

# ON FINITE SIMPLE GROUPS OF ESSENTIAL DIMENSION 3

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ABSTRACT. We show that the only finite simple groups of essential dimension 3 (over  $\mathbb{C}$ ) are  $\mathfrak{A}_6$  and possibly  $\mathrm{PSL}_2(\mathbb{F}_{11})$ . This is an easy consequence of the classification by Prokhorov of rationally connected threefolds with an action of a simple group.

## INTRODUCTION

Let  $G$  be a finite group, and  $X$  a complex projective variety with a faithful action of  $G$ . We will say that  $X$  is a *linearizable* if there exists a complex representation  $V$  of  $G$  and a rational dominant  $G$ -equivariant map  $V \dashrightarrow X$  (such a map is called a *compression* of  $V$ ). The *essential dimension*  $\mathrm{ed}(G)$  of  $G$  (over  $\mathbb{C}$ ) is the minimal dimension of all linearizable  $G$ -varieties. We have to refer to [BR] for the motivation behind this definition; in a very informal way,  $\mathrm{ed}(G)$  is the minimum number of parameters needed to define all Galois extensions  $L/K$  with Galois group  $G$  and  $K \supset \mathbb{C}$ .

The groups of essential dimension 1 are the cyclic groups and the dihedral group  $D_n$ ,  $n$  odd [BR]. The groups of essential dimension 2 are classified in [D2]; the list is already large, and such classification becomes probably intractable in higher dimension. However the *simple* (finite) groups in the list are only  $\mathfrak{A}_5$  and  $\mathrm{PSL}_2(\mathbb{F}_7)$ . In this note we try to go one step further:

**Proposition.** *The simple groups of essential dimension 3 are  $\mathfrak{A}_6$  and possibly  $\mathrm{PSL}_2(\mathbb{F}_{11})$ .*

The result is an easy consequence of the remarkable paper of Prokhorov [P], who classifies all rationally connected threefolds admitting the action of a simple group. We can rule out most of the groups appearing in [P] thanks to a simple criterion [RY]: if a  $G$ -variety  $X$  is linearizable, any abelian subgroup of  $G$  must fix a point of  $X$ . Unfortunately this criterion does not apply to  $\mathrm{PSL}_2(\mathbb{F}_{11})$ , whose only abelian subgroups are cyclic or isomorphic to  $(\mathbb{Z}/2)^2$ .

## 1. PROKHOROV'S LIST

Let  $G$  be a finite simple group with  $\mathrm{ed}(G) = 3$ . By definition there exists a linearizable projective  $G$ -threefold  $X$ . This implies in particular that  $X$  is rationally connected. Such pairs  $(G, X)$  have been classified in [P]: up to conjugation, we have the following possibilities:

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- (1)  $G = \mathrm{SL}_2(\mathbb{F}_8)$  acting on a Fano threefold  $X \subset \mathbb{P}^8$ ;
- (2)  $G = \mathfrak{A}_5, \mathfrak{A}_6, \mathfrak{A}_7, \mathrm{PSL}_2(\mathbb{F}_7), \mathrm{PSL}_2(\mathbb{F}_{11}),$  or  $\mathrm{PSP}_4(\mathbb{F}_3)$ .

The groups  $\mathfrak{A}_5, \mathfrak{A}_6, \mathfrak{A}_7$  have essential dimension 2,3 and 4 respectively, and  $\mathrm{PSL}_2(\mathbb{F}_7)$  has essential dimension 2 [D1]. We are not able to settle the case  $G = \mathrm{PSL}_2(\mathbb{F}_{11})$  (see §3). As for  $\mathrm{PSP}_4(\mathbb{F}_3)$ , we have:

**Proposition 1.** *The essential dimension of  $\mathrm{PSP}(4, \mathbb{F}_3)$  is 4.*

*Proof :* The group  $\mathrm{Sp}(4, \mathbb{F}_3)$  has a linear representation on the space  $W$  of functions on  $\mathbb{F}_3^2$ , the *Weil representation*, for which we refer to [AR], Appendix I. This representation splits as  $W = W^+ \oplus W^-$ , the spaces of even and odd functions; we have  $\dim W^+ = 5$ ,  $\dim W^- = 4$ . The central element  $(-I)$  of  $\mathrm{Sp}(4, \mathbb{F}_3)$  acts on  $W$  by  $(-I)F(x) = F(-x)$ , hence it acts trivially on  $W^+$ , and as  $-\mathrm{Id}$  on  $W^-$ . Thus we get a faithful representation of  $\mathrm{PSP}(4, \mathbb{F}_3)$  on  $W^+$ , with a compression to  $\mathbb{P}(W^+) \cong \mathbb{P}^4$ , hence  $\mathrm{ed}(\mathrm{PSP}(4, \mathbb{F}_3)) \leq 4$ .

