Open Problems and Surveys of Contemporary Mathematics
SMM 6, pp. 99–131

Theta Functions, Old and New

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Introduction

Theta functions are holomorphic functions on $\mathbb{C}^g$, quasi-periodic with respect to a lattice. For $g = 1$ they have been introduced by Jacobi; in the general case they have been thoroughly studied by Riemann and his followers. From a modern point of view they are sections of line bundles on certain complex tori; in particular, the theta functions associated to an algebraic curve C are viewed as sections of a natural line bundle (and of its tensor powers) on a complex torus associated to C, the *Jacobian*, which parametrizes topologically trivial line bundles on C.

Around 1980, under the impulsion of mathematical physics, the idea emerged gradually that one could replace in this definition line bundles by higher rank vector bundles. The resulting sections are called *generalized* (or *non-abelian*) *theta functions*; they turn out to share some (but not all) of the beautiful properties of classical theta functions.

The goal of these lectures is to develop first the modern theory of classical theta functions (complex tori, line bundles, Jacobians), then to explain how it can be generalized by considering higher rank vector bundles. We have tried to make them accessible for students with a minimal background in complex geometry: Chapter 0 of [13] should be more than enough. At a few places, especially in the last chapters, we had to use some more advanced results. Also we have not tried to be exhaustive: sometimes we just give a sketch of proof, or we prove a particular case, or we just admit the result.

These notes come from a series of lectures given at Tsinghua University in April 2011. I am grateful to Tsinghua University and Professor S.-T. Yau for the generous invitation.

1 The cohomology of a torus

1.1 Real tori

Let V be a real vector space, of dimension n. A *lattice* in V is a $\mathbb{Z}$-module $\Gamma \subset V$ such that the induced map $\Gamma \otimes_{\mathbb{Z}} \mathbb{R} \to V$ is an isomorphism; equivalently, any basis of Γ over $\mathbb{Z}$ is a basis of V. In particular $\Gamma \cong \mathbb{Z}^n$.

The quotient $T := V/\Gamma$ is a smooth, compact Lie group, isomorphic to $(\mathbb{S}^1)^n$. The quotient homomorphism $\pi : V \to V/\Gamma$ is the *universal covering* of T. Thus Γ is identified with the fundamental group $\pi_1(T)$.

We want to consider the cohomology algebra $H^*(T, \mathbb{C})$. We think of it as being de Rham cohomology: recall that a smooth p-form ω on T is *closed* if $d\omega = 0$, *exact* if $\omega = d\eta$ for some $(p-1)$-form η. Then

$$H^p(T, \mathbb{C}) = \frac{\{\text{closed } p\text{-forms}\}}{\{\text{exact } p\text{-forms}\}} \cdot$$

Let ℓ be a linear form on V. The 1-form $d\ell$ on V is invariant by translation, hence is the pullback by π of a 1-form on T that we will still denote $d\ell$. Let $(x_1, \ldots, x_n)$ be a system of coordinates on V. The forms $(dx_1, \ldots, dx_n)$ form a basis of the cotangent space $T^*_a(T)$ at each point $a \in T$; thus a p-form ω on T can be written in a unique way

$$\omega = \sum_{i_1 < \cdots < i_p} \omega_{i_1 \cdots i_p}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

where the $\omega_{i_1 \cdots i_p}$ are smooth functions on T (with complex values).

An important role in what follows will be played by the translations $t_a : x \mapsto x + a$ of T. We say that a p-form ω is *constant* if it is invariant by translation, that is, $t_a^* \omega = \omega$ for all $a \in T$; in terms of the above expression for ω, it means that the functions $\omega_{i_1 \ldots i_p}$ are constant. Such a form is determined by its value at 0, which is a skew-symmetric p-linear form on $V = T_0(T)$. We will denote by $\mathrm{Alt}^p(V, \mathbb{C})$ the space of such forms, and identify it to the space of constant p-forms. A constant form is closed, hence we have a linear map $\delta^p : \mathrm{Alt}^p(V, \mathbb{C}) \to H^p(T, \mathbb{C})$. Note that $\mathrm{Alt}^1(V, \mathbb{C})$ is simply $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$, and δ^1 maps a linear form ℓ to the class of $d\ell$.

Proposition 1.1. *The map $\delta^p : \mathrm{Alt}^p(V, \mathbb{C}) \to H^p(T, \mathbb{C})$ is an isomorphism.*

Proof. There are various elementary proofs of this, see for instance [8], III.4. To save time we will use the Künneth formula. We choose our coordinates $(x_1, \ldots, x_n)$ so that $V = \mathbb{R}^n$, $\Gamma = \mathbb{Z}^n$. Then $T = T_1 \times \cdots \times T_n$, with $T_i \cong \mathbb{S}^1$ for each i, and dx_i is a 1-form on T_i, which generates $H^1(T_i, \mathbb{C})$. The Künneth formula gives an isomorphism of graded algebras $H^*(T, \mathbb{C}) \xrightarrow{\sim} \bigotimes_i H^*(T_i, \mathbb{C})$. This means that $H^*(T, \mathbb{C})$ is the exterior algebra on the vector space with basis $(dx_1, \ldots, dx_n)$, and this is equivalent to the assertion of the Proposition. $\square$

What about $H^*(T, \mathbb{Z})$? The Künneth isomorphism shows that it is torsion free, so it can be considered as a subgroup of $H^*(T, \mathbb{C})$. By definition of the de Rham isomorphism the image of $H^p(T,\mathbb{Z})$ in $H^p(T,\mathbb{C})$ is spanned by the closed p-forms ω such that $\int_\sigma \omega \in \mathbb{Z}$ for each p-cycle σ in $H_p(T,\mathbb{Z})$. Write again $T = \mathbb{R}^n/\mathbb{Z}^n$; the closed paths $\gamma_i : t \mapsto te_i$, for $t \in [0,1]$, form a basis of $H_1(T,\mathbb{Z})$, and we have $\int_{\gamma_i} d\ell = \ell(e_i)$. Thus $H^1(T,\mathbb{Z})$ is identified with the subgroup of $H^1(T,\mathbb{C}) = \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C})$ consisting of linear forms $V \to \mathbb{C}$ which take integral values on Γ; it is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z})$. Applying again the Künneth formula gives:

Proposition 1.2. *For each p, the image of $H^p(T,\mathbb{Z})$ in $H^p(T,\mathbb{C}) \cong \mathrm{Alt}^p(V,\mathbb{C})$ is the subgroup of forms which take integral values on Γ; it is isomorphic to* $\mathrm{Alt}^p(\Gamma,\mathbb{Z})$. □

1.2 Complex tori

From now on we assume that V has a complex structure, that is, V is a complex vector space, of dimension g. Thus $V \cong \mathbb{C}^g$ and $\Gamma \cong \mathbb{Z}^{2g}$. Then $T := V/\Gamma$ is a complex manifold, of dimension g, in fact a complex Lie group; the covering map $\pi : V \to V/\Gamma$ is holomorphic. We say that T is a *complex torus*. **Beware:** while all real tori of dimension n are diffeomorphic to $(\mathbb{S}^1)^n$, there are many non-isomorphic complex tori of dimension g – more about that in Sect. 3.3 below.

The complex structure of V provides a natural decomposition

$$\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) = V^* \oplus \overline{V}^* ,$$

where $V^* := \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ and $\overline{V}^* = \mathrm{Hom}_{\mathbb{C}}(\overline{V},\mathbb{C})$ are the subspaces of $\mathbb{C}$-linear and $\mathbb{C}$-antilinear forms respectively. We write the corresponding decomposition of $H^1(T,\mathbb{C})$

$$H^1(T,\mathbb{C}) = H^{1,0}(T) \oplus H^{0,1}(T) .$$

If $(z_1,\ldots,z_g)$ is a coordinate system on V, $H^{1,0}(T)$ is the subspace spanned by the classes of $dz_1,\ldots,dz_g$, while $H^{1,0}(T)$ is spanned by the classes of $d\bar{z}_1,\ldots,d\bar{z}_g$.

The decomposition $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) = V^* \oplus \overline{V}^*$ gives rise to a decomposition

$$\mathrm{Alt}^p(V,\mathbb{C}) \cong \wedge^p V^* \oplus (\wedge^{p-1} V^* \otimes \overline{V}^*) \oplus \cdots \oplus \wedge^p \overline{V}^*$$

which we write

$$H^p(T,\mathbb{C}) = H^{p,0}(T) \oplus \cdots \oplus H^{0,p}(T) .$$

The forms in $\mathrm{Alt}^p(V,\mathbb{C})$ which belong to $H^{p,0}(T)$ (resp. $H^{0,p}(T)$) are those which are $\mathbb{C}$-linear (resp. $\mathbb{C}$-antilinear) in each variable. It is not immediate to characterize those which belong to $H^{q,r}(T)$ for $q,r>0$; for $p=2$ we have:

Proposition 1.3. *Via the identification $H^2(T,\mathbb{C}) = \mathrm{Alt}^2(V,\mathbb{C})$, $H^{2,0}$ is the space of $\mathbb{C}$-bilinear forms, $H^{0,2}$ the space of $\mathbb{C}$-biantilinear forms, and $H^{1,1}$ is the space of $\mathbb{R}$-bilinear forms E such that $E(ix,iy) = E(x,y)$.*

Proof. We have only to prove the last assertion. For $\varepsilon \in \{\pm 1\}$, let $\mathcal{E}_\varepsilon$ be the space of forms $E \in \mathrm{Alt}^2(V,\mathbb{C})$ satisfying $E(ix,iy) = \varepsilon E(x,y)$. We have $\mathrm{Alt}^2(V,\mathbb{C}) = \mathcal{E}_1 \oplus \mathcal{E}_{-1}$, and $H^{2,0}$ and $H^{0,2}$ are contained in $\mathcal{E}_{-1}$.

For $\ell \in V^*$, $\ell' \in \overline{V}^*$, $v, w \in V$, we have

$$(\ell \wedge \ell')(iv, iw) = \ell(iv)\ell'(iw) - \ell(iw)\ell'(iv) = (\ell \wedge \ell')(v, w) \ ,$$

hence $H^{1,1}$ is contained in $\mathcal{E}_1$; it follows that $H^{2,0} \oplus H^{0,2} = \mathcal{E}_{-1}$ and $H^{1,1} = \mathcal{E}_1$. □

2 Line bundles on complex tori

2.1 The Picard group of a manifold

Our next goal is to describe all holomorphic line bundles on our complex torus T. Recall that line bundles on a complex manifold M form a group, the *Picard group* $\mathrm{Pic}(M)$ (the group structure is given by the tensor product of line bundles). It is canonically isomorphic to the first cohomology group $H^1(M, \mathcal{O}_M^*)$ of the sheaf $\mathcal{O}_M^*$ of invertible holomorphic functions on M. To compute this group a standard tool is the *exponential exact sequence* of sheaves

$$0 \to \mathbb{Z}_M \to \mathcal{O}_M \xrightarrow{\mathrm{e}} \mathcal{O}_M^* \to 1$$

where $\mathrm{e}(f) := \exp(2\pi i f)$, and $\mathbb{Z}_M$ denotes the sheaf of locally constant functions on M with integral values. This gives a long exact sequence in cohomology

$$H^1(M,\mathbb{Z}) \longrightarrow H^1(M,\mathcal{O}_M) \longrightarrow \mathrm{Pic}(M) \xrightarrow{c_1} H^2(M,\mathbb{Z}) \longrightarrow H^2(M,\mathcal{O}_M) \qquad (2.1)$$

For $L \in \mathrm{Pic}(M)$, the class $c_1(L) \in H^2(M,\mathbb{Z})$ is the *first Chern class* of L. It is a topological invariant, which depends only on L as a topological complex line bundle (this is easily seen by replacing holomorphic functions by continuous ones in the exponential exact sequence).

When M is a projective (or compact Kähler) manifold, Hodge theory provides more information on this exact sequence.[1] The image of c_1 is the kernel of the natural map $H^2(M,\mathbb{Z}) \to H^2(M,\mathcal{O}_M)$. This map is the composition of the maps $H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{C}) \to H^2(M,\mathcal{O}_M)$ deduced from the injections of sheaves $\mathbb{Z}_M \hookrightarrow \mathbb{C}_M \hookrightarrow \mathcal{O}_M$. Now the map $H^2(M,\mathbb{C}) \to H^2(M,\mathcal{O}_M) \cong H^{0,2}$ is the projection onto the last summand of the Hodge decomposition

$$H^2(M,\mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$$

(for the experts: this can be seen by comparing the de Rham complex with the Dolbeault complex.)

Thus the image of c_1 consists of classes $\alpha \in H^2(M,\mathbb{Z})$ whose image $\alpha_{\mathbb{C}} = \alpha^{0,2} + \alpha^{1,1} + \alpha^{0,2}$ in $H^2(M,\mathbb{C})$ satisfies $\alpha^{0,2} = 0$. But since $\alpha_{\mathbb{C}}$ comes from $H^2(M,\mathbb{R})$ we have $\alpha^{2,0} = \overline{\alpha^{0,2}} = 0$: *the image of* c_1 *consists of the classes in* $H^2(M,\mathbb{Z})$ *whose image in* $H^2(M,\mathbb{C})$ *belongs to* $H^{1,1}$ ("Lefschetz theorem").

[1] In this section and the following we use standard Hodge theory, as explained in [13], 0.6. Note that Hodge theory is much easier in the two cases of interest for us, namely complex tori and algebraic curves.

