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p-Elementary subgroups of the Cremona group

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Abstract

We classify, up to conjugacy, the subgroups of the Cremona group isomorphic to $(\mathbb{Z}/p)^r$, where p is prime and r is maximal.

Keywords: Cremona group; Birational automorphisms; del Pezzo surfaces

Introduction

Let k be an algebraically closed field. The *Cremona group* Cr_k is the group of birational transformations of $\mathbb{P}^2_k$, or equivalently the group of k-automorphisms of the field $k(x,y)$. There is an extensive classical literature about this group, in particular about its finite subgroups—see the introduction of [dF] for a list of references.

The classification of conjugacy classes of elements of prime order p in Cr_k has been given a modern treatment in [B-B] for $p=2$ and in [dF] for $p \geqslant 3$ (see also [B-Bl]). In this note we go one step further and classify p-elementary subgroups—that is, subgroups isomorphic to $(\mathbb{Z}/p)^r$ for p prime. We will mostly describe such a subgroup as a group G of automorphisms of a rational surface S: we identify G to a subgroup of Cr_k by choosing a birational map $\varphi : S \dashrightarrow \mathbb{P}^2$. Then the conjugacy class of G in Cr_k depends only on the data (G, S).

Theorem. *Let G be a subgroup of* Cr_k *of the form* $(\mathbb{Z}/p)^r$ *with p prime* $\neq \mathrm{char}(k)$. *Then*:

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(a) *Assume* $p \geqslant 5$. *Then* $r \leqslant 2$, *and if* $r = 2$ G *is conjugate to the* p*-torsion subgroup of the diagonal torus*[1] *of* $\mathrm{PGL}_3(k) = \mathrm{Aut}(\mathbb{P}^2)$.
(b) *Assume* $p = 3$. *Then* $r \leqslant 3$, *and if* $r = 3$ G *is conjugate to the* 3*-torsion subgroup of the diagonal torus of* $\mathrm{PGL}_4(k)$, *acting on the Fermat cubic surface* $X_0^3 + \cdots + X_3^3 = 0$.
(c) *Assume* $p = 2$. *Then* $r \leqslant 4$, *and if equality holds* G *is conjugate to one of the following subgroups*:
  (c$_1$) *the* 2*-torsion subgroup of the diagonal torus of* $\mathrm{PGL}_5(k)$, *acting on the quartic del Pezzo surface in* $\mathbb{P}^4$ *with equations* $\sum_{i=0}^4 X_i^2 = \sum_{i=0}^4 \lambda_i X_i^2 = 0$ *for some distinct elements* $\lambda_0, \dots, \lambda_4$ *of* k.
  (c$_2$) *the subgroup of* Cr_k *spanned by the involutions*:

$$(x, y) \mapsto (-x, y), \qquad (x, y) \mapsto \left(\frac{1}{x}, y\right), \qquad (x, y) \mapsto \left(x, \frac{\alpha(u)}{y}\right),$$
$$(x, y) \mapsto \left(x, \frac{\lambda(u)y - \alpha(u)}{y - \lambda(u)}\right)$$

for some α, λ *in* $k(u)$ *with* $\alpha \notin \{0, \lambda^2\}$, $u = x^2 + x^{-2}$. *Alternatively, this subgroup acts on the rational surface* $y^2 - \alpha(x)z^2 - \beta(x) = 0$ *in* $k^* \times k^2$, *with* $\beta = \alpha - \lambda^2$, *by changing the sign of* x, y, z *and changing* x *in* x^{-1}.

Note that the question does not make sense when $p = \mathrm{char}(k)$, already in dimension 1: the group $\mathrm{PGL}_2(k)$ contains a subgroup isomorphic to k, hence infinite-dimensional over $\mathbb{Z}/p$.

In the next section we discuss some motivation for this question; then we reduce the problem through standard techniques to the study of p-elementary subgroups $G \subset \mathrm{Aut}(S)$, where S is either a del Pezzo surface or carries a $\mathbb{P}^1$-fibration preserved by G. We will study the latter case in Section 2 and the former in Section 3. Finally in Section 4 we discuss the classification of the conjugacy classes of subgroups isomorphic to $(\mathbb{Z}/2)^4$ (the only case where the conjugacy class is not unique).

As I. Dolgachev pointed out to us, the result (over $\mathbb{C}$) could be deduced from the list of the finite subgroups of the Cremona group established by Kantor [K], and completed by Wiman [W]. However, the results of Sections 2 and 4 would still be needed to decide whether certain subgroups are conjugate or not. Most of the results of Section 3 are contained in those of Kantor and Wiman, but they are so much simpler in our specific situation that we have preferred to give an independent proof.

1. Comments, and beginning of the proof

1.1. Though very large, the Cremona group behaves in some respect like a semi-simple group of rank 2: every maximal torus has dimension 2, and is conjugate to the diagonal torus T of $\mathrm{PGL}_3(k)$ [D1]. In this set-up the analogue of the Weyl group is the whole automorphisms group $\mathrm{GL}_2(\mathbb{Z})$ of T [D1, Corollary 5, p. 522].

Now let H be a semi-simple group over k, and p a prime number $\geqslant 7$ which does not divide the order of $\pi_1(H)$. Then every maximal p-elementary subgroup $G \subset H$ is the p-torsion sub-

[1] By the *diagonal torus* of $\mathrm{PGL}_r(k)$ we mean the subgroup of projective transformations $(X_0, \dots, X_r) \mapsto (t_0 X_0, \dots, t_r X_r)$ for $t_0, \dots, t_r$ in k^*.

group of a maximal torus of H; moreover this torus is unique, in fact it is the centralizer of G in H [Bo].

