})$ defines a linear map $u : L \to \mathcal{S}_r$, which is injective when the forms $\ell_{ij}$ are general enough (observe that $\dim L < \dim \mathcal{S}_r$). Thus we can identify $L$ to its image in $\mathcal{S}_r$, and $X_\ell$ to the hypersurface $\mathcal{X}_r \cap \mathbb{P}(L)$ in $\mathbb{P}(L)$.

Let $G$ be the Grassmann variety of $(n+1)$-dimensional vector subspaces of $\mathcal{S}$, and $F$ the variety of lines contained in $\mathcal{X}_r$. Consider the incidence variety $Z \in F \times G$ of pairs $(\ell, L)$ with $\ell \in \mathbb{P}(L)$. The fibre of the projection $Z \to G$ at a point $L \in G$ is the variety of lines contained in $\mathcal{X}_r \cap \mathbb{P}(L) = X_\ell$.

Put $N := \dim \mathcal{S}_r$. We have $\dim F = 2N - 4 - (r+1)$ by Corollary 3; the projection $Z \to F$ is a fibration of relative dimension $(n-1)(N-n-1)$. This gives $\dim Z = 2N - 4 - (r+1) + (n-1)(N-n-1)$, while $\dim G = (n+1)(N-n-1)$. Thus

$$\dim Z - \dim G = 2n - 3 - r < 0\,,$$

hence the general fibre of the projection $Z \to G$ is empty. □

Note that '$(\ell_{ij})$ general enough' means 'for $(\ell_{ij})$ in a certain Zariski open subset of $(L^*)^N$'. In particular, suppose that our vector space $L$ comes from a vector space $L_0$ over an infinite subfield $k_0$ of $k$; then *the matrices* $(\ell_{ij}) \in (L_0^*)^N$ *such that* $X_\ell$ *contains no lines are Zariski dense in the parameter space* $(L^*)^N$ *for* $r > 2n - 3$.

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