Averaging lemmas with a force term in the transport equation

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Abstract

We obtain several averaging lemmas for transport operator with a force term. These lemmas improve the regularity yet known by not considering the force term as part of an arbitrary right-hand side. We compare the obtained regularities according to the space and velocity variables. Our results are mainly in L^2, and for constant force, in L^p for $1 < p \leq 2$.

Key-words: averaging lemma – force term – kinetic equation – stationary phase – Fourier series – Hardy space

Mathematics Subject Classification:

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1 Introduction

Averaging lemma is a major tool to get compactness from a kinetic equation. ([7], ...). Such results have been used in a great number of papers during these last years. Among this literature, an important result which used an averaging lemma as a key argument is the proof of the hydrodynamic limits of the Boltzmann or BGK equations to the incompressible Euler or Navier-Stokes equations ([13]). Another important application is to get compactness for nonlinear scalar conservation laws in [21] which allows, for instance, to study propagation of high frequency waves ([6]).
Basically, averaging lemma is a result which says that the macroscopic quantities $\int f(t,x,v)\psi(v)\,dv$ have a better regularity with respect to (t,x) than the microscopic quantity $f(t,x,v)$ where f is solution of a kinetic equation.
For example, in [9] and [2], the following result is proved

Theorem [DiPerna, Lions, Meyer – Bézard] *Let f, $g_k \in L^p(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^M)$ with $1 < p \leq 2$ such that*

$$\partial_t f + \mathrm{div}_x[a(v)f] = \sum_{|k|\leq m} \partial_v^k g_k, \tag{1.1}$$

with $a \in W^{m,\infty}(\mathbb{R}^M, \mathbb{R}^N)$ for $m \in \mathbb{N}$. Let $\psi \in W^{m,\infty}(\mathbb{R}^M)$ with compact support. Let $A > 0$ such that the support of ψ is included in $[-A, A]^M$. We assume the following non-degeneracy for a: there exists $0 < \alpha \leq 1$ and $C > 0$ such that for any $(u,\sigma) \in S^N$ and $\varepsilon > 0$,

$$meas\ \Big(\{v \in [-A,A]^M\ ;\ u - \varepsilon < a(v)\cdot\sigma < u + \varepsilon\}\Big) \leq C\varepsilon^\alpha. \tag{1.2}$$

Then $\rho_\psi(t,x) = \displaystyle\int_{\mathbb{R}^M} f(t,x,v)\psi(v)\,dv$ is in $W^{s,p}(\mathbb{R}_t \times \mathbb{R}_x^N)$ where $s = \frac{\alpha}{(m+1)p'}$, p' being the conjugated exponent for p.

For the equation (1.1), the obtained regularity is proved to be optimal, see [19] and [20]. In [11], the gain of a half-derivative in the L^2 context was proved as optimal. A study in the case of a full derivative in x in the second member is done in [17]. We also refer to [10] and [4] for other results about averaging lemmas. Regularity of f itself is also challenging, for example by assuming some regularity in v, see [3], [14] and [1] for some results in this way.

The previous Theorem says for example with $m = 1$ that for the equation

$$\partial_t f + a(v) \cdot \nabla_x f = g - F(t, x, v) \cdot \nabla_v \tilde{g}, \tag{1.3}$$

the obtained regularity is $W^{s,p}(\mathbb{R}_t \times \mathbb{R}_x^N)$ with $s = \frac{\alpha}{2p'}$. When we consider the equation

$$\partial_t f + a(v) \cdot \nabla_x f + F(t, x, v) \cdot \nabla_v f = g, \tag{1.4}$$

that is to say that $\tilde{g} = f$, it is classical to consider the term $F(t, x, v) \cdot \nabla_v f$ as part of the right hand side and to have the regularity $W^{s,p}(\mathbb{R}_t \times \mathbb{R}_x^N)$ with $s = \frac{\alpha}{2p'}$. But for (1.4), the derivation with respect to v leads only on f via the transport equation and not on any disconnected second order term $\tilde{g}$. That is to say, we lose some information because this term is part of characteristics and in the right-hand the term are in L^2 thus the right-hand side would be for $m = 0$ and the obtained regularity should be $W^{s,p}(\mathbb{R}_t \times \mathbb{R}_x^N)$ with $s = \frac{\alpha}{p'}$. This is the first motivation of this paper and one of the result we get.
The notations for (1.4) are $f(t, x, v) \in \mathbb{R}$ with $t \in \mathbb{R}$, $x \in \mathbb{R}^N$, $v \in \mathbb{R}^M$, $a : \mathbb{R}^M \to \mathbb{R}^N$, $F : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^M$ and

