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ORTHOGONAL BUNDLES ON CURVES AND THETA FUNCTIONS

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Abstract. — Let $\mathcal{M}$ be the moduli space of principal SO_r-bundles on a curve C, and $\mathcal{L}$ the determinant bundle on $\mathcal{M}$. We define an isomorphism of $H^0(\mathcal{M},\mathcal{L})$ onto the dual of the space of r-th order theta functions on the Jacobian of C. This isomorphism identifies the rational map $\mathcal{M} \dashrightarrow |\mathcal{L}|^*$ defined by the linear system $|\mathcal{L}|$ with the map $\mathcal{M} \dashrightarrow |r\Theta|$ which associates to a quadratic bundle (E,q) the theta divisor Θ_E. The two components $\mathcal{M}^+$ and $\mathcal{M}^-$ of $\mathcal{M}$ are mapped into the subspaces of even and odd theta functions respectively. Finally we discuss the analogous question for Sp_{2r}-bundles.

Résumé. — Soient $\mathcal{M}$ l'espace des modules des fibrés SO_r-principaux sur une courbe C, et $\mathcal{L}$ le fibré déterminant sur $\mathcal{M}$. Nous définissons un isomorphisme de $H^0(\mathcal{M},\mathcal{L})$ sur le dual de l'espace des fonctions thêta du r-ième ordre sur la Jacobienne de C. Cet isomorphisme identifie l'application rationnelle $\mathcal{M} \dashrightarrow |\mathcal{L}|^*$ définie par le système linéaire $|\mathcal{L}|$ avec l'application $\mathcal{M} \dashrightarrow |r\Theta|$ qui associe à un fibré quadratique (E,q) le diviseur thêta Θ_E. Les deux composantes $\mathcal{M}^+$ et $\mathcal{M}^-$ de $\mathcal{M}$ sont envoyées sur les sous-espaces de fonctions paires et impaires respectivement. Finalement nous discutons le problème analogue pour les fibrés symplectiques.

Introduction

Let C be a curve of genus $g \geqslant 2$, G an almost simple complex Lie group, and $\mathcal{M}_G$ the moduli space of semi-stable G-bundles on C. For each component $\mathcal{M}_G^\bullet$ of $\mathcal{M}_G$, the Picard group is infinite cyclic; its positive generator $\mathcal{L}_G^\bullet$ can be described explicitely as a determinant bundle. Then a natural question, which we will address in this paper for the classical groups, is to describe the space of "generalized theta functions" $H^0(\mathcal{M}_G^\bullet,\mathcal{L}_G^\bullet)$ and the associated rational map $\varphi_G^\bullet : \mathcal{M}_G^\bullet \dashrightarrow |\mathcal{L}_G^\bullet|^*$.

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The model we have in mind is the case $G = \mathrm{SL}_r$. Let J^{g-1} be the component of the Picard variety of C parameterizing line bundles of degree $g-1$; it is isomorphic to the Jacobian of C, and carries a canonical theta divisor Θ consisting of line bundles L in J^{g-1} with $H^0(C,L)\neq 0$. For a general $E\in\mathcal{M}_{\mathrm{SL}_r}$, the locus

$$\Theta_E = \big\{L\in J^{g-1} \mid H^0(C,E\otimes L)\neq 0\big\}$$

is in a natural way a divisor, which belongs to the linear system $|r\Theta|$ on J^{g-1}. We thus obtain a rational map $\vartheta : \mathcal{M}_{\mathrm{SL}_r} \dashrightarrow |r\Theta|$. The main result of [6] is that *there exists an isomorphism* $|\mathcal{L}_{\mathrm{SL}_r}|^* \xrightarrow{\sim} |r\Theta|$ *which identifies the rational maps* φ_{SL_r} *and* ϑ. This gives a reasonably concrete description of φ_{SL_r}, which allows to get some information on the behaviour of this map, at least for small values of r or g (see [2] for a survey of recent results).

Let us consider now the case $G=\mathrm{SO}_r$ with $r\geqslant 3$. The moduli space $\mathcal{M}_{\mathrm{SO}_r}$ parametrizes oriented orthogonal bundles (E,q) on C of rank r; it has two components $\mathcal{M}^+_{\mathrm{SO}_r}$ and $\mathcal{M}^-_{\mathrm{SO}_r}$. Let $\theta : \mathcal{M}_{\mathrm{SO}_r} \dashrightarrow |r\Theta|$ be the map $(E,q)\mapsto \Theta_E$. We will see that θ maps $\mathcal{M}^+_{\mathrm{SO}_r}$ and $\mathcal{M}^-_{\mathrm{SO}_r}$ into the subspaces $|r\Theta|^+$ and $|r\Theta|^-$ corresponding to even and odd theta functions respectively. Our main result is:

THEOREM. — *There are canonical isomorphisms* $|\mathcal{L}^{\pm}_{\mathrm{SO}_r}|^* \xrightarrow{\sim} |r\Theta|^{\pm}$ *which identify* $\varphi^{\pm}_{\mathrm{SO}_r} : \mathcal{M}^{\pm}_{\mathrm{SO}_r} \dashrightarrow |\mathcal{L}^{\pm}_{\mathrm{SO}_r}|^*$ *with the map* $\theta^{\pm} : \mathcal{M}^{\pm}_{\mathrm{SO}_r} \dashrightarrow |r\Theta|^{\pm}$ *induced by* θ.

This is easily seen to be equivalent to the fact that the pull-back map $\theta^* : H^0(J^{g-1},\mathcal{O}(r\Theta))^* \to H^0(\mathcal{M}_{\mathrm{SO}_r},\mathcal{L}_{\mathrm{SO}_r})$ is an *isomorphism*. We will prove that it is injective by restricting to a small subvariety of $\mathcal{M}_{\mathrm{SO}_r}$ (§1). Then we will use the Verlinde formula (§2 and 3) to show that the dimensions are the same. This is somewhat artificial since it forces us for instance to treat separately the cases r even $\geqslant 6$, r odd $\geqslant 5$, $r=3$ and $r=4$. It would be interesting to find a more direct proof, perhaps in the spirit of [6].