1.2. Our theorem (together with [dF]) shows that the first part of this statement also holds for the Cremona group for $p \geqslant 7$. However, *the maximal torus containing G is not unique*. Indeed, let T be the diagonal torus of $\mathrm{PGL}_3(k)$, and G its p-torsion subgroup. The centralizer of G in Cr_k contains the transformations $\sigma_f : (x, y) \mapsto (x, yf(x^p))$ for $f \in k(t)^*$. If f is not a monomial σ_f does not normalize T, and G is also contained in $\sigma_f T \sigma_f^{-1}$.

1.3. Now let us begin the proof of the theorem. Let G be a finite subgroup of Cr_k. Then G can be realized as a group of automorphisms of a rational surface S (see for instance [dF-E, Theorem 1.4]). Moreover we can assume that (G, S) is minimal, that is, every birational G-equivariant morphism of S onto a smooth surface with a G-action is an isomorphism. Then one of the following holds:

- G preserves a fibration $f : S \to \mathbb{P}^1$ with rational fibers;
- $\operatorname{rk}\operatorname{Pic}(S)^G = 1$.

This result goes back to Manin [M], at least in the case (of interest for us) when G is abelian. It is by now a direct consequence of Mori theory, see for instance [Z, Lemma 4.1].

In the former case G embeds in the group of automorphisms of the generic fibre $\mathbb{P}^1_{k(t)}$ of f; in the next section we are going to classify the p-elementary subgroups of $\operatorname{Aut}(\mathbb{P}^1_{k(t)})$. In the latter case S is a del Pezzo surface, and the group $\operatorname{Aut}(S)$ is well known; we will use this information in Section 3 to classify the corresponding p-elementary subgroups. Putting these results together gives the theorem.

2. Subgroups of Aut(P^1_K)

Let K be an extension of k, and p a prime number $\neq \operatorname{char}(k)$. Let us recall the classification of p-elementary subgroups of $\mathrm{PGL}_2(K)$. Let $C_p \subset \mathrm{PGL}_2(k)$ be the cyclic subgroup of homographies $z \mapsto \zeta z$, with $\zeta^p = 1$. Let $\delta : \mathrm{PGL}_2(K) \to K^*/K^{*2}$ be the homomorphism deduced from the determinant.

Lemma 2.1. *Let G be a subgroup of $\mathrm{PGL}_2(K)$ of the form $(\mathbb{Z}/p)^r$, with p prime $\neq \operatorname{char}(k)$.*

(a) *We have $r \leqslant 1$ if p is odd and $r \leqslant 2$ if $p = 2$.*
(b) *If p is odd and G non-trivial, it is conjugate to C_p.*
(c) *Assume $p = 2$, and that the Brauer group of K is trivial. Then $r \leqslant 2$; the homomorphism $\delta : \mathrm{PGL}_2(K) \to K^*/K^{*2}$ induces a bijective correspondence between conjugacy classes of subgroups of $\mathrm{PGL}_2(K)$ isomorphic to $(\mathbb{Z}/2)^2$, and subgroups of order $\leqslant 4$ of K^*/K^{*2}.*

Note that the assumption on K is satisfied when $K = k(t)$ by Tsen's theorem.

Proof. To prove (a) we embed $\mathrm{PGL}_2(K)$ into the group $\mathrm{PGL}_2(\overline{K})$, for which the result is well known.

Assume p is odd. Let σ be an element of G, represented by a matrix $A \in \mathrm{GL}_2(K)$ which satisfies $A^p = \lambda I$ for some scalar $\lambda \in K^*$. Taking determinants give $(\det A)^p = \lambda^2$, so that $\lambda^2 \equiv 1$

(mod.K^{*p}). Since p is odd, this implies $\lambda \in K^{*p}$; thus A is diagonalizable, and σ is conjugate to an element of C_p. This proves (b).

We now assume $p = 2$. Let σ, τ be two distinct commuting involutions in $\mathrm{PGL}_2(K)$; they are represented by matrices $A, B \in \mathrm{GL}_2(K)$ satisfying

$$A^2 = \alpha I, \qquad B^2 = \beta I, \qquad BA = \varepsilon AB \quad \text{for some } \alpha, \beta, \varepsilon \text{ in } K^*.$$

By the Cayley–Hamilton theorem we have $\det A = -\alpha$ and therefore $\delta(\sigma) = -\alpha$ (mod.K^{*2}), and similarly $\delta(\tau) = -\beta$ (mod.K^{*2}). Observe that replacing A by λA, for $\lambda \in K^*$, amounts to multiply α by λ^2.

Taking determinants gives $\varepsilon = \pm 1$; if B commutes with A, it belongs to the subspace of $M_2(K)$ spanned by I and A, and the condition $B^2 = \beta I$ implies that it is proportional to A or I, a contradiction. Thus

$$A^2 = \alpha I, \qquad B^2 = \beta I, \qquad BA = -AB \tag{2.2}$$

so that A and B define an algebra homomorphism $\varphi : Q_{\alpha,\beta} \to M_2(K)$, where $Q_{\alpha,\beta}$ is the quaternion algebra over K of type (α, β). For dimension reasons such a homomorphism is necessarily an isomorphism.