The kernel of c_1, denoted $\mathrm{Pic}^{\mathrm{o}}(M)$, is the group of topologically trivial line bundles. The exact sequence (2.1) shows that it is isomorphic to the quotient of $H^1(M,\mathcal{O}_M)$ by the image of $H^1(M,\mathbb{Z})$. We claim that this image is a lattice in $H^1(M,\mathcal{O}_M)$: this is equivalent to saying that the natural map $H^1(M,\mathbb{R}) \to H^1(M,\mathcal{O}_M)$ is bijective. By Hodge theory, this map is identified with the restriction to $H^1(M,\mathbb{R})$ of the projection of $H^1(M,\mathbb{C}) = H^{1,0} \oplus H^{0,1}$ onto $H^{0,1}$. Since $H^1(M,\mathbb{R})$ is the subspace of classes $\alpha+\bar{\alpha}$ in $H^1(M,\mathbb{C})$, the projection $H^1(M,\mathbb{R}) \to H^{0,1}$ is indeed bijective. Thus $\mathrm{Pic}^{\mathrm{o}}(M)$ is naturally identified with the complex torus $H^1(M,\mathcal{O}_M)/H^1(M,\mathbb{Z})$.

2.2 Flat line bundles

There is another description of $\mathrm{Pic}^{\mathrm{o}}(M)$ which will be of interest for us. Instead of holomorphic line bundles, defined by holomorphic transition functions, we can consider *flat* line bundles, defined by locally constant transition functions; they are parametrized by $H^1(M,\mathbb{C}^*)$.

More important for us will be the *unitary flat* line bundles, defined by locally constant transition functions with values in the unit circle $\mathbb{S}^1$. Let us assume for simplicity that $H^2(M,\mathbb{Z})$ *is torsion free*. In that case the diagram of exact sequences of sheaves

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_M \arrow[r] \arrow[d] & \mathbb{R}_M \arrow[r, "\mathrm{e}"] \arrow[d] & \mathbb{S}^1_M \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & \mathbb{Z}_M \arrow[r] & \mathcal{O}_M \arrow[r, "\mathrm{e}"] & \mathcal{O}^*_M \arrow[r] & 1
\end{tikzcd}$$

gives rise to a diagram in cohomology

$$\begin{tikzcd}
0 \arrow[r] & H^1(M,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^1(M,\mathbb{R}) \arrow[r, "\varepsilon"] \arrow[d, "\pi"] & H^1(M,\mathbb{S}^1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(M,\mathbb{Z}) \arrow[r] & H^1(M,\mathcal{O}_M) \arrow[r] & \mathrm{Pic}^{\mathrm{o}}(M) \arrow[r] & 0
\end{tikzcd}$$

(the homomorphism ε is surjective because $H^2(M,\mathbb{Z})$ is torsion free and therefore injects into $H^2(M,\mathbb{R})$). We have seen in the previous section that π is bijective, so *the map* $H^1(M,\mathbb{S}^1) \to \mathrm{Pic}^{\mathrm{o}}(M)$ *is an isomorphism*. In other words, every line bundle $L \in \mathrm{Pic}^{\mathrm{o}}(M)$ admits a unique unitary flat structure.

2.3 Systems of multipliers

We go back to our complex torus $T = V/\Gamma$.

Lemma 2.1. *Every line bundle on V is trivial.*

Proof . We have $H^2(V,\mathbb{Z}) = 0$ and $H^1(V,\mathcal{O}_V) = 0$ (see [13], p. 46), hence $\mathrm{Pic}(V) = 0$ by the exact sequence (2.1). ☐

Let L be a line bundle on T. We consider the diagram

$$\begin{tikzcd} \pi^*L \arrow[r] \arrow[d] & L \arrow[d] \\ V \arrow[r, "\pi"] & T \end{tikzcd}$$

The action of Γ on V lifts to an action on $\pi^*L = V \times_T L$. We know that π^*L is trivial; we choose a trivialization $\pi^*L \xrightarrow{\sim} V \times \mathbb{C}$. We obtain an action of Γ on $V \times \mathbb{C}$, so that L is the quotient of $V \times \mathbb{C}$ by this action. An element γ of Γ acts linearly on the fibers, hence by

$$\gamma \cdot (z,t) = (z+\gamma, e_\gamma(z)\, t) \quad \text{for} \quad z \in V,\ t \in \mathbb{C}$$

where e_γ is a holomorphic invertible function on V. This formula defines a group action of Γ on $V \times \mathbb{C}$ if and only if the functions e_γ satisfy

$$e_{\gamma+\delta}(z) = e_\gamma(z+\delta)\, e_\delta(z) \quad (\text{“cocycle condition”}).$$

A family $(e_\gamma)_{\gamma\in\Gamma}$ of holomorphic invertible functions on V satisfying this condition is called a system of multipliers. Every line bundle on T is defined by such a system.

A *theta function* for the system $(e_\gamma)_{\gamma\in\Gamma}$ is a holomorphic function $V \to \mathbb{C}$ satisfying

$$\theta(z+\gamma) = e_\gamma(z)\theta(z) \quad \text{for all } \gamma \in \Gamma, z \in V\,.$$

Proposition 2.2. *Let $(e_\gamma)_{\gamma\in\Gamma}$ be a system of multipliers, and L the associated line bundle. The space $H^0(T,L)$ is canonically identified with the space of theta functions for $(e_\gamma)_{\gamma\in\Gamma}$.*

Proof. Any global section s of L lifts to a section $\hat{s} = \pi^*s$ of $\pi^*L = V \times_T L$ over V, defined by $\hat{s}(z) = (z, s(\pi z))$; it is Γ-invariant in the sense that $\hat{s}(z+\gamma) = \gamma \cdot \hat{s}(z)$. Conversely, a Γ-invariant section of π^*L comes from a section of L. Now a section of $\pi^*L \cong V \times \mathbb{C}$ is of the form $z \mapsto (z, \theta(z))$, where $\theta : V \to \mathbb{C}$ is holomorphic. It is Γ-invariant if and only if θ is a theta function for $(e_\gamma)_{\gamma\in\Gamma}$. $\square$

Let $(e_\gamma)_{\gamma\in\Gamma}$ and $(e'_\gamma)_{\gamma\in\Gamma}$ be two systems of multipliers, defining line bundles L and L'. The line bundle $L \otimes L'$ is the quotient of the trivial line bundle $V \times (\mathbb{C}\otimes\mathbb{C})$ by the tensor product action $\gamma\cdot(z, t\otimes t') = (z+\gamma, e_\gamma(z)\, t \otimes e'_\gamma(z)\, t')$; therefore it is defined by the system of multipliers $(e_\gamma e'_\gamma)_{\gamma\in\Gamma}$. In other words, multiplication defines a group structure on the set of systems of multipliers, and we have a surjective group homomorphism

$$\{\text{systems of multipliers}\} \longrightarrow \operatorname{Pic}(T)\ .$$

A system of multipliers $(e_\gamma)_{\gamma\in\Gamma}$ lies in the kernel if and only if the associated line bundle admits a section which is everywhere $\neq 0$; in view of Proposition 2.2, this means that *there exists a holomorphic function $h : V \to \mathbb{C}^*$ such that* $e_\gamma(z) = \dfrac{h(z+\gamma)}{h(z)}$. We will call such systems of multipliers trivial. Thus we can always multiply a given system (e_γ) by a trivial one without changing the associated line bundle.

Remark 2.3. (only for the readers who know group cohomology) Put $H^* := H^0(V, \mathcal{O}_V^*)$. The system of multipliers are exactly the 1-cocycles of Γ with values in H^*, and the trivial systems are the coboundaries. Thus we get a group isomorphism $H^1(\Gamma, H^*) \xrightarrow{\sim} \mathrm{Pic}(T)$ (see [20], §2 for a more conceptual explanation of this isomorphism).

Remark 2.4. The argument in this section apply equally well to *flat* line bundles, since obviously $H^1(V, \mathbb{C}^*) = 0$. The corresponding systems of multipliers are of the form $a(\gamma)_{\gamma\in\Gamma}$, where $a : \Gamma \to \mathbb{C}^*$ is a homomorphism. Similarly, unitary flat line bundles correspond to homomorphisms $\Gamma \to \mathbb{S}^1$. This is nothing but the classical isomorphism $H^1(T, A) \xrightarrow{\sim} \mathrm{Hom}(\pi_1(T), A)$ for $A = \mathbb{C}^*$ or $\mathbb{S}^1$.

2.4 Interlude: hermitian forms

There are many holomorphic invertible functions on V, hence many systems of multipliers giving rise to the same line bundle. Our next goal will be to find a subset of such systems such that each line bundle corresponds exactly to one system of multipliers in that subset. This will involve hermitian forms on V, so let us fix our conventions.

A *hermitian form* H on V will be $\mathbb{C}$-linear in the second variable, $\mathbb{C}$-antilinear in the first. We put $S(x, y) = \mathrm{Re}\, H(x, y)$ and $E(x, y) = \mathrm{Im}\, H(x, y)$. S and E are $\mathbb{R}$-bilinear forms on V, S is symmetric, E is skew-symmetric; they satisfy:

$$S(x, y) = S(ix, iy), \quad E(x, y) = E(ix, iy), \quad S(x, y) = E(x, iy)$$

Using these relations one checks easily that the following data are equivalent:

- The hermitian form H;
- The symmetric $\mathbb{R}$-bilinear form S with $S(x, y) = S(ix, iy)$;
- The skew-symmetric $\mathbb{R}$-bilinear form E with $E(x, y) = E(ix, iy)$.

Moreover,

H non-degenerate $\iff$ E non-degenerate $\iff$ S non-degenerate.

2.5 Systems of multipliers associated to hermitian forms

We denote by $\mathcal{P}$ the set of pairs (H, α), where H is a hermitian form on V, α a map from Γ to $\mathbb{S}^1$, satisfying:

$$E := \mathrm{Im}(H) \ \text{ takes integral values on } \Gamma, \text{ and } \alpha \text{ satisfies}$$

$$\alpha(\gamma + \delta) = \alpha(\gamma)\alpha(\delta)(-1)^{E(\gamma,\delta)} .$$

The law $(H, \alpha) \cdot (H', \alpha') = (H + H', \alpha\, \alpha')$ defines a group structure on $\mathcal{P}$. For $(H, \alpha) \in \mathcal{P}$, we put

$$e_\gamma(z) = \alpha(\gamma) e^{\pi[H(\gamma,z) + \frac{1}{2} H(\gamma,\gamma)]} .$$

We leave as an (easy) exercise to check that this defines a system of multipliers. The corresponding line bundle will be denoted $L(H,\alpha)$. The map $(H,\alpha) \mapsto L(H,\alpha)$ from $\mathcal{P}$ onto $\mathrm{Pic}(T)$ is a group homomorphism; we want to prove that it is an isomorphism.

Proposition 2.5. *The first Chern class $c_1(L(H,\alpha))$ is equal to $E \in \mathrm{Alt}^2(\Gamma,\mathbb{Z}) \cong H^2(T,\mathbb{Z})$.*

Proof. We will use Chern's original definition of the first Chern class of a line bundle L on a compact manifold M (see [13], p. 141). One chooses a $\mathcal{C}^\infty$ metric h on L; this is nothing but a $\mathcal{C}^\infty$ function $L \to \mathbb{R}_+$, which is positive outside the zero section and satisfies $h(\lambda x) = |\lambda|^2 h(x)$ for $x \in L$, $\lambda \in \mathbb{C}$. If s is a local non-vanishing holomorphic section of L, the 2-form $\omega_{L,h} := \frac{1}{2\pi i}\partial\bar\partial \log h(s)$ does not depend on the choice of s; thus $\omega_{L,h}$ is a globally defined closed 2-form, whose class in $H^2(M,\mathbb{C})$ represents $c_1(L)$.

To apply this in our situation, we observe that the metric $\tilde h$ on $V\times\mathbb{C}$ defined by $\tilde h(z,t) = e^{-\pi H(z,z)}|t|^2$ is invariant under Γ; hence it is the pullback of a metric h on $L(H,\alpha)$. The form $\pi^*\omega_{L,h}$ is equal to $\omega_{V\times\mathbb{C},\tilde h}$; to compute it we apply our formula to the section $s : z \mapsto (z,1)$ of $V\times\mathbb{C}$. We find

$$\omega_{V\times\mathbb{C},\tilde h} = \frac{1}{2\pi i}\partial\bar\partial \log e^{-\pi H(z,z)} = \frac{i}{2}\partial\bar\partial\, H(z,z)\ .$$

It remains to prove that $\frac{i}{2}\partial\bar\partial\, H(z,z)$ is the constant 2-form defined by E. It suffices to prove this when $H(x,y) = \bar x_j y_j$; then $\frac{i}{2}\partial\bar\partial\, H(z,z) = \frac{i}{2} dz_j \wedge d\bar z_j$. Let $v = (v_1,\ldots,v_g)$, $w = (w_1,\ldots,w_g)$ two vectors of V; we have

$$(dz_j \wedge d\bar z_j)(v,w) = z_j(v)\bar z_j(w) - z_j(w)\bar z_j(v) = v_j\bar w_j - w_j \bar v_j = -2i\,\mathrm{Im}\, H(v,w)\ ,$$

hence our assertion. $\square$

(See [20], §2 for a proof in terms of group cohomology.)