In the last section we consider the same question for the symplectic group. Here the theta map does not involve the Jacobian of C but the moduli space $\mathcal{N}$ of semi-stable rank 2 vector bundles on C with determinant K_C. Let $\mathcal{L}$ be the determinant bundle on $\mathcal{N}$. For (E,φ) general in $\mathcal{M}_{\mathrm{Sp}_{2r}}$, the reduced subvariety

$$\Delta_E = \big\{F\in\mathcal{N} \mid H^0(E\otimes F)\neq 0\big\}$$

is a divisor on $\mathcal{N}$, which belongs to the linear system $|\mathcal{L}^r|$; this defines a map $\mathcal{M}_{\mathrm{Sp}_{2r}} \dashrightarrow |\mathcal{L}^r|$ which should coincide, up to a canonical isomorphism, with $\varphi_{\mathrm{Sp}_{2r}}$. This is a particular case of the *strange duality* conjecture for the symplectic group, which we discuss in §4. Unfortunately even this particular case is not known, except in a few cases that we explain below.

1. The moduli space $\mathcal{M}_{\mathrm{SO}_r}$

1.1. — Throughout the paper we fix a complex curve C of genus $g \geqslant 2$. For G a semi-simple complex Lie group, we denote by $\mathcal{M}_G$ the moduli space of semi-stable G-bundles on C. It is a normal projective variety, of dimension $(g-1)\dim G$. Its connected components are in one-to-one correspondence with the elements of the group $\pi_1(G)$.

1.2. — Let us consider the case $G = \mathrm{SO}_r$ $(r \geqslant 3)$. The space $\mathcal{M}_{\mathrm{SO}_r}$ is the moduli space of (semi-stable) oriented orthogonal bundles, that is triples (E, q, ω) where E is a semi-stable[(1)] vector bundle of rank r, $q : \mathsf{S}^2 E \to \mathcal{O}_C$ a non-degenerate quadratic form, and ω a section of $\det E$ with $\tilde{q}(\omega) = 1$, where $\tilde{q}$ is the quadratic form on $\det E$ deduced from q. The two components $\mathcal{M}^+_{\mathrm{SO}_r}$ and $\mathcal{M}^-_{\mathrm{SO}_r}$ are distinguished by the parity of the second Stiefel-Whitney class $w_2(E, q) \in H^2(C, \mathbf{Z}/2) \cong \mathbf{Z}/2$. This class has the following property (see e.g. [17, Thm. 2]): for every theta-characteristic κ on C and orthogonal bundle $(E, q) \in \mathcal{M}_{\mathrm{SO}_r}$,

$$w_2(E, q) \equiv h^0(C, E \otimes \kappa) + rh^0(C, \kappa) \pmod 2 \tag{1.3}$$

The involution $\iota : L \mapsto K_C \otimes L^{-1}$ of J^{g-1} preserves Θ, hence lifts to an involution of $\mathcal{O}_{J^{g-1}}(\Theta)$. We denote by $|r\Theta|^+$ and $|r\Theta|^-$ the two corresponding eigenspaces in $|r\Theta|$, and by $\theta : \mathcal{M}_{\mathrm{SO}_r} \dashrightarrow |r\Theta|$ the map $(E, q) \mapsto \Theta_E$.

LEMMA 1.4. — *The rational map $\theta : \mathcal{M}_{\mathrm{SO}_r} \dashrightarrow |r\Theta|$ maps $\mathcal{M}^+_{\mathrm{SO}_r}$ in $|r\Theta|^+$ and $\mathcal{M}^-_{\mathrm{SO}_r}$ in $|r\Theta|^-$.*

Proof. — For any $E \in \mathcal{M}_{\mathrm{SL}_r}$ we have $\iota^*\Theta_E = \Theta_{E^*}$, so $\theta(\mathcal{M}_{\mathrm{SO}_r})$ is contained in the fixed locus $|r\Theta|^+ \cup |r\Theta|^-$ of ι^*. Since $\mathcal{M}^{\pm}_{\mathrm{SO}_r}$ is connected, it suffices to find one element (E, q) of $\mathcal{M}^+_{\mathrm{SO}_r}$ (resp. $\mathcal{M}^-_{\mathrm{SO}_r}$) such that Θ_E is a divisor in $|r\Theta|^+$ (resp. $|r\Theta|^-$).

Let $\kappa \in J^{g-1}$ be an even theta-characteristic of C; a symmetric divisor $D \in |r\Theta|$ is in $|r\Theta|^+$ (resp. $|r\Theta|^-$) if and only if $\mathrm{mult}_\kappa(D)$ is even (resp. odd) – see [13, §2]. Let $J[2]$ be the 2-torsion subgroup of $\mathrm{Pic}(C)$; we take $E = \alpha_1 \oplus \cdots \oplus \alpha_r$, where $\alpha_1, \ldots, \alpha_r \in J[2]$ and $\sum \alpha_i = 0$. We endow E with the diagonal quadratic form q deduced from the isomorphisms $\alpha_i^2 \cong \mathcal{O}_C$.

(1) By [16, 4.2], an orthogonal bundle (E, q) is semi-stable if and only if the vector bundle E is semi-stable.

Then $\Theta_E = \Theta_{\alpha_1} + \cdots + \Theta_{\alpha_r}$. By the Riemann singularity theorem the multiplicity at κ of Θ_α is $h^0(\alpha\otimes\kappa)$. Thus by (1.3)

$$\mathrm{mult}_\kappa(\Theta_E) = \sum_i h^0(\alpha_i\otimes\kappa) = h^0(E\otimes\kappa) \equiv w_2(E,q) \pmod 2.$$

□

1.5. — Let $\mathcal{L}_{\mathrm{SO}_r}$ be the determinant bundle on $\mathcal{M}_{\mathrm{SO}_r}$, that is, the pull back of $\mathcal{L}_{\mathrm{SL}_r}$ by the map $(E,q)\mapsto E$, and let $\mathcal{L}^+_{\mathrm{SO}_r}$ and $\mathcal{L}^-_{\mathrm{SO}_r}$ be its restrictions to $\mathcal{M}^+_{\mathrm{SO}_r}$ and $\mathcal{M}^-_{\mathrm{SO}_r}$. It follows from [5] that for $r\neq 4$, $\mathcal{L}^\pm_{\mathrm{SO}_r}$ generates $\mathrm{Pic}(\mathcal{M}^\pm_{\mathrm{SO}_r})$.