Let (σ', τ') be another pair of commuting involutions with $\delta(\sigma') = \delta(\sigma)$ and $\delta(\tau') = \delta(\tau)$. We can represent them by matrices (A', B') satisfying (2.2), thus corresponding to an isomorphism $\varphi' : Q_{\alpha,\beta} \to M_2(K)$. By the Skolem–Noether theorem φ and φ' are conjugate by an element of $\mathrm{GL}_2(K)$, thus the pairs (σ, τ) and (σ', τ') are conjugate in $\mathrm{PGL}_2(K)$. Moreover by our assumption on K any quaternion algebra over K is isomorphic to $M_2(K)$, thus any pair α, β in K^*/K^{*2} corresponds to a pair of commuting involutions. This proves (c). □

Remarks 2.3. (a) Let us make more explicit the description of the group G in case (c). Consider Eqs. (2.2). In a basis of K^2 of the form (v, Av), we have $A = \left(\begin{smallmatrix} 0 & \alpha \\ 1 & 0 \end{smallmatrix}\right)$, so that σ becomes the homography $z \mapsto \frac{\alpha}{z}$. Write $Bv = \lambda v + \mu A v$; then $BAv = -ABv = -\mu\alpha v - \lambda A v$, so that τ is the homography $z \mapsto \frac{\lambda z - \mu\alpha}{\mu z - \lambda}$. If $\mu \neq 0$ we may suppose $\mu = 1$, so

G is the subgroup $V_{\alpha,\lambda}$ generated by the involutions $z \mapsto \frac{\alpha}{z}$ and $z \mapsto \frac{\lambda z - \alpha}{z - \lambda}$,

where λ is any element of K such that $\alpha - \lambda^2 \equiv \beta$ (mod.K^{*2}).

The case $\mu = 0$ gives the subgroup generated by $z \mapsto \frac{\alpha}{z}$ and $z \mapsto -z$, that is $V_{\alpha,0}$. In particular $V_{1,0}$ is the standard subgroup of $\mathrm{PGL}_2(k)$ generated by $z \mapsto -z$ and $z \mapsto \frac{1}{z}$. We will denote it simply by V.

(b) Using the adjoint action of $\mathrm{PGL}_2(K)$ on the Lie algebra $\mathfrak{sl}_2(K)$ endowed with the Killing form, one can realize G as the 2-torsion subgroup of the diagonal torus of $\mathrm{PGL}_3(k)$, acting on the conic $X^2 - \alpha Y^2 - \beta Z^2$ in $\mathbb{P}^2_K$.

(c) Suppose $K = k(t)$. The homomorphism $\mathrm{div} : K^* \to \mathrm{Div}(\mathbb{A}^1)$ induces an isomorphism $\mathrm{div}_2 : K^*/K^{*2} \xrightarrow{\sim} \mathrm{Div}(\mathbb{A}^1) \otimes_{\mathbb{Z}} \mathbb{Z}/2$. Thus there is a bijective correspondence between classes in K^*/K^{*2} and finite subsets of k.

2.4. We are interested in the automorphism group $\mathrm{Aut}(\mathbb{P}^1_K)$ of the k-scheme $\mathbb{P}^1_K$. Let Γ be the automorphism group of K over k. The action of $\mathrm{Aut}(\mathbb{P}^1_K)$ on the global functions of $\mathbb{P}^1_K$ gives rise to an exact sequence

$$1 \to \mathrm{PGL}_2(K) \to \mathrm{Aut}\big(\mathbb{P}^1_K\big) \xrightarrow{\pi} \Gamma \to 1;$$

the surjection π has a canonical section $s : \Gamma \to \mathrm{Aut}(\mathbb{P}^1_K)$ which maps $\sigma \in \Gamma$ to the automorphism $z \mapsto \sigma(z)$. We will use this section to identify Γ to a subgroup of $\mathrm{Aut}(\mathbb{P}^1_K)$, and view $\mathrm{Aut}(\mathbb{P}^1_K)$ as the semi-direct product $\mathrm{PGL}_2(K) \rtimes \Gamma$. In other words, an element of $\mathrm{Aut}(\mathbb{P}^1_K)$ is a transformation $z \mapsto \frac{a\sigma(z)+b}{c\sigma(z)+d}$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(K)$ and $\sigma \in \Gamma$.

We are interested in the case $K = k(t)$, so that $\Gamma = \mathrm{PGL}_2(k)$. We first need a purely group-theoretical lemma:

Lemma 2.5. *Let A be a group, P and Γ two subgroups of A, with P normal, such that $A = P \rtimes \Gamma$. Let G be a subgroup of A, G_Γ its projection onto Γ, and $G_P := G \cap P$. Assume*:

(i) *The projection $G \to G_\Gamma$ admits a section s.*
(ii) *The cohomology set $H^1(G_\Gamma, P)$ is reduced to one element.*

Then G is conjugate to the semi-direct product $G_P \rtimes G_\Gamma$.

Proof. Replacing Γ by G_Γ we can assume that the projection $G \to \Gamma$ is surjective. Consider the exact sequence

$$1 \to P \to A \xrightarrow{p} \Gamma \to 1.$$

We have two sections of p: the one given by the inclusion $\Gamma \subset A$, and the composition $\Gamma \xrightarrow{s} G \subset A$. These two sections differ by a 1-cocycle of Γ with values in P, hence they are conjugate by (ii). This means that after replacing G by a conjugate subgroup we can assume $\Gamma \subset G$; this implies that the action of Γ on P preserves G_P, and that $G = G_P \rtimes \Gamma$. $\square$

Proposition 2.6. *Let G be a subgroup of* $\mathrm{Aut}(\mathbb{P}^1_{k(t)})$ *of the form $(\mathbb{Z}/p)^r$, with p prime* $\neq \mathrm{char}(k)$.