Theorem 2.6. *The map $(H,\alpha)\mapsto L(H,\alpha)$ defines a group isomorphism $\mathcal{P} \xrightarrow{\sim} \mathrm{Pic}(T)$.*

Proof. Let $\mathcal{Q}$ be the group of hermitian forms H on V such that $\mathrm{Im}(H)$ is integral on Γ. By Proposition 2.5 and Section 2.1 we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(\Gamma,\mathbb{S}^1) & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0\\
 & & \big\downarrow{\scriptstyle L^{o}} & & \big\downarrow{\scriptstyle L} & & \big\downarrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & \mathrm{Pic}^{o}(T) & \longrightarrow & \mathrm{Pic}(T) & \xrightarrow{c_1} & H^2(T,\mathbb{Z}) & &
\end{array}$$

with $\iota(H) = \mathrm{Im}(H) \in \mathrm{Alt}^2(\Gamma,\mathbb{Z}) \cong H^2(T,\mathbb{Z})$.

Let us first prove that ι is bijective onto $\mathrm{Im}(c_1)$. Let $E \in \mathrm{Alt}^2(\Gamma,\mathbb{Z}) \cong H^2(T,\mathbb{Z})$; we have seen in Section 2.1 that E belongs to $\mathrm{Im}(c_1)$ if and only if it belongs to $H^{1,1}$, that is satisfies $E(ix,iy) = E(x,y)$ (Proposition 1.3). By Section

2.4 this is equivalent to $E = \mathrm{Im}(H)$ for a hermitian form $H \in \mathcal{Q}$; moreover H is uniquely determined by E, hence our assertion.

The map L^{o} associates to a unitary character $\alpha : \Gamma \to \mathbb{S}^1$ the unitary flat bundle $L(0,\alpha)$; we have already seen that it is bijective (Section 2.2 and Remark 2.4). Thus the map $(H,\alpha) \mapsto L(H,\alpha)$ is bijective. $\square$

2.6 The theorem of the square

This section is devoted to an important result, Theorem 2.8 below, which is actually an easy consequence of our description of line bundles on T (we encourage the reader to have a look at the much more elaborate proof in [20], §6, valid over any algebraically closed field).

Lemma 2.7. *Let $a \in V$. We have $t^*_{\pi(a)}L(H,\alpha) = L(H,\alpha')$ with $\alpha'(\gamma) = \alpha(\gamma)\mathbb{e}(E(\gamma,a)$.*

Proof. In general, let L be a line bundle on T defined by a system of multipliers $(e_\gamma)_{\gamma\in\Gamma}$. Then $(e_\gamma(z+a))_{\gamma\in\Gamma}$ is a system of multipliers, defining a line bundle L'; the self-map $(z,t) \mapsto (z+a,t)$ of $V \times \mathbb{C}$ is equivariant w.r.t. the actions of Γ defined by $(e_\gamma(z+a))$ on the source and $(e_\gamma(z))$ on the target, so it induces an isomorphism $L' \xrightarrow{\sim} t^*_{\pi(a)}L$.

We apply this to the multiplier $e_\gamma(z) = a(\gamma)e^{\pi[H(\gamma,z)+\frac{1}{2}H(\gamma,\gamma)]}$; we find $e_\gamma(z+a) = e_\gamma(z)\ e^{\pi H(\gamma,a)}$. Recall that we are free to multiply $e_\gamma(z)$ by $\frac{h(z+\gamma)}{h(z)}$ for some holomorphic invertible function h; taking $h(z) = e^{-\pi H(a,z)}$, our multiplier becomes $e_\gamma\ e^{\pi[H(\gamma,a)-H(a,\gamma)]} = e_\gamma\ e^{2\pi i E(\gamma,a)}$. $\square$

Theorem 2.8 (Theorem of the square)**.** *Let L be a line bundle on T.*

1) *The map*

$$\lambda_L : T \to \mathrm{Pic}^{\mathrm{o}}(T), \quad \lambda_L(a) = t^*_a L \otimes L^{-1}$$

is a group homomorphism.

2) *Let $E \in \mathrm{Alt}^2(\Gamma,\mathbb{Z})$ be the first Chern class of L. We have*

$$\mathrm{Ker}\,\lambda_L = \Gamma^\perp/\Gamma\ , \quad \textit{with}\ \ \Gamma^\perp := \{z \in V \mid E(z,\gamma) \in \mathbb{Z}\ \ \textit{for all}\ \gamma \in \Gamma\}\,.$$

3) *If E is non-degenerate, λ_L is surjective and has finite kernel.*

4) *If E is unimodular, λ_L is a group isomorphism.*

Proof. By the Lemma, λ_L is the composition

$$T \xrightarrow{\ \varepsilon\ } \mathrm{Hom}(\Gamma,\mathbb{S}^1) \xrightarrow{\ L^{\mathrm{o}}\ } \mathrm{Pic}^{\mathrm{o}}(T)\ ,$$

where $\varepsilon(a)$, for $a = \pi(\tilde{a}) \in T$, is the map $\gamma \mapsto \mathbb{e}(E(\gamma,\tilde{a})$, and L^{o} is the isomorphism $\alpha \mapsto L(0,\alpha)$ (Theorem 2.6). Therefore we can replace λ_L by ε in the proof. Then 1) and 2) become obvious.

Assume that E is non-degenerate. Let $\chi \in \mathrm{Hom}(\Gamma, \mathbb{S}^1)$. Since Γ is a free $\mathbb{Z}$-module, we can find a homomorphism $u : \Gamma \to \mathbb{R}$ such that $\chi(\gamma) = \mathbb{e}(u(\gamma))$ for each $\gamma \in \Gamma$. Extend u to a $\mathbb{R}$-linear form $V \to \mathbb{R}$; since E is non-degenerate, there exists $a \in V$ such that $u(z) = E(z, a)$, hence $\varepsilon(\pi(a)) = \chi$. Thus ε is surjective.

Let us denote by $e : V \to \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ the $\mathbb{R}$-linear isomorphism associated to E. The dual $\Gamma^* := \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z})$ embeds naturally in $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$, and $\Gamma^{\perp}$ is by definition $e^{-1}(\Gamma^*)$; then e identifies $\Gamma^{\perp}$ with Γ^*, so that the inclusion $\Gamma \subset \Gamma^{\perp}$ corresponds to the map $\Gamma \to \Gamma^*$ associated to $E_{|\Gamma}$. This map has finite cokernel, and it is bijective if E is unimodular; this achieves the proof. □

Remark 2.9. We have seen in Section 2.1 that $\mathrm{Pic}^{\mathrm{o}}(T)$ has a natural structure of complex torus; it is not difficult to prove that the map λ_L is holomorphic. In particular, when E is unimodular, λ_L is an isomorphism of complex tori.

Corollary 2.10. *Assume that $c_1(L)$ is non-degenerate. Any line bundle L' with $c_1(L') = c_1(L)$ is isomorphic to t_a^*L for some a in T.*

Proof . $L' \otimes L^{-1}$ belongs to $\mathrm{Pic}^{\mathrm{o}}(T)$, hence is isomorphic to $t_a^*L \otimes L^{-1}$ for some a in T by 3). □

The following immediate consequence of 1) will be very useful:

Corollary 2.11. *Let $a_1, \dots, a_r$ in T with $\sum a_i = 0$. Then $t_{a_1}^*L \otimes \cdots \otimes t_{a_r}^*L \cong L^{\otimes r}$.* □

3 Polarizations

In this section we will consider a line bundle $L = L(H, \alpha)$ on our complex torus T such that the hermitian form H is *positive definite*. We will first look for a concrete expression of the situation using an appropriate basis.

3.1 Frobenius lemma

The following easy result goes back to Frobenius:

Proposition 3.1. *Let Γ be a free finitely generated $\mathbb{Z}$-module, and $E : \Gamma \times \Gamma \to \mathbb{Z}$ a skew-symmetric, non-degenerate form. There exists positive integers $d_1, \dots, d_g$ with $d_1 \,|\, d_2 \,|\, \cdots \,|\, d_g$ and a basis $(\gamma_1, \dots, \gamma_g; \delta_1, \dots, \delta_g)$ of Γ such that the matrix of E in this basis is $\begin{pmatrix} 0 & \mathbf{d} \\ -\mathbf{d} & 0 \end{pmatrix}$, where $\mathbf{d}$ is the diagonal matrix with entries $(d_1, \dots, d_g)$.*

As a consequence we see that the determinant of E is the square of the integer $d_1 \cdots d_g$, called the *Pfaffian* of E and denoted $\mathrm{Pf}(E)$. The most important case for us will be when $d_1 = \cdots = d_g = 1$, or equivalently $\det(E) = 1$; in that case one says that E is *unimodular*, and that $(\gamma_1, \dots, \gamma_g; \delta_1, \dots, \delta_g)$ is a *symplectic basis* of Γ.

Proof . Let d_1 be the minimum of the numbers $E(\alpha,\beta)$ for $\alpha,\beta\in\Gamma$, $E(\alpha,\beta)>0$; choose γ,δ such that $E(\gamma,\delta)=d_1$. For any $\varepsilon\in\Gamma$, $E(\gamma,\varepsilon)$ is divisible by d_1 – otherwise using Euclidean division we would find ε with $0<E(\gamma,\varepsilon)<d_1$. Likewise $E(\delta,\varepsilon)$ is divisible by d_1. Put $U=\mathbb{Z}\gamma\oplus\mathbb{Z}\delta$; we claim that $\Gamma=U\oplus U^{\perp}$. Indeed, for $x\in\Gamma$, we have

$$x=\frac{E(x,\delta)}{d_1}\gamma+\frac{E(\gamma,x)}{d_1}\delta+(x-\frac{E(x,\delta)}{d_1}\gamma-\frac{E(\gamma,x)}{d_1}\delta)\ .$$

Reasoning by induction on the rank of Γ, we find integers $d_2\,|\,d_3\,|\cdots|\,d_g$ and a basis $(\gamma,\gamma_2,\ldots,\gamma_g;\ \delta,\delta_2,\ldots,\delta_g)$ of Γ, such that the matrix of E is $\begin{pmatrix}0&\mathbf{d}\\-\mathbf{d}&0\end{pmatrix}$. It remains to prove that d_1 divides d_2; otherwise, using Euclidean division again, we can find $k\in\mathbb{Z}$ such that $0<E(\gamma+\gamma_2,k\delta+\delta_2)<d_1$, a contradiction. □

3.2 Polarizations and the period matrix

Going back to our complex torus $T=V/\Gamma$, we assume given a *positive definite* hermitian form H on V, such that $E:=\mathrm{Im}(H)$ takes integral values on Γ. Such a form is called a *polarization* of T; if E is unimodular, we say that H is a *principal polarization.* A complex torus which admits a polarization is classically called a (polarized) *abelian variety*; we will see below that it is actually a projective manifold. It is common to use the abbreviation *p.p.a.v.* for "principally polarized abelian variety".

We choose a basis $(\gamma_1,\ldots,\gamma_g;\delta_1,\ldots,\delta_g)$ as in Proposition 3.1 (note that E is non-degenerate by Section 2.4); we put $\gamma'_j:=\frac{\gamma_j}{d_j}$ for $j=1,\ldots,g$.

Lemma 3.2. *$(\gamma'_1,\ldots,\gamma'_g)$ is a basis of V over $\mathbb{C}$.*

Proof . Let $W=\mathbb{R}\gamma'_1\oplus\cdots\oplus\mathbb{R}\gamma'_g$. Our statement is equivalent to $V=W\oplus iW$. But if $x\in W\cap iW$, we have $H(x,x)=E(x,ix)=0$ since $E_{|W}=0$, hence $x=0$. □

Expressing the δ_j's in this basis gives a matrix $\tau\in M_g(\mathbb{C})$ with $\delta_j=\sum_i\tau_{ij}\gamma'_i$. In the corresponding coordinates, we have

$$\Gamma=\mathbf{d}\mathbb{Z}^g\oplus\tau\mathbb{Z}^g$$

in other words, the elements of Γ are the column vectors $\mathbf{d}p+\tau q$ with $p,q\in\mathbb{Z}^g$. The matrix τ is often called the *period matrix.*

Note that in case the polarization is principal we have $\gamma'_i=\gamma_i$ and $\Gamma=\mathbb{Z}^g\oplus\tau\mathbb{Z}^g$.

Proposition 3.3. *The matrix τ is symmetric, and* $\mathrm{Im}(\tau)$ *is positive definite.*

Proof . Put $\tau=A+iB$, with $A,B\in M_g(\mathbb{R})$. We will compare the bases $(\gamma_1,\ldots,\gamma_g;\delta_1,\ldots,\delta_g)$ and $(\gamma'_1,\ldots,\gamma'_g;i\gamma'_1,\ldots,i\gamma'_g)$ of V over $\mathbb{R}$. The change of basis matrix (expressing the vectors of the first basis in the second one) is $P = \begin{pmatrix} \mathbf{d} & A \\ 0 & B \end{pmatrix}$. Therefore the matrix of E in the second basis is

$$ {}^tP^{-1} \begin{pmatrix} 0 & \mathbf{d} \\ -\mathbf{d} & 0 \end{pmatrix} P^{-1} = \begin{pmatrix} 0 & B^{-1} \\ -{}^tB^{-1} & {}^tB^{-1}(A - {}^tA)B^{-1} \end{pmatrix} $$

Now the condition $E(ix, iy) = E(x, y)$, expressed in the basis $(\gamma'_1, \ldots, \gamma'_g; i\gamma'_1, \ldots, i\gamma'_g)$, is equivalent to $A = {}^tA$ and $B = {}^tB$; we have $H(\gamma'_j, \gamma'_k) = E(\gamma'_j, i\gamma'_k)$, so the matrix of H in the basis $(\gamma'_1, \ldots, \gamma'_g)$ (over $\mathbb{C}$) is B^{-1}, and the positivity of H is equivalent to that of B. □

3.3 The moduli space of p.p.a.v.

In this section we restrict for simplicity to the case the polarization is *principal*; we encourage the reader to adapt the argument to the general case (see for instance [8], VII.1).