Proposition 1.6. — *The map*

$$\theta^* : H^0(J^{g-1},\mathcal{O}(r\Theta))^* \longrightarrow H^0(\mathcal{M}_{\mathrm{SO}_r},\mathcal{L}_{\mathrm{SO}_r})$$

induced by $\theta : \mathcal{M}_{\mathrm{SO}_r} \dashrightarrow |r\Theta|$ *is an isomorphism.*

By Lemma 1.4 θ^* splits as a direct sum $(\theta^+)^*\oplus(\theta^-)^*$, where

$$(\theta^\pm)^* : \big(H^0(J^{g-1},\mathcal{O}(r\Theta))^\pm\big)^* \longrightarrow H^0(\mathcal{M}^\pm_{\mathrm{SO}_r},\mathcal{L}^\pm_{\mathrm{SO}_r}).$$

The Proposition implies that $(\theta^+)^*$ and $(\theta^-)^*$ are isomorphisms, and this is equivalent to the Theorem stated in the introduction.

Proof of the Proposition. — We will show in §3 that the Verlinde formula gives

$$\dim H^0(\mathcal{M}_{\mathrm{SO}_r},\mathcal{L}_{\mathrm{SO}_r}) = \dim H^0(J^{g-1},\mathcal{O}(r\Theta)) = r^g.$$

It is therefore sufficient to prove that θ^* is injective, or equivalently that $\theta(\mathcal{M}_{\mathrm{SO}_r})$ spans the projective space $|r\Theta|$. We consider again the orthogonal bundles $(E,q)=\alpha_1\oplus\cdots\oplus\alpha_r$ for $\alpha_1,\ldots,\alpha_r$ in $J[2]$, $\sum\alpha_i=0$. This bundle has a theta divisor $\Theta_E=\Theta_{\alpha_1}+\cdots+\Theta_{\alpha_r}$. We claim that divisors of this form span $|r\Theta|$. To prove it, let us identify J^{g-1} with the Jacobian J of C (by choosing a divisor class of degree $g-1$). For $a\in J$, the divisor Θ_a is the only element of the linear system $|\mathcal{O}_J(\Theta)\otimes\varphi(a)|$, where $\varphi : J\to\widehat{J}$ is the isomorphism associated to the principal polarization of J. Therefore our assertion follows from the following easy lemma:

Lemma 1.7. — *Let A be an abelian variety, L an ample line bundle on A, $\widehat{A}[2]$ the 2-torsion subgroup of* $\mathrm{Pic}(A)$. *The multiplication map*

$$\sum_{\substack{\alpha_1,\ldots,\alpha_r\in\widehat{A}[2]\\ \alpha_1+\cdots+\alpha_r=0}} H^0(A,L\otimes\alpha_1)\otimes\cdots\otimes H^0(A,L\otimes\alpha_r)\longrightarrow H^0(A,L^r)$$

is surjective.

Proof. — Let 2_A be the multiplication by 2 in A. We have canonical isomorphisms

$$H^0(A, 2_A^* L) \cong \bigoplus_{\alpha\in\widehat{A}[2]} H^0(L\otimes\alpha), \quad H^0(A, 2_A^* L^r) \cong \bigoplus_{\beta\in\widehat{A}[2]} H^0(L^r\otimes\beta);$$

through these isomorphisms the product map $m_r : H^0(A, 2_A^*L)^{\otimes r} \longrightarrow H^0(A, 2_A^*L^r)$ is the direct sum over $\beta\in\widehat{A}[2]$ of the maps

$$m_r^\beta : \sum_{\substack{\alpha_1,\dots,\alpha_r\in\widehat{A}[2]\\ \alpha_1+\dots+\alpha_r=\beta}} H^0(A, L\otimes\alpha_1)\otimes\dots\otimes H^0(A, L\otimes\alpha_r) \longrightarrow H^0(A, L^r\otimes\beta).$$

Since the line bundle 2_A^*L is algebraically equivalent to L^4, the map m_r is surjective [14], hence so is m_r^β for every β. The case $\beta = 0$ gives the lemma. □

2. The Verlinde formula

2.1. — We keep the notation of 1.1; we denote by q the number of simple factors of the Lie algebra of G (we are mainly interested in the case $q = 1$).

To each representation $\rho : G \to \mathrm{SL}_r$ is attached a line bundle $\mathcal{L}_\rho$ on $\mathcal{M}_G$, the pull back of the determinant bundle on $\mathcal{M}_{\mathrm{SL}_r}$ by the morphism $\mathcal{M}_G \to \mathcal{M}_{\mathrm{SL}_r}$ associated to ρ. The Verlinde formula expresses the dimension of $H^0(\mathcal{M}_G, \mathcal{L}_\rho^k)$, for each integer k, in the form

$$\dim H^0(\mathcal{M}_G, \mathcal{L}_\rho^k) = N_{k\mathbf{d}_\rho}(G),$$

where

• $\mathbf{d}_\rho \in \mathbf{N}^q$ is the *Dynkin index* of ρ. For $q = 1$ the number d_ρ is defined and computed in [9, §2]. In the general case the universal cover of G is a product $G_1\times\dots\times G_q$ of almost simple factors, and we put $\mathbf{d}_\rho = (d_{\rho_1}, \dots, d_{\rho_q})$, where ρ_i is the pull back of ρ to G_i.