(a) *When p is odd, we have $r \leqslant 2$; if $r = 2$, G is conjugate to the subgroup $C_p \times C_p$ of* $\mathrm{PGL}_2(K) \rtimes \mathrm{PGL}_2(k)$.
(b) *When $p = 2$ we have $r \leqslant 4$; if equality holds, G is conjugate to the subgroup $V_{\alpha,\lambda} \times V$ of* $\mathrm{PGL}_2(K) \rtimes \mathrm{PGL}_2(k)$ (*see* 2.3(a)) *for some elements α, λ in the V-invariant subfield L of K. The conjugacy class of this subgroup depends only on the classes of α and $\beta = \alpha - \lambda^2$ in L^*/L^{*2}.*

Proof. Let $G' = G \cap \mathrm{PGL}_2(K)$. The bound on r follows from the exact sequence

$$1 \to G' \to G \xrightarrow{\pi} \pi(G) \to 1$$

and Lemma 2.1(a). If equality holds, the above exact sequence splits, so that condition (i) of Lemma 2.5 is satisfied. To check condition (ii), consider the cohomology exact sequence associated to the exact sequence

$$1 \to K^* \to \mathrm{GL}_2(K) \to \mathrm{PGL}_2(K) \to 1;$$

for any finite subgroup H of Γ we have $H^1(H, \mathrm{GL}_2(K)) = \{1\}$ [S, chapitre X, Proposition 3] and $H^2(H, K^*) = \{1\}$ by Tsen's theorem and [S, chapitre X, Proposition 11] and therefore $H^1(H, \mathrm{PGL}_2(K)) = \{1\}$. Thus condition (ii) holds, and Lemma 2.5 implies that G is conjugate to the subgroup $G' \times \pi(G)$ of $\mathrm{PGL}_2(K) \rtimes \mathrm{PGL}_2(k)$. If p is odd, this subgroup is conjugate to $C_p \times C_p$ by Lemma 2.1. If $p = 2$, we can assume $\pi(G) = V$ by Lemma 2.1; then the condition that G' commutes with V implies that α and λ are in the V-invariant subfield L of K, so that G is contained in $\mathrm{PGL}_2(L) \times \mathrm{PGL}_2(k)$. Lemma 2.1 shows that G' is conjugate in $\mathrm{PGL}_2(L)$ to $V_{\alpha,\lambda}$ for some α, λ in L, and that the conjugacy class of G' in $\mathrm{PGL}_2(L)$ depends only of the classes of α and β in L^*/L^{*2}. Since $\mathrm{PGL}_2(L)$ commutes with V, the proposition follows. □

2.7. Thus for p odd and $r = 2$, G is conjugate to the p-torsion subgroup of the diagonal torus of $\mathrm{PGL}_3(k)$. For $p = 2$ and $r = 4$, G is conjugate to the group $V_{\alpha,\lambda} \times V$ generated by

$$(z,t) \mapsto (z,-t), \qquad (z,t) \mapsto \left(z, \frac{1}{t}\right), \qquad (z,t) \mapsto \left(\frac{\alpha}{z}, t\right), \qquad (z,t) \mapsto \left(\frac{\lambda z - \alpha}{z - \lambda}, t\right)$$

with α and $\beta = \alpha - \lambda^2$ in $k(t)^*$. Since the map $t \mapsto t^2 + t^{-2}$ identifies $\mathbb{P}^1/V$ with $\mathbb{P}^1$, the invariance of α and λ under V means that they are rational functions of $t^2 + t^{-2}$. This gives case (c_2) of the theorem. Using Remark 2.3(b) leads to the alternative form given in the theorem.

3. Automorphisms of del Pezzo surfaces

We now consider the case where S is a del Pezzo surface and $G \cong (\mathbb{Z}/p)^r$ a subgroup of $\mathrm{Aut}(S)$ such that $\mathrm{rk}\,\mathrm{Pic}(S)^G = 1$. We first recall the following well-known fact, which is a particular case of the results mentioned in 1.1:

Lemma 3.1. *Let G be a subgroup of $\mathrm{PGL}_n(k)$ of the form $(\mathbb{Z}/p)^r$, with p prime $\neq \mathrm{char}(k)$. Assume that p does not divide n. Then $r \leqslant n - 1$, and if equality holds G is conjugate to the p-torsion subgroup of the diagonal torus.*

Proof. Pulling back G to $\mathrm{SL}_n(k)$ gives a central extension of G by the group $\boldsymbol{\mu}_n$ of nth roots of unity in k. Such extensions are parametrized by the group $H^2(G, \boldsymbol{\mu}_n)$ which is annihilated both by n and p. Thus our extension is trivial, and G lifts to a subgroup of $\mathrm{SL}_n(k)$, isomorphic to $(\mathbb{Z}/p)^r$. Such a subgroup is contained in a maximal torus of $\mathrm{SL}_n(k)$, hence our assertions. □

3.2. Let us start with the case $S = \mathbb{P}^2$. By the above lemma, if $p \neq 3$, we have $r \leqslant 2$, and any subgroup of $\mathrm{PGL}_3(k)$ isomorphic to $(\mathbb{Z}/p)^2$ is conjugate to the p-torsion subgroup of the diagonal torus.