We have seen that the choice of a symplectic basis determines the matrix τ, which in turn completely determines T and H: we have $V = \mathbb{C}^g$ and $\Gamma = \Gamma_\tau := \mathbb{Z}^g \oplus \tau\mathbb{Z}^g$; the hermitian form H is given by the matrix $\mathrm{Im}(\tau)^{-1}$, and its imaginary part E by $E(p + \tau q, p' + \tau q') = {}^tpq' - {}^tqp'$.

The space of symmetric matrices $\tau \in M_g(\mathbb{C})$ with $\mathrm{Im}(\tau)$ positive definite is denoted $\mathbb{H}_g$, and called the *Siegel upper half space*. It is an open subset of the vector space of complex symmetric matrices. From what we have seen it follows that $\mathbb{H}_g$ parametrizes p.p.a.v. $(V/\Gamma, H)$ endowed with a symplectic basis of the lattice Γ.

Now we want to get rid of the choice of the symplectic basis. We have associated to a symplectic basis an isomorphism $V \xrightarrow{\sim} \mathbb{C}^g$ which maps Γ to the lattice Γ_τ. A change of the basis amounts to a linear automorphism M of $\mathbb{C}^g$, inducing a symplectic isomorphism $\Gamma_\tau \xrightarrow{\sim} \Gamma_{\tau'}$. Such an isomorphism is given by $\begin{pmatrix} p' \\ q' \end{pmatrix} = P \begin{pmatrix} p \\ q \end{pmatrix}$, where P belongs to the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$, that is, $P \in M_{2g}(\mathbb{Z})$ and ${}^tPJP = J$, with $J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$.

We have $M(p + \tau q) = p' + \tau' q'$, hence

$$ \begin{pmatrix} \mathbf{1} & \tau' \end{pmatrix} = M \begin{pmatrix} \mathbf{1} & \tau \end{pmatrix} P^{-1} \quad \text{or equivalently} \quad \begin{pmatrix} \mathbf{1} \\ \tau' \end{pmatrix} = {}^tP^{-1} \begin{pmatrix} \mathbf{1} \\ \tau \end{pmatrix} {}^tM \ . $$

If $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $a, b, c, d \in M_g(\mathbb{Z})$, we have ${}^tP^{-1} = -JPJ = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$. We find

$$ \mathbf{1} = (d - c\tau)^tM \ , \quad \tau' = (-b + a\tau)^tM \ , \quad \text{hence} \quad \tau' = (a\tau - b)(-c\tau + d)^{-1} \ . $$

Thus the group $\mathrm{Sp}(2g, \mathbb{Z})$ acts on $\mathbb{H}_g$ by $(P, \tau) \mapsto (a\tau - b)(-c\tau + d)^{-1}$, and two matrices τ, τ' correspond to the same p.p.a.v. with different symplectic bases iff they are conjugated under this action. To get a nicer formula, we observe that

$$\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = tPt \ , \text{ with } \ t = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \ ;$$

since $tJt = -J$, the map $P \mapsto tPt$ is an automorphism of $\mathrm{Sp}(2g,\mathbb{Z})$. Composing our action with this automorphism, we obtain:

Proposition 3.4. *The group* $\mathrm{Sp}(2g,\mathbb{Z})$ *acts on* $\mathbb{H}_g$ *by* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = (a\tau + b)(c\tau + d)^{-1}$. *The quotient* $\mathcal{A}_g := \mathbb{H}_g/\mathrm{Sp}(2g,\mathbb{Z})$ *parametrizes isomorphism classes of* g*-dimensional p.p.a.v.*

It is not difficult to show that the action of $\mathrm{Sp}(2g,\mathbb{Z})$ on $\mathbb{H}_g$ is nice ("properly discontinuous"), so that $\mathcal{A}_g$ is an analytic space ([8], VII.1). A much more subtle result is that $\mathcal{A}_g$ is Zariski open in a projective variety, the *Satake compactification* $\overline{\mathcal{A}_g}$.

We have not made precise in which sense $\mathcal{A}_g$ parametrizes p.p.a.v. It is actually what is called a moduli space; we will give a precise definition in the case of vector bundles (see Section 4.2 below), which can be adapted without difficulty to this case.

3.4 Theta functions

Let H be a polarization on T; we keep the notation of the previous sections. Let $\alpha : \Gamma \to \mathbb{S}^1$ be any map satisfying $\alpha(\gamma+\delta) = \alpha(\gamma)\alpha(\delta)(-1)^{E(\gamma,\delta)}$.

Theorem 3.5. $\dim H^0(T, L(H,\alpha)) = d_1 \cdots d_g = \mathrm{Pf}(E)$.

Proof . We first treat the case $d_1 = \cdots = d_g = 1$. According to Prop. 2.2, we are looking for theta functions satisfying

$$\theta(z+\gamma) = \alpha(\gamma) e^{\pi[H(\gamma,z) + \frac{1}{2}H(\gamma,\gamma)]}\theta(z) \ .$$

Recall that we are free to multiply $e_\gamma(z)$ by $\frac{h(z+\gamma)}{h(z)}$ for some $h \in H^0(V, \mathcal{O}_V^*)$ (this amounts to multiply θ by h). We will use this to make θ *periodic* with respect to the basis elements $\gamma_1, \ldots, \gamma_g$ of Γ.

As before we put $W = \mathbb{R}\gamma_1 \oplus \cdots \oplus \mathbb{R}\gamma_g$. Since $E_{|W} = 0$, the form $H_{|W}$ is a real symmetric form; since $V = W \oplus iW$ (Lemma 3.2), it extends as a $\mathbb{C}$*-bilinear* symmetric form B on V. We put $h(z) = e^{-\frac{\pi}{2}B(z,z)}$: this amounts to replace H in $e_\gamma(z)$ by $H' := H - B$. We have

Lemma 3.6. $H'(p+\tau q, z) = -2i\, {}^t q z$.

Proof . Let $w \in W$. We have $H'(w,y) = 0$ for $y \in W$, hence also for any $y \in V$ because H' is $\mathbb{C}$-linear in y. On the other hand for $z \in V$ we have $H'(z,w) = (H-B)(z,w) = (\bar{H}-B)(w,z) = (\bar{H}-H)(w,z) = 2iE(z,w)$. Thus for $z = \sum z_i\gamma_i \in V$ we have $H'(\gamma_j, z) = 0$ and $H'(\delta_j,z) = \sum_k z_k H'(\delta_j,\gamma_k) = -2iz_j$, hence the lemma. $\square$

Put $L = L(H,\alpha)$. By Cor. 2.10, changing α amounts to replace L by t_a^*L for some $a \in T$. Since the pullback map $t_a^* : H^0(T,L) \to H^0(T,t_a^*L)$ is an isomorphism, it suffices to prove the theorem for a particular value of α; we choose $\alpha(p+\tau q) = (-1)^{{}^tpq}$. Indeed we have mod. 2:

$$ {}^t(p+p')(q+q') \equiv {}^tpq + {}^tp'q' + ({}^tpq' - {}^tp'q) = {}^tpq + {}^tp'q' + E(p+\tau q, p'+\tau q') $$

$$ \text{for } \; p,q,p',q' \in \mathbb{Z}^g \; . $$

Thus our theta functions must satisfy the quasi-periodicity condition

$$ \theta(z+p+\tau q) = \theta(z)\,\mathrm{e}(-{}^tqz - \frac{1}{2}{}^tq\tau q) \qquad \text{for } \; z \in \mathbb{C}^g, \; p,q \in \mathbb{Z}^g \; . $$

In particular, they are periodic with respect to the subgroup $\mathbb{Z}^g \subset \mathbb{C}^g$. This implies that they admit a Fourier expansion of the form $\theta(z) = \sum_{m\in\mathbb{Z}^g} c(m)\mathrm{e}({}^tmz)$. Now let us express the quasi-periodicity condition; we have:

$$ \theta(z+p+\tau q) = \sum_{m\in\mathbb{Z}^g} c(m)\mathrm{e}({}^tm\tau q)\mathrm{e}({}^tmz) $$

and

$$ \begin{aligned} \theta(z)\mathrm{e}(-{}^tqz - \frac{1}{2}{}^tq\tau q) &= \sum_{m\in\mathbb{Z}^g} c(m)\mathrm{e}({}^t(m-q)z - \frac{1}{2}{}^tq\tau q) \\ &= \sum_{m\in\mathbb{Z}^g} c(m+q)\mathrm{e}(-\frac{1}{2}{}^tq\tau q)\mathrm{e}({}^tmz) \; . \end{aligned} $$

Comparing we find $c(m+q) = c(m)\mathrm{e}({}^t(m+\frac{q}{2})\tau q)$. Taking $m=0$ gives $c(q) = c(0)\,\mathrm{e}(\frac{1}{2}{}^tq\tau q)$. Thus our theta functions, if they exist, are all proportional to

$$ \theta(z) = \sum_{m\in\mathbb{Z}^g} \mathrm{e}({}^tmz + \frac{1}{2}{}^tm\tau m) \; . $$

It remains to prove that this function indeed exists, that is that the series converges. But the coefficients $c(m)$ of the Fourier series satisfy $|c(m)| = e^{-q(m)}$, where q is a positive definite quadratic form, and therefore they decrease very fast as $m \to \infty$.

Now we treat the case $d_1 = \cdots = d_g = d$. In this case the form $\frac{1}{d}H$ is a principal polarization, so we can take $L = L_0^{\otimes d}$, where L_0 is the line bundle considered above. The corresponding theta functions satisfy

$$ \theta(z+p+\tau q) = \theta(z)\mathrm{e}(-d{}^tqz - \frac{d}{2}{}^tq\tau q) $$

$$ \text{for } \; z \in \mathbb{C}^g, \; p,q \in \mathbb{Z}^g \qquad (\text{“theta functions of order } d\text{”}). $$

We write again $\theta(z) = \sum_{m\in\mathbb{Z}^g} c(m)\mathrm{e}({}^tmz)$; the quasi-periodicity condition gives

$$ c(m+dq) = c(m)\mathrm{e}({}^t(m+\frac{d}{2})\tau q) = c(m)\mathrm{e}(\frac{-1}{d}{}^tm\tau m)\mathrm{e}(\frac{1}{2d}{}^t(m+dq)\tau(m+dq)) $$

This determines up to a constant all coefficients $c(m)$ for m in a given coset ε of $\mathbb{Z}^g$ modulo $d\mathbb{Z}^g$; the corresponding theta function is

$$\theta[\varepsilon](z) = \sum_{m\in\varepsilon} \mathbb{e}({}^t mz + \frac{1}{2d}{}^t m\tau m) \ . \tag{3.1}$$

By what we have seen the functions $\theta[\varepsilon]$, where ε runs through $\mathbb{Z}^g/d\mathbb{Z}^g$, form a basis of the space of theta functions of order d; in particular, the dimension of this space is d^g.

The proof of the general case is completely analogous but requires more complicated notations. We will not need it in these lectures, so we leave it as an exercise for the reader. □

3.5 Comments

The proof of the theorem gives much more than the dimension of the space of theta functions, namely an explicit basis $(\theta[\varepsilon])_{\varepsilon\in\mathbb{Z}^g/d\mathbb{Z}^g}$ of this space given by formula (3.1). In particular, when the polarization H is principal, the line bundles $L(H,\alpha)$ have a unique non-zero section (up to a scalar); the divisor of this section is called a *theta divisor* of the p.p.a.v. (T,H). By Corollary 2.10 it is well-defined up to translation, so one speaks sometimes of "the" theta divisor. The choice of a symplectic basis gives a particular theta divisor Θ_τ, defined by the celebrated Riemann theta function

$$\theta(z) = \sum_{m\in\mathbb{Z}^g} \mathbb{e}({}^t mz + \frac{1}{2}{}^t m\tau m) \ .$$

It is quite remarkable that starting from a linear algebra data (a lattice Γ in V and a polarization), we get a hypersurface $\Theta \subset T = V/\Gamma$. When the p.p.a.v. comes from a geometric construction (Jacobians, Prym varieties, intermediate Jacobians), this divisor has a rich geometry, which reflects the objects we started with. In particular it is often possible to recover these objects from the data (T,Θ) ("Torelli theorem"), or to characterize the p.p.a.v. obtained in this way ("Schottky problem").

3.6 Reminder: line bundles and maps into projective space

Let M be a projective variety, and L a line bundle on M. The *linear system* $|L|$ is by definition $\mathbb{P}(H^0(M,L))$. Sending a nonzero section to its divisors identifies $|L|$ with the set of effective divisors E on M such that $\mathcal{O}_M(E) \cong L$.

The *base locus* $B(L)$ of L is the intersection of the divisors in $|L|$. Assume that L has no base point, that is, $B(L) = \varnothing$. Then the divisors of $|L|$ passing through a point $m \in M$ form a hyperplane in $|L|$, corresponding to a point $\varphi_L(m)$ in the dual projective space $|L|^*$. This defines a morphism $\varphi_L : M \to |L|^*$. Choosing a basis $(s_0,\ldots,s_n)$ of $H^0(M,L)$ identifies $|L|$, hence also its dual $|L|^*$, to $\mathbb{P}^n$; then $\varphi_L(m) = (s_0(m),\ldots,s_n(m))$, where we have fixed an isomorphism $L_m \xrightarrow{\sim} \mathbb{C}$ to evaluate the s_i at m.