We will need only to know that the Dynkin index is 2 for the standard representation of SO_r ($r \geqslant 5$), 4 for that of SO_3, and $(2,2)$ for that of SO_4.

• $N_{\boldsymbol{\ell}}(G)$ is an integer depending on G, the genus g of C, and $\boldsymbol{\ell}\in\mathbf{N}^q$. We will now explain how this number is computed. Our basic reference is [1].

2.2. The simply connected case

Let us first consider the case where G is simply connected and almost simple (that is, $q = 1$). Let T be a maximal torus of G, and $R = R(G, T)$ the corresponding root system (we view the roots of G as characters of T). We denote by T_ℓ the (finite) subgroup of elements $t \in T$ such that $\alpha(t)^{\ell+h} = 1$ for each *long* root α, and by T_ℓ^{reg} the subset of regular elements $t \in T_\ell$ (that is, such that $\alpha(t) \neq 1$ for each root α). It is stable under the action of the Weyl group W. For $t \in T$, we put $\Delta(t) = \prod_{\alpha\in R}(\alpha(t)-1)$. Then the Verlinde formula is

$$N_\ell(G) = \sum_{t\in T_\ell^{\mathrm{reg}}/W} \Big(\frac{|T_\ell|}{\Delta(t)}\Big)^{g-1}.$$

2.3. — This number can be explicitely computed in the following way. Let $\mathfrak{t}$ be the Lie algebra of T. The character group $P(R)$ of T embeds naturally into $\mathfrak{t}^*$. We endow $\mathfrak{t}^*$ with the W-invariant bilinear form $(|)$ such that $(\alpha \,|\, \alpha) = 2$ for each long root α, and we use this product to identify $\mathfrak{t}^*$ with $\mathfrak{t}$. Let θ be the highest root of R; we denote by P_ℓ the set of dominant weights $\lambda \in P(R)$ such that $(\lambda \mid \theta) \leqslant \ell$. Let $\rho \in P(R)$ be the half-sum of the positive roots. The number $h := (\rho \,|\, \theta) + 1$ is the *dual Coxeter number* of R. We have $|T_\ell| = (\ell+h)^s f\nu$, where s is the rank of R, f the order of the center of G, and ν a number depending on R; it is equal to 1 for R of type D_s and to 2 for B_s ([4, 9.9]).

For $\lambda \in P_\ell$ we put $t_\lambda = \exp 2\pi i\, \frac{\lambda+\rho}{\ell+h}\cdot$ The map $\lambda \mapsto t_\lambda$ is a bijection of P_ℓ onto T_ℓ^{reg}/W ([4, 9.3.c])). For $\lambda \in P_\ell$, we have $\alpha(t_\lambda) = \exp 2\pi i\, \frac{(\alpha\,|\,\lambda+\rho)}{\ell+h}$, and therefore

$$\Delta(t_\lambda) = \prod_{\alpha\in R_+} 4\,\sin^2 \pi\, \frac{(\alpha\,|\,\lambda+\rho)}{\ell+h}\cdot$$

2.4. The non-simply connected case

We now give the formula for a general almost simple group, following [1].

Let Z be the center of G. An element t of T belongs to Z if and only if $\alpha(t) = 1$ for all $\alpha \in R$, or equivalently $w(t) = t$ for all $w \in W$. It follows that Z acts on the set T_ℓ^{reg} by multiplication; this action commutes with that of W and thus defines an action of Z on T_ℓ^{reg}/W. Through the bijection $P_\ell \to T_\ell^{\mathrm{reg}}/W$ the action of Z on P_ℓ is the one deduced from its action on the extended Dynkin diagram (see [15, §3] or [7, 2.3 and 4.3]).

Now let Γ be a subgroup of Z, and let $G' = G/\Gamma$. We denote by P'_ℓ the sublattice of weights $\lambda \in P_\ell$ such that $\lambda_{|\Gamma} = 1$. The action of Γ on P_ℓ preserves P'_ℓ; we denote by $\Gamma\cdot\lambda$ the orbit of a weight λ in P'_ℓ. The Verlinde formula for G' is ([1, Thm. 5.3]):

$$N_\ell(G') = |\Gamma| \sum_{\lambda\in P'_\ell} |\Gamma\cdot\lambda|^{-2g}\Big(\frac{|T_\ell|}{\Delta(t_\lambda)}\Big)^{g-1}.$$

Each term in the sum is invariant under Γ, so we may as well sum over P'_ℓ/Γ provided we multiply each term by $|\Gamma\cdot\lambda|$:

$$N_\ell(G') = |\Gamma| \sum_{\lambda\in P'_\ell/\Gamma} |\Gamma\cdot\lambda|^{1-2g}\Big(\frac{|T_\ell|}{\Delta(t_\lambda)}\Big)^{g-1} \tag{2.5}$$

2.6. The general case

The above formula actually applies to any semi-simple group $G' = G/\Gamma$, where G is a product of simply connected groups $G_1,\dots,G_q$ [1].

We choose a maximal torus $T^{(i)}$ in G_i for each i and put $T = T^{(1)}\times\cdots\times T^{(q)}$. Let $\boldsymbol{\ell} := (\ell_1,\dots,\ell_q)$ be a q-uple of nonnegative integers. We put $T_{\boldsymbol{\ell}} = T^{(1)}_{\ell_1}\times\cdots\times T^{(q)}_{\ell_q}$; the subset $T^{\mathrm{reg}}_{\boldsymbol{\ell}}$ of regular elements in $T_{\boldsymbol{\ell}}$ is the product of the subsets $(T^{(i)}_{\ell_i})^{\mathrm{reg}}$. For each i, let $P^{(i)}_{\ell_i}$ be the set of dominant weights of $T^{(i)}$ associated to G_i and ℓ_i as in 2.3, and let $P_{\boldsymbol{\ell}} = P^{(1)}_{\ell_1}\times\cdots\times P^{(q)}_{\ell_q}$. For $\lambda = (\lambda_1,\dots,\lambda_q)\in P_{\boldsymbol{\ell}}$, we put $t_\lambda = (t_{\lambda_1},\dots,t_{\lambda_q})\in T_{\boldsymbol{\ell}}$; this defines a bijection of $P_{\boldsymbol{\ell}}$ onto $T^{\mathrm{reg}}_{\boldsymbol{\ell}}/W$. The elements of $P_{\boldsymbol{\ell}}$ are characters of T, and we denote by $P'_{\boldsymbol{\ell}}$ the subset of characters which are trivial on Γ. The group Γ is contained in the center $Z_1\times\cdots\times Z_q$ of G, which acts naturally on $P_{\boldsymbol{\ell}}$ and $P'_{\boldsymbol{\ell}}$. Then