The case $p = 3$ is classical (see e.g. [Bo, 6.4], for a more general statement): we have again $r \leqslant 2$, and a subgroup isomorphic to $(\mathbb{Z}/3)^2$ is conjugate either to the diagonal subgroup, or to the subgroup spanned by the automorphisms $(X_0, X_1, X_2) \mapsto (X_1, X_2, X_0)$ and $(X_0, X_1, X_2) \mapsto (X_0, \alpha X_1, \alpha^2 X_2)$ for $\alpha \in \boldsymbol{\mu}_3$.

3.3. If S is obtained from $\mathbb{P}^2$ by blowing up one or two points, the group $\mathrm{Aut}(S)$ is a subgroup of $\mathrm{PGL}_3(k)$, so 3.2 applies. Suppose that S is the blow up of $\mathbb{P}^2$ at three non-collinear points. Let $N \subset \mathrm{PGL}_3(k)$ be the subgroup of automorphisms preserving this three-points subset. The group $\mathrm{Aut}(S)$ is the semi-direct product of N and a subgroup of order 2. Let G be a subgroup of $\mathrm{Aut}(S)$ isomorphic to $(\mathbb{Z}/p)^r$. If $p \neq 2$, G is contained in N, so that 3.2 applies. If $p = 2$, we have $r \leqslant 3$ again by 3.2.

We now consider the case $S = \mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 3.4. *Let G be a group of automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$, isomorphic to $(\mathbb{Z}/p)^r$, such that $\operatorname{rk}\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)^G = 1$. Then $p = 2$ and $r \leqslant 3$.*

Proof. The automorphism group of $\mathbb{P}^1 \times \mathbb{P}^1$ is the semi-direct product

$$\big(\mathrm{PGL}_2(k) \times \mathrm{PGL}_2(k)\big) \rtimes \mathbb{Z}/2,$$

where $\mathbb{Z}/2$ acts on $\mathrm{PGL}_2(k) \times \mathrm{PGL}_2(k)$ by exchanging the factors. If $p \neq 2$, our subgroup G is contained in $\mathrm{PGL}_2(k) \times \mathrm{PGL}_2(k)$, hence $\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)^G = \operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ has rank 2.

Thus we have $p = 2$. The subgroup $G'$of G preserving the two $\mathbb{P}^1$-fibrations is contained in $\mathrm{PGL}_2(k) \times \mathrm{PGL}_2(k)$, thus in $C_2 \times C_2$ up to conjugacy, and G is conjugate to a subgroup of the semi-direct product $(C_2 \times C_2) \rtimes \mathbb{Z}/2$. But the elements of order 2 in this group not contained in $C_2 \times C_2$ are contained in the (direct) product of $\mathbb{Z}/2$ by the diagonal subgroup $C_2 \subset C_2 \times C_2$. Therefore G must be contained in $C_2 \times \mathbb{Z}/2$, hence $r \leqslant 3$. □

3.5. It remains to consider the case when S is obtained from $\mathbb{P}^2$ by blowing up ℓ points in general position, with $4 \leqslant \ell \leqslant 8$. We start by recalling some classical facts about such surfaces, which can be found for instance in [D2]. The primitive cohomology $H^2(S,\mathbb{Z})_{\mathrm{prim}}$ (the orthogonal of the canonical class in $H^2(S,\mathbb{Z})$) is the root lattice of a root system R. The group $\mathrm{Aut}(S)$ acts faithfully on $H^2(S,\mathbb{Z})_{\mathrm{prim}}$, hence can be identified with a subgroup of the automorphism group of the root system R; it is actually contained in the Weyl group $W \subset \mathrm{Aut}(R)$ [Do]. The root systems which appear are the following (see [B]):

- $\ell = 4$: $R = A_4$, $W = \mathfrak{S}_5$;
- $\ell = 5$: $R = D_5$, $W = (\mathbb{Z}/2)^4 \rtimes \mathfrak{S}_5$;
- $\ell = 6$: $R = E_6$, $|W| = 2^7.3^4.5$;
- $\ell = 7$: $R = E_7$, $|W| = 2^{10}.3^4.5.7$;
- $\ell = 8$: $R = E_8$, $|W| = 2^{14}.3^5.5^2.7$.

3.6. For $\ell = 7$ (respectively $\ell = 8$), the linear system $|-K_S|$ (respectively $|-2K_S|$) defines a degree 2 morphism onto $\mathbb{P}^2$ (respectively a quadric cone in $\mathbb{P}^3$), branched along a canonically embedded smooth curve C of genus 3 (respectively 4). This gives rise to an exact sequence

$$1 \to \mathbb{Z}/2 \to \mathrm{Aut}(S) \to \mathrm{Aut}(C) \tag{3.7}$$

(the right-hand side map is actually surjective, but we will not need this). To control subgroups of $\mathrm{Aut}(C)$ the following lemma will be useful:

Lemma 3.8. *Let p be a prime number, r an integer, and C a curve of genus g with a faithful action of the group $(\mathbb{Z}/p)^r$. Then p^{r-1} divides $2g - 2$.*

Proof. Put $G = (\mathbb{Z}/p)^r$, and consider the covering $\pi : C \to C/G$. Since the stabilizer of any point of C is cyclic, the fibre of π above a branch point consists of p^{r-1} points with ramification index p. Thus Hurwitz's formula gives the result. $\square$

Proposition 3.9. *For p prime $\geqslant 5$, the group $(\mathbb{Z}/p)^2$ does not act faithfully on a del Pezzo surface.*