$$a(v) \cdot \nabla_x f = \sum_{i=1}^{N} a_i(v)\, \partial_{x_i} f, \quad F(t, x, v) \cdot \nabla_v f = \sum_{i=1}^{M} F_i(t, x, v)\, \partial_{v_i} f.$$

In this paper, we will prove the following averaging lemmas on equation (1.4).

Theorem 1 (L^2 result)
Let $a \in C^{N+3}(\mathbb{R}_v^M, \mathbb{R}_x^N)$, $F \in C^{N+3}(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^M, \mathbb{R}_v^M)$, $f, g \in L^2(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^M)$, satisfying (1.4). Let $A > 0$ and $\psi \in C_c^{N+2}(\mathbb{R}_v^M)$ such that the support of ψ is included in $[-A, A]^M$. We assume that there exists $0 < \alpha \leq 1$ and $C > 0$ such that for any $(u, \sigma) \in S^N$ and $\varepsilon > 0$,

$$meas\ \Big(\{v \in [-A, A]^M\, ;\, u - \varepsilon < a(v) \cdot \sigma < u + \varepsilon\}\Big) \leq C\varepsilon^\alpha. \tag{1.5}$$

Then the averaging

$$\rho_\psi(t, x) = \int_{\mathbb{R}^M} f(t, x, v)\psi(v)\, dv$$

is in $H_{loc}^{\alpha/2}(\mathbb{R}_t \times \mathbb{R}_x^N)$.

Remark 1.1 We notice that we well obtain $\alpha/2$ instead of $\alpha/4$.

Remark 1.2 For Vlasov equation, the classical application of averaging lemma is: the DiPerna, Lions, Meyer Theorem gives the compactness for ρ_ψ with an operator of the kind (1.4) by applying the result with $g_1 = -F \cdot f$ when $F \in L_{loc}^\infty$. More precisely, if f^n, g_0^n and $g_1^n = -F_n \cdot f^n$ are solutions of (1.1) *with some bounds in L^p, then ρ_ψ^n is bounded in $W^{s,p}(\mathbb{R}_t \times \mathbb{R}_x^N)$ with $s = \dfrac{\alpha}{2p'}$, and thus is compact in $W^{s',p}(\mathbb{R}_t \times \mathbb{R}_x^N)$ with $s' < s$. For $p = 2$, it is compact in $H^{s'}(\mathbb{R}_t \times \mathbb{R}_x^N)$ with $s' < \frac{\alpha}{4}$. By this way, the paper [8] prove the existence of weak solutions for Vlasov-Maxwell. With the first Theorem of this paper, the obtained compactness is in $H^{s'}_{loc}(\mathbb{R}_t \times \mathbb{R}_x^N)$ with $s' < \frac{\alpha}{2}$.*

When the force is constant, we obtain a global regularity result with a less smooth test function.

Theorem 2 (L^2 result with F constant)
Let $a \in C^\gamma(\mathbb{R}_v^M, \mathbb{R}_x^N)$, $F(t,x,v) = F \in \mathbb{R}^M$, $F \neq 0$, f, $g \in L^2(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^M)$ satisfying (1.4) where we assume that the function a satisfies the following condition with γ, which is a positive integer, such that
$\forall (v,\sigma) \in \mathbb{R}^M \times S^N, \quad \sigma = (\sigma_0, \sigma_1, \cdots, \sigma_N), \quad \widetilde{\sigma} = (\sigma_1, \cdots, \sigma_N),$

$$|\sigma_0 + a(v).\widetilde{\sigma}| + \sum_{k=1}^{\gamma-1} \left|(F \cdot \nabla_v)^k a(v) \cdot \widetilde{\sigma}\right| > 0. \qquad (\gamma ND) \qquad (1.6)$$

Let $\psi \in C_c^1(\mathbb{R}_v^M)$, then the averaging

$$\rho_\psi(t,x) = \int_{\mathbb{R}^M} f(t,x,v)\psi(v)\,dv$$

is in $H^{1/\gamma}(\mathbb{R}_t \times \mathbb{R}_x^N)$.