$$N_{\boldsymbol{\ell}}(G') = |\Gamma| \sum_{\lambda\in P'_{\boldsymbol{\ell}}/\Gamma} |\Gamma\cdot\lambda|^{1-2g}\Big(\frac{|T_{\boldsymbol{\ell}}|}{\Delta(t_\lambda)}\Big)^{g-1} \tag{2.7}$$

with

$$\frac{|T_{\boldsymbol{\ell}}|}{\Delta(t_\lambda)} = \prod_{i=1}^{q}\frac{|T_{\ell_i}|}{\Delta_i(t_{\lambda_i})},\qquad \Delta_i(t) = \prod_{\alpha\in R(G_i,T^{(i)})}(\alpha(t)-1)$$

for $t\in T^{(i)}$.

3. The Verlinde formula for SO_r

We now apply the previous formulas to the case $G' = \mathrm{SO}_r$. We will rest very much on the computations of [15]. We will borrow their notation as well as that of [7].

3.1. The case $G' = \mathrm{SO}_{2s}$, $s \geqslant 3$

The root system R is of type D_s. Let $(\varepsilon_1, \ldots, \varepsilon_s)$ be the standard basis of $\mathbf{R}^s$. The weight lattice $P(R)$ is spanned by the fundamental weights

$$\varpi_j = \varepsilon_1 + \cdots + \varepsilon_j \ (1 \leqslant j \leqslant s-2),$$

$$\varpi_{s-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{s-1} - \varepsilon_s),\ \varpi_s = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{s-1} + \varepsilon_s).$$

For $\lambda \in P(R)$, we write $\lambda + \rho = \sum_i t_i \varpi_i = \sum_i u_i \varepsilon_i$ with

$$u_1 = t_1 + \cdots + t_{s-2} + \frac{1}{2}(t_{s-1} + t_s), \ldots, u_{s-2} = t_{s-2} + \frac{1}{2}(t_{s-1} + t_s)$$

$$u_{s-1} = \frac{1}{2}(t_{s-1} + t_s), \quad u_s = \frac{1}{2}(-t_{s-1} + t_s)$$

$$(u_i \in \frac{1}{2}\mathbf{Z}, \quad u_i - u_{i+1} \in \mathbf{Z}).$$

Put $k = \ell + 2s - 2$. The condition $\lambda \in P_\ell$ becomes: $u_1 > \cdots > u_s$, $u_1 + u_2 < k$ and $u_{s-1} + u_s > 0$; the condition $\lambda \in P'_\ell$ imposes moreover $t_{s-1} \equiv t_s \pmod 2$, that is, $u_i \in \mathbf{Z}$ for each i. Thus we find a bijection between P'_ℓ and the subsets $U = \{u_1, \ldots, u_s\}$ of $\mathbf{Z}$ satisfying the above conditions.

The group Z is canonically isomorphic to $P(R)/Q(R)$ (note that $R = R^\vee$ in this case); its nonzero elements are the classes of ϖ_1, ϖ_{s-1} and ϖ_s. The nonzero element γ which vanishes in SO_{2s} is represented by the only weight in this list which comes from SO_{2s}, namely ϖ_1. It corresponds to the automorphism of the extended Dynkin diagram which exchanges α_0 with α_1 and α_{s-1} with α_s (see [7, Table D_l]); it acts on P_ℓ by $\gamma(u_1, \ldots, u_s) = (k - u_1, u_2, \ldots, u_{s-1}, -u_s)$. Thus the subsets U as above with $u_s \geqslant 0$, and moreover $u_1 \leqslant \frac{k}{2}$ if $u_s = 0$, form a system of representatives of P'_ℓ/Γ. The corresponding orbit has one element if $u_1 = \frac{k}{2}$ and $u_s = 0$, and 2 otherwise.

For a subset U corresponding to the weight λ we have [15]

$$\Delta(t_\lambda) = \Pi_k(U) = \prod_{1 \leqslant i < j \leqslant s} 4 \sin^2 \frac{\pi}{k}(u_i - u_j)\ 4 \sin^2 \frac{\pi}{k}(u_i + u_j).$$

Now we restrict ourselves to the case $\ell = 2$, so that $k = r = 2s$. Put $V = \{s, s-1, \ldots, 0\}$. The subsets U to consider are those of the form $U_j := V - \{j\}$ for $0 \leqslant j \leqslant s$. We have

- $\Pi_r(U_j) = 4r^{s-1}$ for $1 \leqslant j \leqslant s-1$ by Corollary 1.7 (ii) in [15];
- $\Pi_r(U_0) = \Pi_r(U_s) = r^{s-1}$ by Corollary 1.7 (iii) in [15].