Proof. A glance at the list 3.5 shows that the only case where the order of the Weyl group is divisible by p^2, with $p \geqslant 5$, is $\ell = 8$. In this case the exact sequence (3.7) shows that $(\mathbb{Z}/p)^2$ acts faithfully on a smooth curve C of genus 4, and this contradicts Lemma 3.8. $\square$

Proposition 3.10. *Let S be a del Pezzo surface admitting a faithful action of $(\mathbb{Z}/3)^r$ with $r \geqslant 3$. Then $r = 3$, S is isomorphic to the Fermat cubic $X_0^3 + \cdots + X_3^3 = 0$, and $(\mathbb{Z}/3)^3$ acts as the 3-torsion subgroup of the diagonal torus in $\mathrm{PGL}_4(k)$.*

Proof. The list 3.5 gives $\ell \geqslant 6$. On the other hand, 3.6 and Lemma 3.8 give $\ell \leqslant 6$. Thus S is a cubic surface in $\mathbb{P}^3$, and G is a subgroup of $\mathrm{PGL}_4(k)$. By Lemma 3.1 we have $r = 3$, and there is a coordinate system $(X_0, \ldots, X_3)$ on $\mathbb{P}^3$ such that G is the diagonal subgroup $(\boldsymbol{\mu}_3)^4/\boldsymbol{\mu}_3$ of $\mathrm{PGL}_4(k)$.

The surface S is defined by an element F of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$ which is semi-invariant with respect to the action of $(\boldsymbol{\mu}_3)^4$. Under this action the space $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$ is the direct sum of the invariant subspace spanned by $X_0^3, \ldots, X_3^3$ and of 16 1-dimensional subspaces spanned by monomials, corresponding to 16 different characters of $(\boldsymbol{\mu}_3)^4$. Since S is smooth, F must be of the form $a_0X_0^3 + \cdots + a_3X_3^3$ with $a_i \neq 0$ for each i, hence the result. $\square$

Proposition 3.11. *Let S be a del Pezzo surface and G a subgroup of $\mathrm{Aut}(S)$ isomorphic to $(\mathbb{Z}/2)^r$ with $r \geqslant 4$ and $\mathrm{rk}\,\mathrm{Pic}(S)^G = 1$. Then $r = 4$, S is a quartic del Pezzo surface in $\mathbb{P}^4$ with equations $\sum_{i=0}^4 X_i^2 = \sum_{i=0}^4 \lambda_i X_i^2 = 0$ for some distinct elements $\lambda_0, \ldots, \lambda_4$ of k, and G is the 2-torsion subgroup of the diagonal torus in $\mathrm{PGL}_5(k)$.*

Proof. Once again the list 3.5 gives $\ell \geqslant 5$, and 3.6 and Lemma 3.8 give $\ell \neq 8$. If $\ell = 7$ we get from 3.6 a subgroup $(\mathbb{Z}/2)^{r-1}$ in $\mathrm{Aut}(C) \subset \mathrm{PGL}_3(k)$, hence $r \leqslant 3$ by Lemma 3.1.

Suppose S is a cubic surface. Then G acts on the set of lines in S; since 27 is odd, there must be one orbit with one element, that is, one line stable under G. This contradicts the assumption on $\mathrm{Pic}(S)^G$.

Suppose S is an intersection of two quadrics in $\mathbb{P}^4$. Then G is a subgroup of $\mathrm{PGL}_5(k)$; by Lemma 3.1 we have $r = 4$, and there is a coordinate system $(X_0, \ldots, X_4)$ on $\mathbb{P}^4$ such that G is the diagonal subgroup $(\boldsymbol{\mu}_2)^5/\boldsymbol{\mu}_2$ of $\mathrm{PGL}_5(k)$.

The representation of $(\boldsymbol{\mu}_2)^5$ on $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$ splits as

$$H^0\big(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)\big) = I \oplus \sum_{i<j} k \cdot (X_iX_j),$$

where I is the invariant subspace spanned by $X_0^2, \ldots, X_4^2$. The 2-dimensional subspace of quadratic forms vanishing on S must be contained in I, since otherwise it would contain some X_iX_j. After a change of coordinates we find the form given in the proposition. $\square$

Remark 3.12. The same method gives the subgroups of type $(\mathbb{Z}/3)^2$ of the Cremona group: besides the two conjugacy classes of subgroups of $\mathrm{PGL}_3(k)$ described in 3.2, one gets the automorphisms groups of certain cubic surfaces and del Pezzo surfaces of degree 1. The subgroups of type $(\mathbb{Z}/2)^3$ and $(\mathbb{Z}/2)^2$ can also be described with analogous, but more tedious, methods.

4. Conjugacy classes of 2-elementary subgroups

It follows from the theorem that the p-elementary subgroups of Cr_k of maximal order form only one conjugacy class, except in the case $p = 2$. We are going to analyze the latter case. We assume $\mathrm{char}(k) \neq 2$ throughout this section.

4.1. To study subgroups of type (c_1), let us first recall some classical facts about quartic del Pezzo surfaces—one possible reference is [H, lecture 22].

A quartic del Pezzo surface $S \subset \mathbb{P}^4$ is contained in a pencil $(Q_\lambda)_{\lambda \in \mathbb{P}^1}$ of quadrics. There are exactly 5 singular quadrics $Q_{\lambda_0}, \dots, Q_{\lambda_4}$ in this pencil; the map $S \mapsto \{\lambda_0, \dots, \lambda_4\}$ is an isomorphism from the moduli space of quartic del Pezzo surfaces onto the moduli space of 5-points subsets of $\mathbb{P}^1$ (modulo the action of PGL_2). The quadrics $Q_{\lambda_0}, \dots, Q_{\lambda_4}$ have rank 4; their singular points $p_0, \dots, p_4$ span $\mathbb{P}^4$.

