QUOTIENTS OF JACOBIANS

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ABSTRACT. Let C be a curve of genus g, and let G be a finite group of automorphisms of C. The group G acts on the Jacobian J of C; we prove that for $g \geq 21$ the quotient J/G has canonical singularities and Kodaira dimension 0. On the other hand we give examples with $g \leq 4$ for which J/G is uniruled.

1. INTRODUCTION

Let C be a (smooth, projective) curve of genus g, and let G be a finite group of automorphisms of C. The group G acts on the Jacobian J of C. There are two rather different possibilities for the quotient J/G [K-L]: either it has canonical singularities and Kodaira dimension 0, or it is uniruled. The aim of this note is to show that the latter case is rather exceptional – in fact, it does not occur for $g \geq 21$. This bound is rough, and can certainly be lowered. On the other hand we give examples of uniruled J/G for $g \leq 4$. When $g = 3$ this is an essential ingredient in our proof that the cycle $[C] - [(-1)^*C]$ on J is torsion modulo algebraic equivalence [B-S]; in fact this note grew out of the observation that already in genus 3, the case where J/G is uniruled is quite rare.

2. J/G CANONICAL

Let C be a curve of genus g and J its Jacobian. Let G be a subgroup of $\mathrm{Aut}(C)$, hence also of $\mathrm{Aut}(J)$. By [K-L, Theorem 2], there are two possibilities:

- Either J/G has canonical singularities, and Kodaira dimension 0;
- or J/G is uniruled.

Proposition 1. *Assume $g \geq 21$. The quotient variety J/G has canonical singularities, and Kodaira dimension 0.*

Proof : We first observe that the fixed locus $\mathrm{Fix}(\sigma) \subset J$ of any element $\sigma \neq 1$ of G has codimension ≥ 2. Indeed the dimension of $\mathrm{Fix}(\sigma)$ is the multiplicity of the eigenvalue 1 for the action of σ on $H^0(C, K_C)$, that is, the genus of $C/\langle\sigma\rangle$, which is $\leq g-2$ as soon as $g \geq 4$ by the Hurwitz formula.

Let σ be an element of order r in G, and let $p \in J$ be a fixed point of σ. The action of σ on the tangent space $T_p(J)$ is isomorphic to the action on $T_0(J) = H^{0,1}(C)$. Let $\zeta = e^{\frac{2\pi i}{r}}$; we write the eigenvalues of σ acting on $H^{0,1}(C)$ in the form $\zeta^{a_1}, \ldots, \zeta^{a_g}$, with $0 \leq a_i < r$. By Reid's criterion [R, Theorem 3.1], J/G has canonical singularities if and only if $\sum a_i \geq r$ for all σ in G.

The eigenvalues of σ acting on $H^1(C,\mathbb{C})$ are $\zeta^{a_1}, \ldots, \zeta^{a_g}; \zeta^{-a_1}, \ldots, \zeta^{-a_g}$. Thus $\mathrm{Tr}\,\sigma^*_{|H^1(C,\mathbb{C})} = 2\sum \cos \dfrac{2\pi a_i}{r} \cdot$ By the Lefschetz formula, this trace is equal to $2 - f$, where f is the number of fixed points of σ; in particular, it is ≤ 2. Using $\cos x \geq 1 - \dfrac{x^2}{2}$, we find

$$1 \geq \sum \cos \frac{2\pi a_i}{r} \geq g - \frac{2\pi^2}{r^2} \sum a_i^2 \,,$$

hence $(\sum a_i)^2 \geq \sum a_i^2 \geq \dfrac{g-1}{2\pi^2}\, r^2$. Thus if $g \geq 1 + 2\pi^2 = 20.739\ldots$, we get $\sum a_i > r$. ■

Remark.– The proposition does not extend to the case of an arbitrary abelian variety: indeed we give below examples where J/G is uniruled; then if A is any abelian variety, the quotient of $J \times A$ by G acting trivially on A is again uniruled.

I am indebted to A. Höring for pointing out the reference [K-L].

3. J/G UNIRULED

We will now give examples of (low genus) curves C such that J/G is uniruled. The genus 2 case is quite particular (and probably well known). We put $\rho := e^{\frac{2\pi i}{3}}$.

Proposition 2. *If $g(C)=2$, J/G has Kodaira dimension 0 if and only if $G=\langle\sigma\rangle$ and σ is the hyperelliptic involution, or an automorphism of order 3 with eigenvalues (ρ,ρ^2) on $H^0(C,K_C)$, or an automorphism of order 6 with eigenvalues $(-\rho,-\rho^2)$ on $H^0(C,K_C)$.*

Proof : Let σ be an element of order r in G, and let ζ be a primitive r-th root of unity. The eigenvalues of σ acting on $H^0(C,K_C)$ are ζ^a,ζ^b, with $0\le a,b<r$. Suppose that J/G has canonical singularities. By Reid's criterion we must have $a+b\ge r$; replacing ζ by ζ^{-1} gives $2r-a-b\ge r$, hence $a+b=r$. Since σ has order r, σ^k acts non-trivially on $H^0(C,K_C)$ for $0<k<r$; therefore a and r are coprime. Replacing ζ by ζ^a we may assume that the eigenvalues of σ are ζ and ζ^{-1}.