We have $|T_2| = 4r^s$ (2.3). Multiplying the terms U_0 and U_s by 2^{1-2g} and summing, we find:

$$N_2(\mathrm{SO}_{2s}) = 2\big[(s-1)\cdot r^{g-1}\big] + 2^{1-2g}\big[2\cdot (4r)^{g-1}\big] = r^g.$$

3.2. The case $G' = \mathrm{SO}_{2s+1}$, $s \geqslant 2$

Then R is of type B_s. Denoting again by $(\varepsilon_1, \ldots, \varepsilon_s)$ the standard basis of $\mathbf{R}^s$, the weight lattice $P(R)$ is spanned by the fundamental weights

$$\varpi_1 = \varepsilon_1, \varpi_2 = \varepsilon_1 + \varepsilon_2, \ldots, \varpi_{s-1} = \varepsilon_1 + \cdots + \varepsilon_{s-1}, \varpi_s = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_s).$$

For $\lambda \in P(R)$, we write $\lambda + \rho = \sum_i t_i \varpi_i = \sum_i u_i \varepsilon_i$ with

$$u_1 = t_1 + \cdots + t_{s-1} + \frac{t_s}{2}, \ldots, u_{s-1} = t_{s-1} + \frac{t_s}{2}, u_s = \frac{t_s}{2},$$

with $u_i \in \frac{1}{2}\mathbf{Z}$ and $u_i - u_{i+1} \in \mathbf{Z}$ for each i. Put $k = \ell + 2s - 1$. The condition $\lambda \in P_\ell$ becomes $u_1 > \cdots > u_s > 0$ and $u_1 + u_2 < k$. Since ϖ_s is the only fundamental weight which does not come from SO_{2s+1}, the condition $\lambda \in P'_\ell$ is equivalent to t_s odd, that is, $u_s \in \mathbf{Z} + \frac{1}{2}$. Thus we find a bijection between P'_ℓ and the subsets $U = \{u_1, \ldots, u_s\}$ of $\mathbf{Z} + \frac{1}{2}$ satisfying

$$u_1 > \cdots > u_s > 0, \quad u_1 + u_2 < k.$$

The non-trivial element γ of Γ acts on P_ℓ by $\gamma(u_1, \ldots, u_s) = (k - u_1, u_2, \ldots, u_s)$. Thus the elements U as above with $u_1 \leqslant \frac{k}{2}$ form a system of representatives of P'_ℓ/Γ. The corresponding orbit has one element if $u_1 = \frac{k}{2}$ and 2 otherwise.

For a subset U corresponding to the weight λ we have [15]

$$\Delta(t_\lambda) = \Phi_r(U) = \prod_{1 \leqslant i < j \leqslant s} 4 \sin^2 \frac{\pi}{r}(u_i - u_j)\ 4 \sin^2 \frac{\pi}{r}(u_i + u_j) \prod_{i=1}^{s} 4 \sin^2 \frac{\pi}{r} u_i.$$

Now we restrict ourselves to the case $\ell = 2$, so that $k = r = 2s + 1$. Put $V = \{s + \frac{1}{2}, s - \frac{1}{2}, \ldots, \frac{1}{2}\}$. The subsets U to consider are the subsets $U_j := V - \{j + \frac{1}{2}\}$ for $0 \leqslant j \leqslant s$. We have

- $\Phi_r(U_j) = 4r^{s-1}$ for $0 \leqslant j \leqslant s-1$ by Corollary 1.9 (ii) in [15];
- $\Phi_r(U_s) = r^{s-1}$ by Corollary 1.9 (ii) in [15].

We have again $|T_2| = 4r^s$ (2.3). Multiplying the term U_s by 2^{1-2g} and summing, we find:

$$N_2(\mathrm{SO}_{2s+1}) = 2\big[s \cdot r^{g-1} + 2^{1-2g}(4r)^{g-1}\big] = r^g.$$

3.3. The case $G' = \mathrm{SO}_3$

In that case $G = \mathrm{SL}_2$ has a unique fundamental weight ρ, and a unique positive root $\theta = 2\rho$. The Dynkin index of the standard representation of SO_3 is 4, so we want to compute $N_4(\mathrm{SO}_3)$. We have $|T_4| = 12$ (2.3).

The set P_4 contains the weights $k\rho$ with $0 \leqslant k \leqslant 4$; the weights with k even come from SO_3, and Γ exchanges $k\rho$ and $(4-k)\rho$. Thus a system of representatives of P'_4/Γ is $\{0, 2\rho\}$, with $|\Gamma \cdot 0| = 2$ and $|\Gamma \cdot 2\rho| = 1$. Formula (2.5) gives:

$$N_2(\mathrm{SO}_3) = 2 \cdot \left[2^{1-2g} 12^{g-1} + 3^{g-1}\right] = 3^g.$$

3.4. The case $G' = \mathrm{SO}_4$

In that case $G = \mathrm{SL}_2 \times \mathrm{SL}_2$ and the nontrivial element of Γ is $(-I, -I)$. The Dynkin index of the standard representation of SO_4 is $(2,2)$. We have $|T_2| = 8$ for SL_2, hence $|T_{(2,2)}| = 8^2$. The set $P_{(2,2)}$ contains the weights $(k\rho, l\rho)$ with $0 \leqslant k, l \leqslant 2$, and $P'_{(2,2)}$ is defined by the condition $k \equiv l \pmod 2$. The element $(-I, -I)$ exchanges $(k\rho, l\rho)$ with $((2-k)\rho, (2-l)\rho)$. Thus $P'_{(2,2)}/\Gamma$ consists of the classes of $(0,0)$, $(0, 2\rho)$ and (ρ, ρ), the latter being the only one with a nontrivial stabilizer. Formula (2.7) gives

$$N_{(2,2)}(\mathrm{SO}_4) = 2\,[\,2 \cdot 2^{1-2g} \cdot 4^{2g-2} + 2^{2g-2}\,] = 4^g.$$

Therefore for each $r \geqslant 3$ we have obtained $\dim H^0(\mathcal{M}_{\mathrm{SO}_r}, \mathcal{L}_{\mathrm{SO}_r}) = r^g$. This achieves the proof of Proposition 1.6, and therefore of the Theorem stated in the introduction.