To prove that we have equality, we observe<sup>1</sup> that  $\mathrm{PSP}(4, \mathbb{F}_3)$  contains a subgroup isomorphic to  $(\mathbb{Z}/2)^4$ . One way to see this is to use the isomorphism  $\mathrm{PSP}(4, \mathbb{F}_3) \cong \mathrm{SO}^+(5, \mathbb{F}_3)$ : the group of diagonal matrices with entries  $\pm 1$  and determinant 1 is contained in  $\mathrm{SO}^+(5, \mathbb{F}_3)$ , and isomorphic to  $(\mathbb{Z}/2)^4$ . By [BR] we have

$$\mathrm{ed}(\mathrm{PSP}(4, \mathbb{F}_3)) \geq \mathrm{ed}((\mathbb{Z}/2)^4) = 4. \quad \blacksquare$$

## 2. THE GROUP $\mathrm{SL}_2(\mathbb{F}_8)$

It remains to prove that the pair  $(\mathrm{SL}_2(\mathbb{F}_8), X)$  mentioned in (1) is not linearizable. To do this we will use the following criterion ([RY], Appendix):

**Lemma 1.** *If  $(G, X)$  is linearizable, every abelian subgroup of  $G$  has a fixed point in  $X$ .*

**Proposition 2.** *The essential dimension of  $\mathrm{SL}_2(\mathbb{F}_8)$  is  $\geq 4$ .*

The group  $\mathrm{SL}_2(\mathbb{F}_8)$  has a representation of dimension 7, hence its essential dimension is  $\leq 6$  – we do not know its precise value.

*Proof :* The group  $\mathrm{SL}_2(\mathbb{F}_8)$  acts on a rational Fano threefold  $X \subset \mathbb{P}^8$  in the following way [P]. Let  $U$  be an irreducible 9-dimensional representation of  $\mathrm{SL}_2(\mathbb{F}_8)$ ; there exists a non-degenerate invariant quadratic form  $q$  on  $U$ , unique up to a scalar. Then  $\mathrm{SL}_2(\mathbb{F}_8)$  acts on the orthogonal Grassmannian  $\mathbb{G}_{\mathrm{iso}}(4, U)$  of 4-dimensional isotropic subspaces of  $U$ . This Grassmannian admits a  $O(q)$ -equivariant embedding into  $\mathbb{P}^{15}$ , given by the half-spinor representation [M]. The threefold  $X$  is the intersection of  $\mathbb{G}_{\mathrm{iso}}(4, U)$  with a subspace  $\mathbb{P}^8 \subset \mathbb{P}^{15}$  invariant under  $\mathrm{SL}_2(\mathbb{F}_8)$ .

Let  $N \subset \mathrm{SL}_2(\mathbb{F}_8)$  be the subgroup of matrices  $\begin{pmatrix} I & a \\ 0 & I \end{pmatrix}$ ,  $a \in \mathbb{F}_8$ . We will show that  $N$  has no fixed point in  $\mathbb{G}_{\mathrm{iso}}(4, U)$ , and therefore in  $X$ .

<sup>1</sup>I am indebted to A. Duncan for this observation.

Let  $\chi_U$  be the character of the representation  $U$ . We have  $\chi_U(n) = 1$  for  $n \in N, n \neq 1$  (see for instance [C], 2.7). It follows that  $U$  restricted to  $N$  is the sum of the regular representation and the trivial one; in other words, as a  $N$ -module we have

$$U = \mathbb{C}_1^2 \oplus \sum_{\substack{\lambda \in \hat{N} \\ \lambda \neq 1}} \mathbb{C}_\lambda,$$

where  $\mathbb{C}_\lambda$  is the one-dimensional representation associated to the character  $\lambda$ . The subspaces  $\mathbb{C}_\alpha$  and  $\mathbb{C}_\beta$  must be orthogonal for  $\alpha \neq \beta$ ; since  $q$  is non-degenerate, its restriction to each  $\mathbb{C}_\lambda$  ( $\lambda \neq 1$ ) and to  $\mathbb{C}_1^2$  must be non-degenerate.

Now any vector subspace  $L \subset U$  fixed by  $N$  must be the sum of some of the  $\mathbb{C}_\lambda$ , for  $\lambda \neq 1$ , and of some subspace of  $\mathbb{C}_1^2$ ; this implies that  $L$  cannot be isotropic as soon as  $\dim L \geq 2$ . Hence  $N$  has no fixed point on  $\mathbb{G}_{\text{iso}}(4, U)$ , and  $X$  is not linearizable by Lemma 1. ■

### 3. ABOUT $\text{PSL}_2(\mathbb{F}_{11})$

The Weil representation  $W^-$  of  $\text{SL}_2(\mathbb{F}_{11})$  factors through  $\text{PSL}_2(\mathbb{F}_{11})$ , hence provides a 5-dimensional representation of the latter group; thus its essential dimension is 3 or 4. According to [P] there are two rationally connected threefolds with an action of  $\text{PSL}_2(\mathbb{F}_{11})$ , the Klein cubic  $X^k \subset \mathbb{P}^4$  given by  $\sum_{i \in \mathbb{Z}/5} X_i^2 X_{i+1} = 0$  and a Fano threefold  $X^a \subset \mathbb{P}^9$  of degree 14, birational to  $X^k$ . The group  $\text{PSL}_2(\mathbb{F}_{11})$  has order  $660 = 2^2 \cdot 3 \cdot 5 \cdot 11$ ; its abelian subgroups are cyclic, except the 2-Sylow subgroups which are isomorphic to  $(\mathbb{Z}/2)^2$ . A finite order automorphism of a rationally connected variety has always a fixed point (for instance by the holomorphic Lefschetz formula); one checks easily that a 2-Sylow subgroup of  $\text{PSL}_2(\mathbb{F}_{11})$  has a fixed point on both  $X^k$  and  $X^a$ . So lemma 1 does not apply, and another approach is needed.

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