Put $\zeta=e^{i\alpha}$; the trace of σ acting on $H^1(C,\mathbb{Z})$ is $2(\zeta+\zeta^{-1})=4\cos\alpha$, and it is an integer. This implies $\cos\alpha=0$ or $\pm\dfrac{1}{2}$, hence $\alpha\in\mathbb{Z}\cdot\dfrac{2\pi}{6}$ and $r\in\{2,3,6\}$. If $r=2$ we must have $\zeta=\zeta^{-1}=-1$, hence σ is the hyperelliptic involution; if $r=3$ or 6 we have $\{\zeta,\zeta^{-1}\}=\{\rho,\rho^2\}$ or $\{-\rho,-\rho^2\}$.

Thus G contains elements of order 3 and 6, and at most one element of order 2. Since the order of $\mathrm{Aut}(C)$ is not divisible by 9 [B-L, 11.7], G is isomorphic to either $\mathbb{Z}/3$, or a central extension of $\mathbb{Z}/3$ by $\mathbb{Z}/2$ – that is, $\mathbb{Z}/6$. This proves the Proposition. ∎

The last two cases are realized by the curves C of the form $y^2=x^6+kx^3+1$, with the automorphisms $\sigma_1:(x,y)\mapsto(\rho x,y)$ and $\sigma_2:(x,y)\mapsto(\rho x,-y)$. The surfaces $S_i:=J/\langle\sigma_i\rangle$ are (singular) K3 surfaces, each depending on one parameter. The surface S_1 is studied in detail in [B-V]: it has 9 singular points of type A_2. The surface S_2 is the quotient of S_1 by the involution induced by $(-1)_J$; it has one singular point of type A_5 (the image of $0\in J$), 4 points of type A_2 corresponding to pairs of singular points of S_1, and 5 points of type A_1 corresponding to triples of points of order 2 of J.

It is more subtle to give examples with $g\ge 3$. By Reid's criterion, we must exhibit an automorphism σ of C, of order r, such that the eigenvalues of σ acting on $H^0(C,K_C)$ are of the form $\zeta^{a_1},\ldots,\zeta^{a_g}$, with ζ a primitive r-th root of unity, $a_i\ge 0$ and $\sum a_i<r$. In the following tables we give the curve C (in affine coordinates), the order r of σ and ζ, the expression of σ, a basis of eigenvectors for σ acting on $H^0(C,K_C)$, the exponents $a_1,\ldots,a_g$; in the last column we check the inequality $\sum a_i<g$, which implies that $J/\langle\sigma\rangle$ is uniruled.

$g=3$

curve	r	$\sigma(x,y)=$	basis of $H^0(K_C)$	$a_1,\ldots,a_3$	$\sum a_i$
$y^2=x^8-1$	8	$(\zeta x,y)$	$\frac{dx}{y},\frac{xdx}{y},\frac{x^2dx}{y}$	$1,2,3$	$6<8$
$y^2=x(x^7-1)$	14	$(\zeta^2x,\zeta y)$	$\frac{dx}{y},\frac{xdx}{y},\frac{x^2dx}{y}$	$1,3,5$	$9<14$
$y^3=x(x^3-1)$	9	$(\zeta^3x,\zeta y)$	$\frac{dx}{y^2},\frac{xdx}{y^2},\frac{dx}{y}$	$1,4,2$	$7<9$
$y^3=x^4-1$	12	$(\zeta^{-3}x,\zeta^{-4}y)$	$\frac{dx}{y^2},\frac{xdx}{y^2},\frac{dx}{y}$	$5,2,1$	$8<12$

The fact that $J/\langle\sigma\rangle$ is uniruled for the third curve C is an important ingredient of the proof in [B-S] that the Ceresa cycle $[C]-[(-1)^*C]$ in J is torsion modulo algebraic equivalence. The same property for the fourth curve is observed in [L-S].

$g = 4$

curve	r	$\sigma(x,y) =$	basis of $H^0(K_C)$	$a_1, \ldots, a_4$	$\sum a_i$
$y^2 = x(x^9-1)$	18	$(\zeta^2 x, \zeta y)$	$\frac{dx}{y}, \frac{xdx}{y}, \frac{x^2dx}{y}, \frac{x^3dx}{y}$	$1, 3, 5, 7$	$16 < 18$
$y^3 = x^5 - 1$	15	$(\zeta^{-3}x, \zeta^{-5}y)$	$\frac{dx}{y^2}, \frac{xdx}{y^2}, \frac{x^2dx}{y^2}, \frac{dx}{y}$	$7, 4, 1, 2$	$14 < 15$

In genus 5 there is no example:

Proposition 3. *Assume $g = 5$. The quotient variety J/G has canonical singularities, and Kodaira dimension 0.*

Proof : The paper [K-K] contains the list of possible automorphisms of genus 5 curves, and gives for each automorphism of order r its eigenvalues $e^{\frac{2\pi i a_1}{r}}, \ldots, e^{\frac{2\pi i a_5}{r}}$ on $H^0(K)$, with $0 \leq a_i < 5$. We want to prove $\sum a_i \geq r$.

We first observe that if an eigenvalue and its inverse appear in the list, we have $\sum a_i \geq r$; this eliminates all cases with $r \leq 10$. For $r \geq 11$ the numbers $a_1, \ldots, a_5$ are always distinct and nonzero; therefore $\sum a_i \geq 1+2+\cdots+5 = 15$. The remaining cases are $r = 20$ and 22; one checks immediately that for all k coprime to r we have $\sum b_i \geq r$, where $b_i \equiv k a_i \pmod{r}$ and $0 < b_i < r$. This proves the Proposition. ■

I know no example in genus ≥ 6, and I suspect there are none.

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