4. The moduli space $\mathcal{M}_{\mathrm{Sp}_{2r}}$

4.1. — Let r be an integer $\geqslant 1$. The space $\mathcal{M}_{\mathrm{Sp}_{2r}}$ is the moduli space of (semi-stable) symplectic bundles, that is pairs (E, φ) where E is a semi-stable(2) vector bundle of rank $2r$ and trivial determinant and $\varphi : \mathsf{\Lambda}^2 E \to \mathcal{O}_C$ a non-degenerate alternate form. It is connected. To alleviate the notation we will denote it by $\mathcal{M}_r$. The determinant bundle $\mathcal{L}_r$ generates $\mathrm{Pic}(\mathcal{M}_r)$ ([10, 12]).

To describe the "strange duality" in an intrinsic way we need a variant of this space, namely the moduli space $\mathcal{M}'_r$ of semi-stable vector bundles F of rank $2r$ and determinant K_C^r, endowed with a symplectic form $\psi : \mathsf{\Lambda}^2 F \to K_C$. If κ is a theta-characteristic on C, the map $E \mapsto E \otimes \kappa$ induces an isomorphism $\mathcal{M}_r \xrightarrow{\sim} \mathcal{M}'_r$. We denote by $\mathcal{L}'_r$ the line bundle corresponding to $\mathcal{L}_r$ under any of these isomorphisms.

(2) By the same argument as in the orthogonal case (footnote 1), a symplectic bundle (E, φ) is semi-stable if and only if E is semi-stable as a vector bundle.

Similarly, we will consider for t even the moduli space $\mathcal{M}'_{\mathrm{SO}_t}$ of semi-stable vector bundles E of rank t and determinant $K_C^{t/2}$, endowed with a quadratic form $q : \mathsf{S}^2E \to K_C$. It has two components $\mathcal{M}'^{\pm}_{\mathrm{SO}_t}$ depending on the parity of $h^0(E)$; if κ is a theta-characteristic on C, the map $E \mapsto E \otimes \kappa$ induces isomorphisms $\mathcal{M}^{\pm}_{\mathrm{SO}_t} \xrightarrow{\sim} \mathcal{M}'^{\pm}_{\mathrm{SO}_t}$ (1.3). The space $\mathcal{M}'^{+}_{\mathrm{SO}_t}$ carries a canonical Weil divisor, the reduced subvariety

$$\mathcal{D} = \big\{(E,q) \in \mathcal{M}'^{+}_{\mathrm{SO}_t} \mid H^0(C,E) \neq 0\big\};$$

$2\mathcal{D}$ is a Cartier divisor, defined by a section of the generator $\mathcal{L}'_{\mathrm{SO}_t}$ of $\operatorname{Pic}(\mathcal{M}'^{+}_{\mathrm{SO}_t})$ ([12, §7]).

4.2. The strange duality for symplectic bundles

Let r, s be integers $\geqslant 2$, and $t = 4rs$. Consider the map

$$\pi : \mathcal{M}_r \times \mathcal{M}'_s \longrightarrow \mathcal{M}'_{\mathrm{SO}_t}$$

which maps $((E,\varphi),(F,\psi)$ to $(E\otimes F, \varphi\otimes\psi)$. Since $\mathcal{M}_r$ is connected and contains the trivial bundle $\mathcal{O}^{2r}$ with the standard symplectic form, the image lands in $\mathcal{M}'^{+}_{\mathrm{SO}_t}$.

For $(E,\varphi) \in \mathcal{M}_r$, the pull back of $\mathcal{L}_{\mathrm{SO}_t}$ to $\{(E,\varphi)\} \times \mathcal{M}'_s$ is the line bundle associated to $2r$ times the standard representation, that is $\mathcal{L}'^{2r}_s$; similarly its pull back to $\mathcal{M}_r \times \{(F,\psi)\}$, for $(F,\psi) \in \mathcal{M}'_s$, is $\mathcal{L}_r^{2s}$. It follows that

$$\pi^*\mathcal{L}_{\mathrm{SO}_t} \cong \mathcal{L}_r^{2s} \boxtimes \mathcal{L}'^{2r}_s .$$

If κ is a theta-characteristic on C with $h^0(\kappa) = 0$, we have $\pi(\mathcal{O}_C^{2r}, \kappa^{2s}) \notin \mathcal{D}$ ($\mathcal{O}_C^{2r}$ and κ^{2s} are endowed with the standard alternate forms). Thus $\Delta := \pi^*\mathcal{D}$ is a Weil divisor on $\mathcal{M}_r \times \mathcal{M}'_s$, whose double is a Cartier divisor defined by a section of $(\mathcal{L}_r^s \boxtimes \mathcal{L}'^r_s)^2$; but this moduli space is locally factorial ([18, Thm. 1.2]), so that Δ is actually a Cartier divisor, defined by a section δ of $\mathcal{L}_r^s \boxtimes \mathcal{L}'^r_s$, well-defined up to a scalar. Via the Künneth isomorphism we view δ as an element of $H^0(\mathcal{M}_r, \mathcal{L}_r^s) \otimes H^0(\mathcal{M}'_s, \mathcal{L}'^r_s)$. The *strange duality* conjecture for symplectic bundles is

CONJECTURE 4.3. — *The section δ induces an isomorphism*

$$\delta^\sharp : H^0(\mathcal{M}_r, \mathcal{L}_r^s)^* \xrightarrow{\sim} H^0(\mathcal{M}'_s, \mathcal{L}'^r_s).$$

If the conjecture holds, the rational map $\varphi_{\mathcal{L}_r^s} : \mathcal{M}_r \dashrightarrow |\mathcal{L}_r^s|^*$ is identified through $\delta^\sharp$ to the map $\mathcal{M}_r \dashrightarrow |\mathcal{L}'^r_s|$ given by $E \mapsto \Delta_E$, where Δ_E is the trace of Δ on $\{E\} \times \mathcal{M}'_s$; set-theoretically:

$$\Delta_E = \big\{(F,\varphi) \in \mathcal{M}'_s \mid H^0(C, E\otimes F) \neq 0\big\}.$$

By [15], we have $\dim H^0(\mathcal{M}_r, \mathcal{L}_r^s) = \dim H^0(\mathcal{M}_s, \mathcal{L}_s^r)$. Therefore the conjecture is equivalent to:

4.4. — *The linear system $|\mathcal{L}_r'^s|$ is spanned by the divisors Δ_E, for $E \in \mathcal{M}_r$*

We now specialize to the case $s = 1$. The space $\mathcal{M}_1'$ is the moduli space $\mathcal{N}$ of semi-stable rank 2 vector bundles on C with determinant K_C; its Picard group is generated by the determinant bundle $\mathcal{L}$. The conjecture becomes:

Conjecture 4.5. — *The isomorphism $\delta^\sharp : H^0(\mathcal{M}_r, \mathcal{L}_r)^* \xrightarrow{\sim} H^0(\mathcal{N}, \mathcal{L}^r)$ identifies the map $\varphi_{\mathcal{L}_r} : \mathcal{M}_r \dashrightarrow |\mathcal{L}_r|^*$ with the rational map $E \mapsto \Delta_E$ of $\mathcal{M}_r$ into $|\mathcal{L}|$.*

By 4.4 this is equivalent to saying that the linear system $|\mathcal{L}^r|$ on $\mathcal{N}$ is spanned by the divisors Δ_E for $E \in \mathcal{M}_r$.

4.6. — Let G be a semi-stable vector bundle of rank r and degree 0. To G is associated a divisor $\Theta_G \in |\mathcal{L}^r|$, supported on the set

$$\Theta_G = \big\{F \in \mathcal{N} \mid H^0(C, G \otimes F) \neq 0\big\}$$

provided this set is $\neq \mathcal{N}$ [8]. Put $E = G \oplus G^*$, with the standard symplectic form. We have $\Theta_G = \Theta_{G^*}$ by Serre duality, hence $\Delta_E = \frac{1}{2}\Theta_E = \frac{1}{2}(\Theta_G + \Theta_{G^*}) = \Theta_G$; thus conjecture 4.5 holds if the linear system $|\mathcal{L}^r|$ on $\mathcal{N}$ is spanned by the divisors Θ_G for G semi-stable of degree 0. In particular, it suffices to prove that $|\mathcal{L}^r|$ is spanned by the divisors $\Theta_{L_1} + \cdots + \Theta_{L_r}$, for $L_1, \ldots, L_r \in J$. As a consequence of [6], the divisors Θ_L for L in J span $|\mathcal{L}|$, so *Conjecture 4.5 holds if the multiplication map $m_r : \mathsf{S}^r H^0(\mathcal{N}, \mathcal{L}) \to H^0(\mathcal{N}, \mathcal{L}^r)$ is surjective.*

Proposition 4.7. — *Conjecture 4.5 holds in the following cases:*

(i) *$r = 2$ and C has no vanishing thetanull;*
(ii) *$r \geqslant 3g - 6$ and C is general enough;*
(iii) *$g = 2$, or $g = 3$ and C is non-hyperelliptic.*

Proof. — In each case the multiplication map $m_r : \mathsf{S}^r H^0(\mathcal{N}, \mathcal{L}) \to H^0(\mathcal{N}, \mathcal{L}^r)$ is surjective. This follows from [3, Prop. 2.6 c)], in case (i), and from the explicit description of $\mathcal{M}_{\mathrm{SL}_2}$ in case (iii). When C is generic, the surjectivity of m_r for r even $\geqslant 2g - 4$ follows from that of m_2 together with [11]. We have $H^i(\mathcal{N}, \mathcal{L}^j) = 0$ for $i \geqslant 1$ and $j \geqslant -3$ by [10, Thm. 2.8]. By [14] this implies that the multiplication map

$$H^0(\mathcal{N}, \mathcal{L}) \otimes H^0(\mathcal{N}, \mathcal{L}^k) \longrightarrow H^0(\mathcal{N}, \mathcal{L}^{k+1})$$

is surjective for $k \geqslant \dim \mathcal{N} - 3 = 3g - 6$. Together with the previous result this implies the surjectivity of m_r for $r \geqslant 3g - 6$, and therefore by 4.6 the Proposition. □

COROLLARY 4.8. — *Suppose C has no vanishing thetanull. There is a canonical isomorphism $|\mathcal{L}_2|^* \xrightarrow{\sim} |4\Theta|^+$ which identify the maps $\varphi_{\mathcal{L}_2} : \mathcal{M}_2 \dashrightarrow |\mathcal{L}_2|^*$ with $\theta : \mathcal{M}_2 \dashrightarrow |4\Theta|^+$ such that $\theta(E,\varphi) = \Theta_E$.*

Proof. — Let $i : J^{g-1} \to \mathcal{N}$ be the map $L \mapsto L \oplus \iota^* L$. The composition

$$H^0(\mathcal{M}_2, \mathcal{L}_2)^* \xrightarrow{\delta^\sharp} H^0(\mathcal{N}, \mathcal{L}^2) \xrightarrow{i^*} H^0(J^{g-1}, \mathcal{O}(4\Theta))^+$$

is an isomorphism by Prop. 4.7, (i) and Prop. 2.6 c) of [3]; it maps $\varphi_{\mathcal{L}_2}(E,\varphi)$ to $i^*\Delta_E$. Using Serre duality again we find $i^*\Delta_E = \frac{1}{2}(\Theta_E + \Theta_{E^*}) = \Theta_E$, hence the Corollary. □

Remarks 4.9.

1) The corollary does not hold if C has a vanishing thetanull: the image of θ is contained in that of i^*, which is a proper subspace of $|4\Theta|^+$.

2) The analogous statement for $r \geqslant 3$ does not hold: the Verlinde formula implies $\dim H^0(\mathcal{N}, \mathcal{L}^r) > \dim H^0(J^{g-1}, \mathcal{O}(2r\Theta))^+$ for $g \geqslant 3$, or $g = 2$ and $r \geqslant 4$.

Added in proof. — P. Belkale has announced a proof of the strange duality conjecture for vector bundles on a generic curve of given genus (preprint `math.AG/0602018`). As explained in 4.6, this implies Conjecture 4.5 for a generic curve.

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