If $E \in |L|$, we also denote the linear system $|L|$ by $|E|$, and the map φ_L by φ_E. Thus $|E|$ is the set of effective divisors linearly equivalent to E.

3.7 The Lefschetz theorem

Theorem 3.7 (Lefschetz)**.** *Let L be a line bundle on T.*

1) *Assume $H^0(T,L) \neq 0$. For $k \geq 2$, the linear system $|L^{\otimes k}|$ has no base points.*

2) *Assume that the hermitian form associated to L is positive definite. For $k \geq 3$, the map $\varphi_{L^{\otimes k}} : T \to |L^{\otimes k}|^*$ is an embedding.*

Proof. Let us prove 1) in the case $k = 2$ – the proof in the general case is identical. Let $D \in |L|$. A simple but crucial observation is that

$$x \in t_a^* D \iff a \in t_x^* D \ .$$

By Corollary 2.11 we have $t_a^* D + t_{-a}^* D \in |L^{\otimes 2}|$ for all a in T. Given $x \in T$, we choose a outside the divisors $t_x^* D$ and $t_{-x}^* D^-$, where D^- denotes the image of D by the involution $z \mapsto -z$; then $x \notin t_a^* D + t_{-a}^* D$, which proves 1).

We will prove only a part of 2), namely the *injectivity* of $\varphi_{L^{\otimes k}}$; the proof that its tangent map at each point is injective is analogous but requires some more preparation. We will do it for $k = 3$ (the same proof works for all k) and assume moreover that the polarization is *principal* – again the general case requires more work, see [20], §17.

Let x, y in T such that $\varphi_{L^{\otimes 3}}(x) = \varphi_{L^{\otimes 3}}(y)$. This means that any divisor $E \in |L^{\otimes 3}|$ passing through x passes through y. Let Θ be the unique element of $|L|$. Let $a \in t_x^* \Theta$; we choose $b \in T$ outside $t_y^* \Theta$ and $t_{a-y}^* \Theta^-$, and take $E = t_a^* \Theta + t_b^* \Theta + t_{-a-b}^* \Theta$. We have $x \in E$, hence $y \in E$, but $y \notin t_b^* \Theta$ and $y \notin t_{-a-b}^* \Theta$, so $y \in t_a^* \Theta$, that is, $a \in t_y^* \Theta$. We conclude that the divisors $t_x^* \Theta$ and $t_y^* \Theta$ have the same support. But Θ has no multiple component, since by 1) this would imply $\dim H^0(T,L) > 1$. Thus $t_x^* \Theta = t_y^* \Theta$, and by Theorem 2.8.4) this implies $x = y$. ∎

Remark 3.8. A line bundle L such that $\varphi_{L^{\otimes k}}$ is an embedding for k large enough is said to be *ample*. The celebrated (and difficult) Kodaira embedding theorem states that this is the case if and only if the class $c_1(L)$ can be represented by a $(1,1)$-form which is everywhere positive definite (see [13], Section I.4, for a precise statement and a proof). The Lefschetz theorem gives a much more elementary version for complex tori. It is also more precise, since it says that $k \geq 3$ is enough for $L^{\otimes k}$ to give an embedding. We are now going to discuss the map defined by $L^{\otimes 2}$ in the case of a principal polarization.

3.8 The linear system $|2\Theta|$

Let us again focus on the case of a principal polarization. The Riemann theta function is even, so its divisor Θ_τ is symmetric – that is, $\Theta_\tau^- = \Theta_\tau$. From Theorem 2.8.4) one deduces that the symmetric theta divisors are the translates $t_a^* \Theta_\tau$ where

a runs through the 2^{2g} points of order 2 of T. Note that by Theorem 2.8.1), all divisors 2Θ, where Θ is a symmetric theta divisor, are linearly equivalent. Thus the linear system $|2\Theta|$ is canonically associated to the principal polarization.

We will denote by i_T the involution $z \mapsto -z$. The quotient $K := T/i_T$ is called the *Kummer variety* of T; it has 2^{2g} singular points, which are the images of the points of order 2 in T.

Proposition 3.9. *Let* Θ *be an* irreducible *symmetric theta divisor on* T. *The map* $\varphi_{2\Theta} : T \to |2\Theta|^*$ *factors through* i_T *and embeds* $K = T/i_T$ *into* $|2\Theta|^*$.

(See Remark 3.10 below for the irreducibility hypothesis.)

Proof. By Theorem 3.7 the map is everywhere defined. Recall that a basis of $H^0(T, \mathcal{O}_T(2\Theta))$ is given by the theta functions

$$\theta[\varepsilon](z) = \sum_{m\in\varepsilon} \mathrm{e}({}^t mz + \frac{1}{4}{}^t m\tau m) \ ,$$

where ε runs through the cosets of $\mathbb{Z}^g/2\mathbb{Z}^g$. Thus $\varphi_{2\Theta}$ maps $\pi(z) \in T$ to $\big(\theta[\varepsilon](z)\big)_{\varepsilon\in\mathbb{Z}^g/2\mathbb{Z}^g}$ in $\mathbb{P}^{2^g-1}$. Since each $\varepsilon \in \mathbb{Z}^g/2\mathbb{Z}^g$ is stable under the involution $m \mapsto -m$, the functions $\theta[\varepsilon]$ are even; therefore $\varphi_{2\Theta}$ factors through i_T, and induces a map $K \to |2\Theta|^*$.

Let us prove that this map is injective. Let $x \neq y$ in T with $\varphi_{2\Theta}(x) = \varphi_{2\Theta}(y)$. Let a be a general point of $t_x^*\Theta$. The divisor $t_a^*\Theta + t_{-a}^*\Theta$ belongs to $|2\Theta|$ and contains x, hence also y.

Since $t_x^*\Theta \neq t_y^*\Theta$, a does not belong to $t_y^*\Theta$; thus $y \notin t_a^*\Theta$, and therefore $y \in t_{-a}^*\Theta$. This means $y - a \in \Theta$, and since Θ is symmetric $a \in t_{-y}^*\Theta$. We conclude that $t_x^*\Theta = t_{-y}^*\Theta$, hence $x = -y$, which proves our assertion.

The injectivity of the tangent map at the smooth points of K is proved in the same way; the analysis at the singular points is more delicate, see [18]. □

Remark 3.10. What if Θ is reducible? It is not difficult to show that T must be a product of lower-dimensional p.p.a.v.; that is, $T = T_1 \times \cdots \times T_p$ and $\Theta = \Theta_1 \times T_2 \times \cdots \times T_p + \cdots + T_1 \times \cdots \times T_{p-1} \times \Theta_p$. In that case the geometry of (T, Θ) is determined by that of the (T_i, Θ_i).

Example 3.11. Suppose $g = 2$. Then $\varphi_{2\Theta}$ embeds $K = T/i_T$ in $\mathbb{P}^3$. It is easy to see that K has degree 4 (hint: use $K_T = \mathcal{O}_T = \varphi_{2\Theta}^*\mathcal{O}_{\mathbb{P}^3}(\deg(K) - 4)$); it has 16 double points corresponding to the 16 points of order 2 in T. This is the celebrated Kummer quartic surface, found by Kummer in 1864.

The following remarkable formula explains (in part) the particular role of the linear system $|2\Theta|$. We use the notations of the proof of Theorem 3.5.

Proposition 3.12 (Addition formula)**.**

$$\theta(z+a)\,\theta(z-a) = \sum_{\varepsilon\in\mathbb{Z}^g/2\mathbb{Z}^g} \theta[\varepsilon](z)\,\theta[\varepsilon](a) \qquad \textit{for} \ \ z, a \ \ \textit{in}\ \mathbb{C}^g \ .$$

Proof . We have

$$\theta(z+a)\,\theta(z-a) = \sum_{p,q\in\mathbb{Z}^g} \mathbb{e}\big({}^t(p+q)z + {}^t(p-q)a + \frac{1}{2}({}^tp\tau p + {}^tq\tau q)\big)\ .$$

Putting $r = p+q$, $s = p-q$ defines a bijection of $\mathbb{Z}^g \times \mathbb{Z}^g$ onto the set of pairs (r,s) in $\mathbb{Z}^g \times \mathbb{Z}^g$ with $r \equiv s$ (mod. $2\mathbb{Z}^g$). This set is the union of the subsets $\varepsilon \times \varepsilon \subset \mathbb{Z}^g \times \mathbb{Z}^g$ for $\varepsilon \in \mathbb{Z}^g/2\mathbb{Z}^g$; thus

$$\begin{aligned}\theta(z+a)\,\theta(z-a) &= \sum_{\varepsilon\in\mathbb{Z}^g/2\mathbb{Z}^g} \sum_{r,s\in\varepsilon} \mathbb{e}({}^trz + \frac{1}{4}{}^tr\tau r)\,\mathbb{e}({}^tsa + \frac{1}{4}{}^ts\tau s)\\ &= \sum_{\varepsilon\in\mathbb{Z}^g/2\mathbb{Z}^g} \theta[\varepsilon](z)\,\theta[\varepsilon](a)\ . \qquad \square\end{aligned}$$

The addition formula has the following geometric interpretation:

Corollary 3.13. *Let* Θ *be a symmetric theta divisor on* T. *For* $a \in T$, *put* $\kappa(a) := t_a^*\Theta + t_{-a}^*\Theta \in |2\Theta|$. *There is a commutative diagram:*

$$\begin{tikzcd} & {|2\Theta|^*} \arrow[dd, "\wr"] \\ T \arrow[ur, "\varphi_{2\Theta}"] \arrow[dr, "\kappa"'] & \\ & {|2\Theta|} \end{tikzcd}$$

Proof . After a translation by a point of order 2 we can assume $\Theta = \Theta_\tau$ for some symplectic basis of Γ. We identify both $|2\Theta_\tau|$ and its dual to $\mathbb{P}^{2^g-1}$ using the basis $(\theta[\varepsilon])_{\varepsilon\in\mathbb{Z}^g/2\mathbb{Z}^g}$. Then, by the addition formula,

$$\kappa(a) = \big(\theta[\varepsilon](a)\big)_{\varepsilon\in\mathbb{Z}^g/2\mathbb{Z}^g} = \varphi_{2\Theta}(a)\ . \quad \square$$

(For a more intrinsic description of the isomorphism $|2\Theta| \xrightarrow{\sim} |2\Theta|^*$, see [21], p. 555.)

4 Curves and their Jacobians

In this section we denote by C a smooth projective curve (= compact Riemann surface) of genus g.

4.1 Hodge theory for curves

We first recall briefly Hodge theory for curves, which is much easier than in the general case. We start from the exact sequence of sheaves

$$0 \to \mathbb{C}_C \longrightarrow \mathcal{O}_C \xrightarrow{d} K_C \to 0\ ,$$

where $\mathbb{C}_C$ is the sheaf of locally constant complex functions, and K_C (also denoted Ω^1_C or ω_C) is the sheaf of holomorphic 1-forms. Taking into account $H^0(C,\mathcal{O}_C) = \mathbb{C}$ and $H^1(C,K_C) \cong \mathbb{C}$ (Serre duality), we obtain an exact sequence

$$0 \to H^0(C,K_C) \xrightarrow{\partial} H^1(C,\mathbb{C}) \xrightarrow{p} H^1(C,\mathcal{O}_C) \to 0 \ .$$

By definition $g = \dim H^0(C,K_C)$; by Serre duality we have also $\dim H^1(C,\mathcal{O}_C) = g$, hence $\dim H^1(C,\mathbb{C}) = 2g$.

We put $H^{1,0} := \operatorname{Im}\partial$ and $H^{0,1} := \overline{H^{1,0}}$; $H^{1,0}$ is the subspace of classes in $H^1(C,\mathbb{C})$ which can be represented by holomorphic forms, and $H^{0,1}$ by antiholomorphic forms.

Lemma 4.1. *Let $\alpha \neq 0$ in $H^0(C,K_C)$; then $i\int_C \alpha\wedge\bar\alpha > 0$.*

Proof. Let $z = x+iy$ be a local coordinate in an open subset U of C. We can write $\alpha = f(z)dz$ in U, so that

$$i\int_U \alpha\wedge\bar\alpha = \int_U |f(z)|^2\, i\, dz\wedge d\bar z = \int_U |f(z)|^2\, 2\, dx\wedge dy > 0 \ . \quad \square$$

Proposition 4.2. *$H^1(C,\mathbb{C}) = H^{1,0}\oplus H^{0,1}$; the map p induces an isomorphism $H^{0,1} \to H^1(C,\mathcal{O}_C)$.*

Proof. The second assertion follows from the first and from the above exact sequence. For dimension reasons it suffices to prove that $H^{1,0}\cap H^{0,1} = (0)$. Let $x \in H^{1,0}\cap H^{0,1}$. There exists $\alpha,\beta \in H^0(C,K_C)$ such that $x = [\alpha] = [\bar\beta]$, hence $\alpha - \bar\beta = df$ for some C^∞ function f on C. Then $\beta\wedge\bar\beta = df\wedge\beta = d(f\beta)$, hence $\int_C \beta\wedge\bar\beta = 0$ by Stokes theorem. By the Lemma this implies $\beta = 0$ hence $x = 0$. $\square$

Proposition 4.3. *$p(H^1(C,\mathbb{Z}))$ is a lattice in $H^{0,1}$; the hermitian form H on $H^{0,1}$ defined by $H(\alpha,\beta) := 2i\int_C \bar\alpha\wedge\beta$ induces a principal polarization on the complex torus $H^{0,1}/p(H^1(C,\mathbb{Z}))$.*

Proof. The first assertion has already been proved (Section 2.1). Lemma 4.1 shows that the form H is positive definite on $H^{0,1} = \overline{H^{1,0}}$. Let $a,b \in H^1(C,\mathbb{Z})$; we have

$$a = \bar\alpha + \alpha\ ,\quad b = \bar\beta + \beta \quad \text{with} \quad \alpha = p(a)\ ,\ \beta = p(b)\ .$$

Their cup-product in $H^2(C,\mathbb{Z}) = \mathbb{Z}$ is given by

$$a\cdot b = \int_C (\bar\alpha+\alpha)\wedge(\bar\beta+\beta) = \frac{1}{2i}\big(H(\alpha,\beta) - H(\beta,\alpha)\big) = \operatorname{Im}(H)(\alpha,\beta)\ ;$$

thus $\operatorname{Im}(H)$ induces on $H^1(C,\mathbb{Z})$ the cup-product, which is unimodular by Poincaré duality. $\square$

The g-dimensional abelian variety $JC := H^{0,1}/p(H^1(C,\mathbb{Z}))$ with the principal polarization H is called the *Jacobian* of C; it plays an essential role in the study of the curve.