The group $\mathrm{Aut}(S)$ contains a normal, canonical subgroup G_S isomorphic to $(\mathbb{Z}/2)^4$. Indeed for $0 \leqslant \ell \leqslant 4$, there is a unique involution σ_ℓ of $\mathbb{P}^4$ whose fixed locus consists of p_ℓ and the hyperplane H_ℓ spanned by the points p_i for $i \neq \ell$; these involutions span the group G_S. In more concrete terms, choose the coordinates on $\mathbb{P}^1$ so that $\lambda_0, \dots, \lambda_4 \in k$. There exists a system of coordinates on $\mathbb{P}^4$ such that the equations of Q_∞ and Q_0 are respectively

$$\sum_{i=0}^{4} X_i^2 = 0, \qquad \sum_{i=0}^{4} \lambda_i X_i^2 = 0.$$

Then G_S is the 2-torsion subgroup of the diagonal torus in $\mathrm{PGL}_5(k)$; the involution σ_ℓ maps $(X_0, \dots, X_\ell, \dots, X_4)$ to $(X_0, \dots, -X_\ell, \dots, X_4)$. As before we view G_S as a subgroup of Cr_k, well defined up to conjugacy.

Proposition 4.2. *The map $S \mapsto G_S$ induces a bijection between the moduli space of quartic del Pezzo surfaces and the set of conjugacy classes of subgroups of* Cr_k *of type* (c_1).

As pointed out by the referee, the proposition can be deduced from the more general results of Iskovskikh [I]. For the convenience of the reader we will give the proof in our particular case, because it is much simpler than in the general framework considered in [I].

Let S and S' be two quartic del Pezzo surfaces. If G_S is conjugate to $G_{S'}$, there exists a birational map $\varphi : S \dashrightarrow S'$ which is equivariant with respect to the action of $(\mathbb{Z}/2)^4$. Thus the proposition is a consequence of the following result, which is a particular case of Theorem 3.3 in [I]:

Proposition 4.3. *Any $(\mathbb{Z}/2)^4$-equivariant birational map $\varphi : S \dashrightarrow S'$ between quartic del Pezzo surfaces is an isomorphism.*

Proof. Put $G := (\mathbb{Z}/2)^4$. We have a G-equivariant diagram

$$\begin{array}{ccc} & \hat{S} & \\ {}^{\varepsilon}\swarrow & & \searrow^{\eta} \\ S & \overset{}{\underset{\varphi}{\dashrightarrow}} & S' \end{array}$$

where ε and η are birational morphisms. As usual we write

$$\operatorname{Pic}(\hat{S}) = \varepsilon^* \operatorname{Pic}(S) \oplus \sum_{i \in I} \mathbb{Z}[E_i] \quad \text{with } E_i^2 = -1,\ E_i.E_j = 0 \text{ for } i \neq j;$$

each E_i is an effective (possibly reducible) divisor contracted by ε. Let us put $H = -K_S$ and $H' = -K_{S'}$. We have $\varepsilon_* \eta^* H' \in \operatorname{Pic}(S)^G = \mathbb{Z}H$, hence[2]

$$\eta^* H' \equiv m\varepsilon^* H - \sum_{i \in I} r_i E_i$$

for some positive integers m, $(r_i)_{i \in I}$ satisfying

$$4 = 4m^2 - \sum_{i \in I} r_i^2. \tag{4.4}$$

Assume that φ is not an isomorphism; then $I \neq \emptyset$, so $m \geqslant 2$ by (4.4). Since $K_{\hat{S}} \equiv \varepsilon^* K_S + \sum_{i \in I} E_i$, we have

$$\eta^* H' + mK_{\hat{S}} \equiv \sum_{i \in I} (m - r_i) E_i.$$

On the other hand, we have $\eta_*(\eta^* H' + mK_{\hat{S}}) \equiv H' + mK_{S'}$, and this linear system is empty since $m \geqslant 2$. This implies $r_i > m$ for some $i \in I$.

The group G acts on the finite set I, and the function $i \mapsto r_i$ is G-invariant. Now the key point is that every orbit of G in I has at least 4 elements: indeed the stabilizer of $i \in I$ fixes the point $\varepsilon(E_i)$ of S, and the stabilizer of a point p of S has order $\leqslant 4$ (this stabilizer is $(\mathbb{Z}/2)^\nu$, where ν is the number of coordinates of p which are zero). Thus there are at least 4 elements i of I with $r_i > m$, and this contradicts (4.4). Therefore we have $m = 1$, $I = \emptyset$, and φ is an isomorphism. □

Remark 4.5. Proposition 4.3 implies that the normalizer of G_S in Cr_k is the finite group $\operatorname{Aut}(S)$. The same argument shows that the normalizer of $(\mathbb{Z}/3)^3$ in Cr_k is the group of automorphisms of the Fermat cubic surface, that is, $(\mathbb{Z}/3)^3 \rtimes \mathfrak{S}_4$.

4.6. For $\sigma \in \mathrm{Cr}_k$, we denote by $NF(\sigma)$ the *normalized fixed locus* of σ, that is, the normalization of the union of the non-rational curves in $\mathbb{P}^2$ fixed by σ. The isomorphism class of $NF(\sigma)$ is an

[2] The sign $\equiv$ denotes linear equivalence.

invariant of the conjugacy class of σ, and this is the basic invariant we will use to distinguish conjugacy classes.