4.2 Line bundles on C

To study line bundles on C we use again the exact sequence (2.1):

$$0 \to H^1(C,\mathbb{Z}) \xrightarrow{i} H^1(C,\mathcal{O}_C) \longrightarrow \mathrm{Pic}(C) \xrightarrow{c_1} H^2(C,\mathbb{Z}) \cong \mathbb{Z} \to 0 \ .$$

Here for a line bundle L on C, $c_1(L)$ is simply the degree $\deg(L)$ (through the canonical isomorphism $H^2(C,\mathbb{Z}) \cong \mathbb{Z}$): $\deg(L) = \deg(D)$ for any divisor D such that $\mathcal{O}_C(D) \cong L$.

Note that i is the composition of the maps $H^1(C,\mathbb{Z}) \to H^1(C,\mathbb{C}) \xrightarrow{p} H^1(C,\mathcal{O}_C)$ deduced from the inclusions of sheaves $\mathbb{Z}_C \subset \mathbb{C}_C \subset \mathcal{O}_C$. Hence:

Proposition 4.4. *We have an exact sequence* $0 \to JC \longrightarrow \mathrm{Pic}(C) \xrightarrow{\deg} \mathbb{Z} \to 0$. □

Thus JC is identified with $\mathrm{Pic}^{\mathrm{o}}(C)$, the group of isomorphism classes of degree 0 line bundles on C – or the group of degree 0 divisors modulo linear equivalence. More precisely, one can show that JC is a *moduli space* for degree 0 line bundles on C. This means the following. Let S be a complex manifold (or analytic space), and let $\mathcal{L}$ be a line bundle on $C \times S$. For $s \in S$, put $\mathcal{L}_s := \mathcal{L}_{C\times\{s\}}$. We say that $(\mathcal{L}_s)_{s\in S}$ is a *holomorphic family* of line bundles on C parametrized by S. If the line bundles $\mathcal{L}_s$ have degree 0, we get a map $S \to JC$; we want this map to be holomorphic.

In fact we have even more: the line bundles L in JC form a holomorphic family. Namely, there exists a line bundle $\mathcal{P}$ on $C \times JC$ such that $\mathcal{P}_L = L$ for each $L \in JC$. Such a line bundle is called a *Poincaré line bundle*. It is unique up to tensor product by the pullback of a line bundle on JC.

4.3 The Abel-Jacobi maps

As an illustration, choose a divisor D_1 of degree 1 on C and define $\alpha : C \to JC$ by $\alpha(p) = \mathcal{O}_C(p - D_1)$. It is holomorphic (hence algebraic, since both C and JC are projective manifolds): indeed it is defined by the line bundle $\mathcal{O}_{C\times C}(\Delta - p^*D_1)$ on $C \times C$, where Δ is the diagonal and p the first projection.

More generally, let $C^{(d)}$ denote the d-th symmetric power of the curve C, that is, the quotient of C^d by the symmetric group $\mathfrak{S}_d$. This is a *smooth* variety: indeed since this a local question it suffices to prove it for the affine line $\mathbb{C}$; but the map $(z_1,\ldots,z_d) \mapsto (s_1,\ldots,s_d)$, where s_i is the i-th elementary symmetric function of $z_1,\ldots,z_d$, identifies $\mathbb{C}^{(d)}$ to $\mathbb{C}^d$. Using the map $(p_1,\ldots,p_d) \mapsto p_1 + \cdots + p_d$ we will view the elements of $C^{(d)}$ as effective divisors of degree d on C.

Now we choose a divisor D_d of degree d and define a map $\alpha_d : C^{(d)} \to JC$ by $\alpha_d(E) = \mathcal{O}_C(E - D_d)$. Again this is holomorphic, for instance because $\alpha_d(p_1 + \cdots + p_d) = \alpha(p_1) + \cdots + \alpha(p_d)$ up to a constant. For $E \in C^{(d)}$, the fiber $\alpha_d^{-1}(\alpha_d(E))$ is the linear system $|E|$ (Sect. 3.6).

Proposition 4.5. *For $d \le g$ the map α_d is generically injective.*

Proof. By the observation preceding the Proposition we must prove[2] $h^0(E) = 1$ for a general $E \in C^{(d)}$. If D is an effective divisor, we have $h^0(D-p) = h^0(D)-1$

[2] We use the standard notations $h^0(\mathcal{F}) := \dim H^0(C,\mathcal{F})$ for a sheaf $\mathcal{F}$ on C, and $h^0(D) := h^0(\mathcal{O}_C(D))$ for a divisor D.

for p general in C, hence by induction $h^0(D-E) = h^0(D) - d$ for E general in $C^{(d)}$ with $d \le h^0(D)$. Taking $D = K$ gives $h^0(K-E) = g - d$ for $d \le g$, hence $h^0(E) = 1$ by Riemann-Roch. $\square$

Corollary 4.6. *$\alpha_g : C^{(g)} \to JC$ is birational.*

Another consequence is that the image of $\alpha_{g-1} : C^{(g-1)} \to JC$ is a divisor in JC. In fact:

Theorem 4.7 (Riemann)**.** *The image of $\alpha_{g-1} : C^{(g-1)} \to JC$ is a theta divisor of JC.*

We have to refer to [1], p. 23 for the proof.

Remark 4.8. 1) Recall that the map α_{g-1} depends on the choice of a divisor D, or equivalently of the line bundle $L = \mathcal{O}_C(D)$, of degree $g-1$. We will denote by Θ_L the corresponding theta divisor; explicitly:

$$\Theta_L = \{M \in JC \mid H^0(M \otimes L) \neq 0\} .$$

2) There is a way to avoid the inelegant choice of a divisor D_d in the definition of α_d. Let J^d denote the set of isomorphism classes of line bundles of degree d on C. Choosing a line bundle L_d of degree d defines a bijection $JC \to J^d$ (by $M \mapsto M \otimes L_d$). This provides a structure of projective variety on J^d which does not depend on the choice of L_d. By construction J^d is isomorphic to JC, but there is no canonical isomorphism.

Now we have a canonical map $\alpha_d : C^{(d)} \to J^d$ defined simply by $\alpha_d(E) = \mathcal{O}_C(E)$. In particular we have a canonical divisor $\Theta \subset J^{g-1}$, which is the locus of the line bundles L in J^{g-1} with $H^0(L) \neq 0$.

3) A consequence of the Riemann theorem is that the theta divisor is *irreducible*, so a Jacobian cannot be a product of non-trivial p.p.a.v. (Remark 3.10).

5 Vector bundles on curves

As explained in the introduction, generalized theta functions appear when we replace JC, the moduli space of degree 0 line bundles on C, by the analogous moduli spaces for higher rank vector bundles. We will now explain what this means.

5.1 Elementary properties

Let E be a vector bundle on C, of rank r. The maximum wedge power $\wedge^r E$ is a line bundle on C, denoted $\det(E)$. Its degree is denoted by $\deg(E)$. It has the following properties:

- In an exact sequence $0 \to F \to E \to G \to 0$ we have $\det(E) \cong \det(F) \otimes \det(G)$;

- For any line bundle L on C, we have $\det(E\otimes L)=\det(E)\otimes L^{\otimes r}$.
- (Riemann-Roch) $h^0(E)-h^1(E)=\deg(E)+r(1-g)$.

It will be convenient to introduce the *slope* $\mu(E):=\dfrac{\deg(E)}{r}\in\mathbb{Q}$. Thus Riemann-Roch can be written $h^0(E)-h^1(E)=r(\mu(E)+1-g)$.

5.2 Moduli spaces

We have seen that the Jacobian of C parametrizes line bundles of degree 0, in the sense that for any holomorphic family $(\mathcal{L}_s)_{s\in S}$ the corresponding map $S\to JC$ is holomorphic. Unfortunately such a nice moduli space does not exist in higher rank. Indeed we will show the following:

Lemma 5.1. *Let L be a non-trivial line bundle on C with no base point* (*Sect.* 3.6)*. There exists a holomorphic family of vector bundles $(\mathcal{E}_t)_{t\in\mathbb{C}}$ on C such that:*

$$\mathcal{E}_t\cong\mathcal{O}_C\oplus\mathcal{O}_C\quad\textit{for}\quad t\neq 0\qquad \mathcal{E}_0\cong L\oplus L^{-1}\ .$$

Proof. We first take $C=\mathbb{P}^1$ and $L=\mathcal{O}_{\mathbb{P}^1}(2)$. Put $F:=\mathbb{P}^1\times\mathbb{C}$, $\mathcal{O}_F(k):=\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1}(k)$. Consider the homomorphism

$$u:\mathcal{O}_F^{\oplus 3}\longrightarrow\mathcal{O}_F(2)\quad\text{given by}\quad (X^2,Y^2,tXY)\ ,$$

where t is the coordinate on $\mathbb{C}$ and (X,Y) the homogeneous coordinates on $\mathbb{P}^1$. The map u is surjective, so its kernel is a rank 2 bundle $\mathcal{F}$ on $\mathbb{P}^1\times\mathbb{C}$. We claim that

$$\mathcal{F}_t\cong\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}\quad\text{for}\quad t\neq 0\qquad \mathcal{F}_0\cong\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-2)\ .$$

There is a variety of ways to prove this. Perhaps the easiest is to observe that any vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles. Since $\mathcal{F}_t$ is a sub-bundle of $\mathcal{O}_{\mathbb{P}^1}^{\oplus 3}$ with determinant $\mathcal{O}_{\mathbb{P}^1}(-2)$, it is either $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ or $\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-2)$; the first case occurs if and only if $h^0(\mathcal{F}_t)=0$, that is, if and only if $t\neq 0$. Then the vector bundle $\mathcal{F}(1)$ on F has the required properties.

Now we consider the general case. Let s be a nonzero section of L; we can find a section t of L which does no vanish at the zeroes of s. Then (s,t) defines a map $u:C\to\mathbb{P}^1$ such that $u^*\mathcal{O}_{\mathbb{P}^1}(1)=L$. The vector bundle $\mathcal{E}:=(u,\mathrm{Id})^*\mathcal{F}(1)$ on $C\times\mathbb{C}$ has the required properties. □

This implies that there is no reasonable moduli space $\mathcal{M}$ containing both $\mathcal{O}_C^{\oplus 2}$ and $L\oplus L^{-1}$: the family constructed in the lemma would give rise to a holomorphic map $\mathbb{C}\to\mathcal{M}$ mapping $\mathbb{C}\smallsetminus\{0\}$ to a point, and 0 to a different point. There are two ways to deal with this problem. The sophisticated one, which we will not discuss here, replaces moduli spaces by a more elaborate notion called *moduli stacks*. The reader interested by this point of view may look at [12].

Instead we will follow the classical (by now) approach, which eliminates certain vector bundles, for instance those of the form $L\oplus L^{-1}$ which appear in the lemma; this is done as follows:

Definition 5.2. *A vector bundle E on C is* stable *if $\mu(F) < \mu(E)$ for every sub-bundle $0 \subsetneq F \subsetneq E$. It is* polystable *if it is a direct sum of stable sub-bundles of slope $\mu(E)$.*

Theorem 5.3. *There exists a moduli space $\mathcal{M}^s(r,d)$ for stable vector bundles of rank r and degree d. It is a smooth connected quasi-projective manifold; it admits a projective compactification $\mathcal{M}(r,d)$ whose points correspond to isomorphism classes of polystable bundles.*

Note that we do not claim that $\mathcal{M}(r,d)$ is a moduli space for polystable bundles; the situation is more complicated. We refer to [19] for a precise statement as well as the proof.

An important by-product of the proof is the fact that stability is an open condition: if S is a variety and $\mathcal{E}$ a vector bundle on $C \times S$, the set of $s \in S$ such that $\mathcal{E}_s$ is stable is Zariski open in S ([19], Prop. 7.2.6).

5.3 The moduli space $\mathcal{M}(r)$

We will in fact focus on a slightly different moduli space. The map $\det : \mathcal{M}(r,d) \to J^d$ which associates to a vector bundle its determinant is holomorphic. Let L be a line bundle of degree d; the fiber $\det^{-1}(L)$ is denoted $\mathcal{M}(r,L)$. We denote by J_r the subgroup (isomorphic to $(\mathbb{Z}/r\mathbb{Z})^{2g}$) of line bundles $\alpha \in JC$ with $\alpha^{\otimes r} \cong \mathcal{O}_C$.

Proposition 5.4. *The map $\mathcal{M}(r,L) \times JC \to \mathcal{M}(r,d)$ given by $(E,\lambda) \mapsto E \otimes \lambda$ identifies $\mathcal{M}(r,d)$ with the quotient of $\mathcal{M}(r,L) \times JC$ by J_r acting by $\alpha \cdot (E,\lambda) = (E \otimes \alpha, \lambda \otimes \alpha^{-1})$.*

Proof. Let E in $\mathcal{M}(r,d)$. The pairs (F,λ) with $F \in \mathcal{M}(r,L)$, $\lambda \in JC$ and $E \cong F \otimes \lambda$ are obtained by taking $\lambda \in JC$ with $\lambda^{\otimes r} = \det(E) \otimes L^{-1}$ and $F = E \otimes \lambda^{-1}$. We can always find such a λ, hence a pair (F,λ), and two such pairs differ by the action of J_r. $\square$

Thus $\mathcal{M}(r,d)$ is determined by JC and $\mathcal{M}(r,L)$; from now on we will focus on the latter space. Note that for $N \in \mathrm{Pic}(C)$ the map $E \mapsto E \otimes N$ induces an isomorphism $\mathcal{M}(r,L) \xrightarrow{\sim} \mathcal{M}(r, L \otimes N^{\otimes r})$; thus up to isomorphism, $\mathcal{M}(r,L)$ depends only of the degree d of L (mod. r). When r and d are coprime $\mathcal{M}(r,L) = \mathcal{M}^s(r,L)$ is smooth, and is a nice moduli space; however the most interesting case for us will be $d = 0$, and the moduli space $\mathcal{M}(r,\mathcal{O}_C)$, which we will denote simply $\mathcal{M}(r)$. This is also the moduli space of principal $\mathrm{SL}(r)$-bundles, so its study fits into the more general theory of principal G-bundles for a semisimple group G.

Let us summarize in the next Proposition some elementary properties of $\mathcal{M}(r)$, which follow from its construction (see [19]). From now on we will assume that the genus g of C is ≥ 2 (for $g \leq 1$ there are no stable bundles of degree 0 and rank > 1).

Proposition 5.5. *$\mathcal{M}(r)$ is a projective normal irreducible variety, of dimension $(r^2-1)(g-1)$, with mild singularities (so-called* rational *singularities). Except when $r = g = 2$, its singular locus is the locus of non-stable bundles.*

As algebraic varieties, the moduli spaces $\mathcal{M}(r,L)$ are very different from complex tori:

Proposition 5.6. *The moduli space* $\mathcal{M}(r,L)$ *is* unirational*; that is, there exists a rational dominant map*[3] $\mathbb{P}^N \dashrightarrow \mathcal{M}(r,L)$.

Proof . Using the isomorphism $\mathcal{M}(r,L) \xrightarrow{\sim} \mathcal{M}(r, L\otimes N^{\otimes r})$ we may assume $\deg(L) > r(2g-1)$, so $\mu(E) > 2g-1$ for $E \in \mathcal{M}(r,L)$. Since E is polystable this implies $H^0(E^* \otimes K_C(p)) = 0$ for any $p \in C$, hence by Serre duality $H^1(E(-p)) = 0$. Then the exact sequence

$$0 \to E(-p) \to E \to E_p \to 0$$

gives for each p a surjection $\mathrm{ev}_p : H^0(E) \twoheadrightarrow E_p$; that is, the global sections of E generate E at p.

Now we claim that a general subspace of dimension $r+1$ of $H^0(E)$ still generates E at each point. For $p \in C$, let Z_p be the subvariety of the Grassmannian $\mathbb{G}(r+1, H^0(E))$ consisting of subspaces V which do not span E_p. This is equivalent to $\dim V \cap \mathrm{Ker}(\mathrm{ev}_p) \geq 2$, so Z_p has codimension 2 (exercise!). Thus $Z = \cup_{p\in C} Z_p$ has codimension 1 in the Grassmannian; any V in the complement of Z generates E at each point. For such a V the evaluation map $V \otimes \mathcal{O}_C \to E$ is surjective. Its kernel is a line bundle; taking determinants we see that it is L^{-1}. Thus E^* is the kernel of a surjective map $V^* \otimes \mathcal{O}_C \to L$.

Conversely, let $\mathbb{G}_0$ be the open subset of the Grassmannian $\mathbb{G}(r+1, H^0(L))$ parametrizing subspaces which span L at each point. For $W \in \mathbb{G}_0$, we have an exact sequence

$$0 \to F_W \longrightarrow W \otimes_{\mathbb{C}} \mathcal{O}_C \xrightarrow{\mathrm{ev}} L \to 0\ ;$$

The dual $E_W := F_W^*$ is a rank r vector bundle with determinant L; we obtain in this way an algebraic family of such bundles, parametrized by $\mathbb{G}_0$, such that every element of $\mathcal{M}^s(r,L)$ appears in the family. The subspaces $W \in \mathbb{G}_0$ such that E_W is stable form a Zariski open subset $\mathbb{G}_1 \subset \mathbb{G}_0$ (Sect. 5.2), and we have a surjective map $f : \mathbb{G}_1 \to \mathcal{M}^s(r,L)$ such that $f(W) = E_W$. Since Grassmannians are rational varieties, composing f with a birational map $\mathbb{P}^N \dashrightarrow \mathbb{G}_1$ gives the required rational dominant map. □

Corollary 5.7. *Any rational map from* $\mathcal{M}(r,L)$ *to a complex torus is constant.*

Proof . Let $T = V/\Gamma$ be a complex torus. In view of the proposition, it suffices to show that any rational map $\varphi : \mathbb{P}^N \dashrightarrow T$ is constant. Let p,q be two general points of $\mathbb{P}^N$. The restriction of φ to the line $\langle p,q\rangle$ defines a map $\mathbb{P}^1 \to T$, which factors through V since $\mathbb{P}^1$ is simply connected, hence is constant. Thus $\varphi(p) = \varphi(q)$. □

[3]In the rest of this section we assume some familiarity with the notion of rational maps – see e.g. [13], p. 490.

5.4 Rationality

The *Lüroth problem* asks whether an unirational variety X is necessarily rational. The answer is positive when X is a curve (Lüroth, 1876) or a surface (Castelnuovo, 1895), but not in higher dimension (see for instance [9]).

While $\mathcal{M}(r,L)$ is known to be rational when $\deg(L)$ is prime to r [17], the rationality of $\mathcal{M}(r)$ is an open problem, already for $r=2$ and $g=3$ – despite the fact that in this case we have an explicit description of $\mathcal{M}(2)$ as a quartic hypersurface in $\mathbb{P}^7$ (Sect. 6.5).

6 Generalized theta functions

6.1 The theta divisor

Since $\mathcal{M}(r)$ is simply connected, there is no hope to describe its line bundles by systems of multipliers as for complex tori. However we may try to mimic the definition of the theta divisor: for $L \in J^{g-1}$, we put

$$\Delta_L := \{E \in \mathcal{M}(r) \mid H^0(E \otimes L) \neq 0\}.$$

Theorem 6.1 ([10])**.** 1) *Δ_L is a Cartier divisor on $\mathcal{M}(r)$.*

2) *The line bundle $\mathcal{L} = \mathcal{O}(\Delta_L)$ is independent of L, and* $\mathrm{Pic}(\mathcal{M}(r)) = \mathbb{Z}[\mathcal{L}]$.

Recall that an effective Cartier divisor is a subvariety locally defined by an equation – or, globally, as the zero locus of a section of a line bundle. On a singular variety (as is $\mathcal{M}(r)$) this is stronger than having codimension 1.

Proof . We will only show why Δ_L is a divisor on the stable locus $\mathcal{M}^s(r)$, referring to [10] for the rest of the proof. It is a consequence of the following lemma:

Lemma 6.2. *Let S be a complex variety, $(\mathcal{E}_s)_{s\in S}$ a family of vector bundles on C, with $\mu(\mathcal{E}_s) = g-1$ for all $s \in S$. Then the locus*

$$\{s \in S \mid H^0(C, \mathcal{E}_s) \neq 0\}$$

is defined locally by one equation (possibly trivial).

Proof . We will use a general fact about cohomology of coherent sheaves (see [20], §5) : locally on S there exist vector bundles F, G and a homomorphism $u : F \to G$ such that we have for each s in S an exact sequence

$$0 \to H^0(C, \mathcal{E}_s) \longrightarrow F(s) \xrightarrow{u(s)} G(s) \longrightarrow H^1(C, \mathcal{E}_s) \to 0 \ .$$

By Riemann-Roch we have $h^0(\mathcal{E}_s) = h^1(\mathcal{E}_s)$, hence F and G have the same rank. We see that $H^0(C, \mathcal{E}_s) \neq 0$ if and only if $\det(u(s)) = 0$, that is, the section $\det(u)$ of $\det(G) \otimes \det(F)^{-1}$ vanishes at s, hence the lemma. □

Coming back to $\mathcal{M}(r)$, the construction of the moduli space implies that locally for the complex topology, there is a “Poincaré bundle”, that is a rank r

vector bundle $\mathcal{E}$ on $C \times V$ such that $\mathcal{E}_{|C\times\{E\}} \cong E$ for E in V. Applying the lemma to $\mathcal{E} \otimes L$ shows that Δ_L is a divisor on $\mathcal{M}^s(r)$, unless $\Delta_L = \mathcal{M}(r)$. But this cannot hold: if α is a general element of JC, we have $H^0(L \otimes \alpha) = 0$, hence $\alpha^{\oplus r} \notin \Delta_L$. □

6.2 Generalized theta functions

By analogy with the case of Jacobians, the sections of $H^0(\mathcal{M}(r), \mathcal{L}^{\otimes k})$ are called *generalized* (or *non-abelian*) *theta functions* of order k. They are associated to the group $\mathrm{SL}(r)$ (there are more general theta functions associated to each complex reductive group, but we will not discuss them in these notes).

Like for complex tori, the first question we can ask about these theta functions is the dimension of the space $H^0(\mathcal{M}(r), \mathcal{L}^{\otimes k})$. The answer, much more intricate than Theorem 3.5 for complex tori, is known as the *Verlinde formula*; it has been first found by E. Verlinde using physics arguments, then proved mathematically in many different ways – see e.g. [29]. The formula is as follows:

$$\dim H^0(\mathcal{M}(r), \mathcal{L}^{\otimes k}) = \Big(\frac{r}{r+k}\Big)^g \sum_{\substack{S \amalg T = [1, r+k] \\ |S| = r}} \prod_{\substack{s\in S \\ t \in T}} \Big|2 \sin \pi \frac{s-t}{r+k}\Big|^{g-1} \tag{6.1}$$

For $r = 2$ it reduces (after some trigonometric manipulations) to:

$$\dim H^0(\mathcal{M}(2), \mathcal{L}^{\otimes k}) = (\frac{k}{2}+1)^{g-1} \sum_{i=1}^{k+1} \frac{1}{(\sin \frac{i\pi}{k+2})^{2g-2}} \ .$$

Even in rank 2, it is not at all obvious that the right hand side is an integer!

6.3 Linear systems and rational maps in $\mathbb{P}^n$

This section is the logical continuation of Section 3.6; we again assume some familiarity with the notion of rational map ([13], p. 490). We keep our projective variety M and a line bundle L on M; we do not assume $B(L) = \varnothing$. We still have a map $M \smallsetminus B(L) \to |L|^*$, which we see as a rational map $\varphi_L : M \dashrightarrow |L|^*$.

Conversely, suppose given a rational map φ of M to a projective space $\mathbb{P}(V)$. We assume that M is *normal*; then the indeterminacy locus B of φ has codimension ≥ 2. We assume moreover that the line bundle $\varphi^*\mathcal{O}_{\mathbb{P}(V)}(1)$ on $M \smallsetminus B$ extends to a line bundle L on M. By Hartogs theorem the restriction map $H^0(M, L) \to H^0(M \smallsetminus B, L)$ is bijective, so we get a pullback homomorphism $\varphi^* : V^* \to H^0(M, L)$. We have a commutative diagram

$$\begin{tikzcd} & {|L|^*} \arrow[dd, dashed, "\mathbb{P}({}^t\varphi^*)"] \\ M \arrow[ur, dashed, "\varphi_L"] \arrow[dr, dashed, "\varphi"'] & \\ & \mathbb{P}(V) \end{tikzcd}$$

Indeed for m general in M, $\varphi_L(m)$ is the hyperplane of $|L|$ formed by the divisors passing through m; its image under $\mathbb{P}({}^t\varphi^*)$ is the hyperplane of $\mathbb{P}(V)^*$ formed by the hyperplanes of $\mathbb{P}(V)$ passing through $\varphi(m)$, and this corresponds by duality to the point $\varphi(m) \in \mathbb{P}(V)$.