Let S be a quartic del Pezzo surface. The 5 elements σ_ℓ of G_S (4.1) fix the elliptic curve $H_\ell \cap S$, while the other elements of G_S have no fixed curve. On the other hand, for a subgroup $G \subset \mathrm{Cr}_k$ of type (c$_2$), at most 3 elements of G have a non-trivial normalized fixed locus—namely, with the notation of 2.6, the non-trivial elements of $V_{\alpha,\lambda} \times \{1\}$. Therefore:

Proposition 4.7. *A subgroup of type* (c$_2$) *is not conjugate to one of type* (c$_1$).

We now consider subgroups of type (c$_2$). Such a subgroup is the image of an embedding $(\mathbb{Z}/2)^4 \hookrightarrow \mathrm{Cr}_k$, which maps the elements $e_1, \ldots, e_4$ of the canonical basis to the involutions

$$(x, y) \mapsto (-x, y), \qquad (x, y) \mapsto \left(\frac{1}{x}, y\right), \qquad (x, y) \mapsto \left(x, \frac{\alpha(u)}{y}\right),$$
$$(x, y) \mapsto \left(x, \frac{\lambda(u)y - \alpha(u)}{y - \lambda(u)}\right)$$

with α, λ in $k(u)$, $\alpha \neq 0, \lambda^2$, $u = x^2 + x^{-2}$. The conjugacy class of this embedding depends only on the classes of α and $\beta = \alpha - \lambda^2$ in $k(u)^*/k(u)^{*2}$. We will represent these classes by polynomials in u with simple roots.

Proposition 4.8. *Assume that α or β has a zero outside* $\{-2, 2\}$. *Then if the embeddings associated to the pairs* (α, β) *and* (α', β') *are conjugate, we have $\alpha = \alpha'$ and $\beta = \beta'$ in* $k(u)^*/k(u)^{*2}$.

Proof. To prove the proposition, we must reconstruct from the embedding $\varphi : (\mathbb{Z}/2)^4 \hookrightarrow \mathrm{Cr}_k$ the classes α and β in $k(u)/k(u)^{*2}$.

Consider the normalized fixed locus $C = NF(\varphi(e_3))$. It is the normalization of the hyperelliptic curve $y^2 = \alpha(x^2 + x^{-2})$, provided this curve has genus $\geqslant 1$. The subgroup V of $(\mathbb{Z}/2)^4$ spanned by e_1 and e_2 acts on C and commutes with the hyperelliptic involution σ, so it acts on the rational curve $R = C/\langle\sigma\rangle$. There is a coordinate x on R such that e_1 acts by $x \mapsto -x$ and e_2 by $x \mapsto x^{-1}$; it is unique up to the action of V (because V is its own centralizer in $\mathrm{PGL}_2(k)$). In particular the map $\pi : R \to \mathbb{P}^1$ such that $\pi(x) = x^2 + x^{-2}$ is well defined; it induces an isomorphism $R/V \xrightarrow{\sim} \mathbb{P}^1$.

From the hyperelliptic curve C we get the fixed locus of σ, which corresponds to the class of $\pi^*\alpha$ in $k(x)^*/k(x)^{*2}$ by the isomorphism div_2 (Remark 2.3(c)). This is not enough, however, because the homomorphism $\pi^* : k(u)^*/k(u)^{*2} \to k(x)^*/k(x)^{*2}$ is not injective: it has a kernel of order 4, spanned by $u - 2$ and $u + 2$. In other words, if P is a non-constant polynomial in u such that $P(\pm 2) \neq 0$, the polynomials

$$P(u), \qquad (u-2)P(u), \qquad (u+2)P(u), \qquad (u-2)(u+2)P(u)$$

give rise to the same hyperelliptic curve C with equation $z^2 = P(x^2 + x^{-2})$.

To distinguish these cases we consider the action of V on C. If $\alpha = P$, e_1 acts by $(x, z) \mapsto (-x, z)$; this involution has 4 fixed points, above the points $x = 0$ and $x = \infty$ of R (observe that the fixed locus of σ does not intersect the ramification locus $\{0, \infty, \pm 1, \pm i\}$ of π). On the other hand, if $\alpha(u) = (u-2)P(u)$, we have $z = y(x - x^{-1})^{-1}$, so that e_1 gives the involution $(x, z) \mapsto (-x, -z)$, which is fixed-point free. Similar computations show that, if $\alpha(u) = (u-2)^{\varepsilon}(u+2)^{\eta}P(u)$, with $\varepsilon, \eta \in \{0, 1\}$, e_1 (respectively e_2) gives a fixed-point free involution if and only if $\varepsilon + \eta = 1$ (respectively $\varepsilon = 1$). Therefore if $\pi^*\alpha \neq 1$ in $k(x)^*/k(x)^{*2}$, the 4 elements $\alpha_i \in k(u)^*/k(u)^{*2}$ such that $\pi^*\alpha_i = \pi^*\alpha$ are distinguished by the action of V on C.

This works provided $NF(\varphi(e_3))$ is non-empty, that is, provided α vanishes outside $\{-2, 2\}$; similarly, if β vanishes outside $\{-2, 2\}$, we recover β from the curve $NF(\varphi(e_4))$ with the action of V. If, say, α has a zero outside $\{-2, 2\}$ but β does not, then the same argument determines the classes of α and $\alpha\beta$ in $k(u)^*/k(u)^{*2}$, hence also that of β. The proposition follows. □

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