6.4 The theta map

We go back to our moduli space $\mathcal{M}(r)$ and the generator $\mathcal{L}$ of its Picard group. The next step is to ask for the map defined by the linear systems $|\mathcal{L}^{\otimes k}|$. In fact we will concentrate on the simplest one, namely $\varphi_{\mathcal{L}}$. Our task will be to give a geometric description of this map. In order to do this we associate to each vector bundle $E \in \mathcal{M}(r)$ the locus

$$\theta(E) := \{L \in J^{g-1} \mid H^0(E \otimes L) \neq 0\}$$

Proposition 6.3. *$\theta(E)$ is either equal to J^{g-1}, or is a divisor in J^{g-1}, belonging to the linear system $|r\Theta|$.*

Proof. Consider the vector bundle $E \otimes \mathcal{P}$ on $C \times J^{g-1}$, where $\mathcal{P}$ is a Poincaré line bundle (Section 4.2). It defines the family of vector bundles $(E \otimes L)_{L \in J^{g-1}}$ on C. These bundles have slope $g-1$, hence we can apply Lemma 6.2, which shows that $\theta(E)$ is defined locally by one (possibly trivial) equation.

Let S be an irreducible variety, and $\mathcal{E}$ a vector bundle on $C \times S$, with $\deg(\mathcal{E}_s) = 0$ for each s. Lemma 6.2, applied to the vector bundle $\mathcal{E} \otimes \mathcal{P}$ on $C \times S \times J^{g-1}$, gives a line bundle $\mathcal{N}$ on $J^{g-1} \times S$ and a section τ of $\mathcal{N}$ with zero locus $Z = \cup_s \theta(\mathcal{E}_s)$. Put

$$S^{\mathrm{o}} := \{s \in S \mid \theta(\mathcal{E}_s) \neq J^{g-1}\} ;$$

S^{o} is the projection on S of the complement of Z in $J^{g-1} \times S$, so it is a Zariski open subset of S.

Applying this locally to our moduli space $\mathcal{M}(r)$, we see that the vector bundles $E \in \mathcal{M}(r)$ with $\theta(E) = J^{g-1}$ form a closed analytic (and therefore algebraic) subset of $\mathcal{M}(r)$. Let $\mathcal{M}(r)^{\mathrm{o}}$ be the complement of this subset. When E runs through $\mathcal{M}(r)^{\mathrm{o}}$, the Chern class $c_1(\theta(E))$ is constant. So if we fix $E_0 \in \mathcal{M}(r)^{\mathrm{o}}$, we have a rational map $\mathcal{M}(r) \dashrightarrow \mathrm{Pic}^{\mathrm{o}}(J^{g-1})$ mapping $E \in \mathcal{M}(r)^{\mathrm{o}}$ to $\mathcal{O}_J(\theta(E) - \theta(E_0))$. By Corollary 5.7 this map is constant, hence $\mathcal{O}_J(\theta(E))$ is independent of E.

Let $\alpha_1, \ldots, \alpha_r$ be distinct elements of JC. We have

$$\theta(\alpha_1 \oplus \cdots \oplus \alpha_r) = t^*_{\alpha_1}\Theta + \cdots + t^*_{\alpha_r}\Theta \in |r\Theta|$$

(see Corollary 2.11). Thus whenever $\theta(E) \neq J^{g-1}$ we have $\theta(E) \in |r\Theta|$. □

Thus we have a rational map $\theta : \mathcal{M}(r) \dashrightarrow |r\Theta|$.

Theorem 6.4 ([6])**.** *There is a natural isomorphism*

$$H^0(\mathcal{M}(r), \mathcal{L}) \xrightarrow{\sim} H^0(J^{g-1}, \mathcal{O}(r\Theta))^*$$

making the following diagram commutative:

$$\begin{tikzcd} & {|\mathcal{L}|^*} \arrow[dd, "\wr"] \\ \mathcal{M}(r) \arrow[ur, dashed, "\varphi_{\mathcal{L}}"] \arrow[dr, dashed, "\theta"'] & \\ & {|r\Theta|} \end{tikzcd}$$

Sketch of proof : For $L \in J^{g-1}$, let H_L be the hyperplane in $|r\Theta|$ consisting of the divisors passing through L. By definition the pullback of H_L under θ is the divisor Δ_L. Thus, as explained in Section 6.3, we get a commutative diagram

$$\begin{tikzcd} & {|\mathcal{L}|^*} \arrow[dd, dashed, "\lambda"] \\ \mathcal{M}(r) \arrow[ur, dashed, "\varphi_{\mathcal{L}}"] \arrow[dr, dashed, "\theta"'] & \\ & {|r\Theta|} \end{tikzcd}$$

with $\lambda := \mathbb{P}({}^t\theta^*)$. It remains to prove that λ is bijective. Surjectivity is not difficult, let us prove it in the case $r = 2$. If λ is not surjective, the image of θ is contained in a hyperplane of $|2\Theta|$. But this image contains all the divisors $\theta(\alpha \oplus \alpha^{-1}) = t_\alpha^*\Theta + t_{-\alpha}^*\Theta$ for $\alpha \in JC$; and Corollary 3.13 implies that these divisors span $|2\Theta|$ (otherwise the image of $\varphi_{2\Theta}$ would be contained in a hyperplane).

We have $\dim|r\Theta| = r^g - 1$ by Theorem 3.5, so the crucial point is to prove the same equality for $\dim|\mathcal{L}|$. Of course this follows (in a non-trivial way) from the Verlinde formula (6.1); in [6], since the Verlinde formula was not yet available, we constructed a rational dominant map from a certain abelian variety to the moduli space, and applied Theorem 3.5 to get the result. $\square$

Corollary 6.5. *The base locus of the linear system $|\mathcal{L}|$ on $\mathcal{M}(r)$ is the set of vector bundles $E \in \mathcal{M}(r)$ such that $\theta(E) = J^{g-1}$.* $\square$

Thus the rather mysterious map $\varphi_{\mathcal{L}}$ is identified with the more concrete map θ; one usually refers to θ, or $\varphi_{\mathcal{L}}$, as the *theta map*. We will now see that this explicit description allows a good understanding of the theta map in the rank 2 case.

6.5 Rank 2

In rank 2 the theta map is by now fairly well understood. We summarize what is known in one theorem:

Theorem 6.6. 1) *The theta map* $\theta : \mathcal{M}(2) \to |2\Theta|$ *is a morphism.*

2) *If* C *is not hyperelliptic or* $g = 2$, θ *is an embedding.*

3) *If* C *is hyperelliptic of genus* ≥ 3, θ *is* 2*-to-*1 *onto its image in* $|2\Theta|$, *and this image admits an explicit description.*

This is the conjunction of various results. Part 1) is due to Raynaud [28], part 3) to Bhosle-Ramanan [11]. In case 2), the fact that θ is generically injective was proved in [2]; from this Brivio and Verra deduced that θ embeds $\mathcal{M}^s(2)$, and this was extended to $\mathcal{M}(2)$ in [14].

Recall that $\mathcal{M}(2)$ has dimension $3g-3$. In particular:

Corollary 6.7 ([23])**.** *For* $g = 2$, $\theta : \mathcal{M}(2) \to |2\Theta| \cong \mathbb{P}^3$ *is an isomorphism.* □

Consider the map $k : JC \to \mathcal{M}(2)$ given by $k(L) = L \oplus L^{-1}$. The composition $\theta \circ k$ is the map κ studied in Section 3.8; thus k embeds the Kummer variety K of JC into $\mathcal{M}(2)$, and the restriction of θ to K is the natural embedding of K into $|2\Theta|$. For $g > 2$ K is the singular locus of $\mathcal{M}(2)$ (Proposition 5.5); when C is not hyperelliptic, we obtain a variety in $|2\Theta|$ which is singular along the Kummer variety.

For $g = 3$ and C not hyperelliptic, a very nice application appears in [24]. In that case $\dim \mathcal{M}(2) = 6$, so θ embeds $\mathcal{M}(2)$ as a hypersurface in $|2\Theta| \cong \mathbb{P}^7$. It is not difficult to prove that it has degree 4 (for instance by computing its canonical bundle). Now Coble had found long ago that there is a *unique* quartic hypersurface in $|2\Theta|$ which is singular along the Kummer variety, for which he had written down an explicit equation (see [3] for a modern account). Therefore this hypersurface is $\mathcal{M}(2)$.

We will illustrate the methods used to prove the above results by giving the proof of 1).

6.6 Raynaud's theorem

We will prove part 1) of Theorem 6.6 in the following form:

Proposition 6.8. *Let* $E \in \mathcal{M}(2)$. *Then* $\theta(E) \neq J^{g-1}$.

Proof . If $E = L \oplus L^{-1}$, we have $\theta(E) = \Theta_L + \Theta_{L^{-1}} \neq J^{g-1}$. Therefore we may assume that E is stable.

Suppose $\theta(E) = J^{g-1}$. Put $F = E \otimes L$ for some L in J^{g-1}; our hypothesis becomes $h^0(F \otimes \alpha) > 0$ for all $\alpha \in JC$. Put $h := \min_{\alpha \in JC} h^0(F \otimes \alpha)$. Replacing F by $F \otimes \alpha$ for an appropriate α we may assume $h^0(F) = h$. We will use the semi-continuity theorem in cohomology, which implies that there is a Zariski open subset $U \subset JC$ (containing 0) such that $h^0(F \otimes \alpha) = h$ for $\alpha \in U$ ([20], §5).

Put $F' := F^* \otimes K_C$; let $p \in C$. The Riemann-Roch theorem gives

$$\big(h^0(F(p)) - h^0(F)\big) + \big(h^0(F') - h^0(F'(-p)\big) = 2 \ .$$

For p general we have $h^0(F') - h^0(F'(-p) \geq 1$, hence $h^0(F(p)) - h^0(F) \leq 1$. But $h^0(F(p)) = h$ would imply $h^0(F(p-q)) < h$ for q general, contradicting the definition of h. Thus $h^0(F(p)) = h + 1$.

Put $G := F(p)$. We have $h^0(G) = h+1$, and $h^0(G(-q)) = h^0(F(p-q)) = h$ for q general in C. From the exact sequence

$$0 \to G(-q) \to G \to G_q \to 0$$

we see that the global sections of G generate a rank 1 subsheaf L_0 of G. This is not necessarily a sub-line bundle because the quotient G/L_0 may have torsion; but it is contained in a unique sub-line bundle L, the kernel of the projection from G to the torsion-free quotient of G/L (sometimes called the *saturation* of L_0 in G). This sub-line bundle L has the property that *any rank* 1 *subsheaf* $M \subset G$ *with* $h^0(M) > 0$ *is contained in* L. Indeed if s is a nonzero section of M, at a general point x of C $s(x)$ generates M_x and L_x; therefore the map $M \to G/L$ is zero generically, hence everywhere.

Now take q, r general in C and consider $G(q-r)$. As before its global sections generate a sub-line bundle $L' \subset G(q-r)$. But we have $L(q-r) \subset G(q-r)$, and $h^0(L(q-r)) > 0$ since $h^0(L) = h^0(G) \geq 2$. Hence $L(q-r) \subset L'$. But symmetrically we have $L'(r-q) \subset L$, hence $L(q-r) = L'$. In particular we find $h^0(L(q-r)) = h = h^0(L)$ for q, r general in C. This implies $h^0(L(q)) = h^0(L)+1$, hence by Riemann-Roch $h^0(K \otimes L^{-1}) = h^0(K \otimes L^{-1}(-q))$ for q general; this is possible only if $h^0(K \otimes L^{-1}) = 0$. Applying again Riemann-Roch and using $h^0(L) \geq 2$, we get $\deg(L) \geq g+1 > \mu(G)$, a contradiction. □

6.7 Higher rank

In contrast with the rank 2 case, not much is known in higher rank. It is known since [28] that there exist stable bundles E with $\theta(E) = J^{g-1}$ – that is, base points for the linear system $|\mathcal{L}|$; in fact, they exist as soon as $r \geq g+2$, and even $r \geq 4$ if C is hyperelliptic [26]. On the other hand, in rank 3 there are no base points for $g = 2$ [28], $g = 3$ [4], or if C is general enough [28].

The situation is somewhat particular when $g = 2$, since $\dim \mathcal{M}(r) = \dim |r\Theta| = r^2 - 1$.

Proposition 6.9. *Let* $g = 2$*.*

1) $\theta : \mathcal{M}(r) \dashrightarrow |r\Theta|$ *is generically finite.*
2) *Its degree is* 1 *for* $r = 2$*,* 2 *for* $r = 3$*,* 30 *for* $r = 4$*.*

Part 1) is proved in [4]. The rank 2 case has been discussed in Corollary 6.7. In rank three $\theta : \mathcal{M}(3) \to |3\Theta| \cong \mathbb{P}^8$ is a double covering, branched along a sextic hypersurface which can be explicitly described [25]. The case $r = 4$ is due to Pauly [27].

Let us conclude with a

Conjecture 6.10. *For* $g \geq 3$*, the theta map* $\theta : \mathcal{M}(r) \dashrightarrow |r\Theta|$ *is generically* 2*-to-*1 *onto its image if* C *is hyperelliptic, and generically injective otherwise.*

This is unknown even for $r = g = 3$.

6.8 Further reading

There are a number of topics which I would have liked to cover in these lectures but could not by lack of time. Here are a few of them, with references to the literature:

- The beautiful interplay between curves and their Jacobians: Torelli theorem, Schottky problem, etc. A nice overview can be found in [22]; some of the topics are developed in [1].
- The heat equation and its extension to generalized theta functions. The original paper [16] of Hitchin is of course somewhat advanced, but still quite readable.
- Higgs bundles. Though not directly related to generalized theta functions, this is an important subject with many applications. Here again one can look at the original paper [15] of Hitchin; see also [7] for a short introduction.
- Principal bundles. This amounts to replace the group $\mathrm{SL}(r)$ by any semi-simple group. Essentially all we have said extends to this set-up. There are few results on the theta map, see [5] for the orthogonal and symplectic groups.

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