Remark 1.3 The proof is not valid when $F = 0$.

Remark 1.4 [$M = 1$**, one dimensional velocity**]

1. *The Sobolev estimates for ρ_ψ comes from optimal bounds in stationary phase lemmas. Then, with only $f, g \in L^2$ and $M = 1$, we expect that the Theorem 2 gives the best Sobolev's exponent.*

2. *Since $\gamma \geq N+1$, with only $f, g \in L^2$, we expect that ρ_ψ belongs at most in $H^{1/(N+1)}(\mathbb{R}_X^{N+1})$ when $M = 1$.*

The following Theorem is a comparison between the two previous results. It shows that Theorem 1 does not give the best Sobolev exponent when $M = 1$ and that Theorem 2 is not optimal for $M > 1$.

Theorem 3 *For $N \geq 2$ and $M = 1$, Theorem 2 gives a stronger smoothing effect than Theorem 1 for the best $\gamma = \gamma_{opt}$ compared with the best $\alpha = \alpha_{opt}$ since*

$$\frac{1}{\gamma_{opt}} = \frac{1}{N+1} \quad > \quad \frac{\alpha_{opt}}{2} = \frac{1}{2N}.$$

Conversely, for $N = M$, Theorem 1 can give one half derivative with the best $\alpha = 1$.

Finally, we have two resuls in the L^p framework.

Theorem 4 (First L^p result with F constant)
Let $a \in C^{N+3}(\mathbb{R}_v^M, \mathbb{R}_x^N)$, $F(t,x,v) = F \in \mathbb{R}_v^M$, $f, g \in L^p(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^M)$, satisfying (1.4). Let $A > 0$ and $\psi \in C_c^{N+2}(\mathbb{R}_v^M)$ such that the support of ψ is included in $[-A, A]^M$. We assume that there exists $0 < \alpha \leq 1$ and $C > 0$ such that for any $(u, \sigma) \in S^N$ and $\varepsilon > 0$,

$$meas\ \Big(\{v \in [-A, A]^M\, ;\, u - \varepsilon < a(v)\cdot\sigma < u + \varepsilon\}\Big) \leq C\varepsilon^\alpha. \qquad (1.7)$$

Then the averaging

$$\rho_\psi(t,x) = \int_{\mathbb{R}^M} f(t,x,v)\psi(v)\,dv$$

is in $W_{loc}^{\alpha/p',p}(\mathbb{R}_t \times \mathbb{R}_x^N)$.

Theorem 5 (Second L^p result with F constant)
Let $a \in C^\gamma(\mathbb{R}_v^M, \mathbb{R}_x^N)$, $F(t,x,v) = F \in \mathbb{R}_v^M$, $F \neq 0$, f, $g \in L^p(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^M)$ with $1 < p \leq 2$, satisfying (1.4), where we assume that a satisfies the following condition with γ, which is a positive integer, such that

$$\forall (v,\sigma) \in \mathbb{R}^M \times S^N, \quad \sigma = (\sigma_0, \sigma_1, \cdots, \sigma_N), \quad \tilde{\sigma} = (\sigma_1, \cdots, \sigma_N),$$

$$|\sigma_0 + a(v).\tilde{\sigma}| + \sum_{k=1}^{\gamma-1} \Big|(F\cdot\nabla_v)^k a(v)\cdot\tilde{\sigma}\Big| > 0. \qquad (\gamma ND)$$

Let $\psi \in C_c^1(\mathbb{R}_v^M)$, then the averaging

$$\rho_\psi(t,x) = \int_{\mathbb{R}} f(t,x,v)\psi(v)\,dv$$

is in $W^{s,p}(\mathbb{R}_t \times \mathbb{R}_x^N)$ with $s = \frac{2}{\gamma p'}$.

Remark 1.5 These results are presented with the time dependence because it is more useful for applications. In the proof of next sections, we take the following notations. We set $X = (t,x)$ and $b(v) = (1, a(v))$. Then (1.4) can be rewritten in the following way

$$b(v)\cdot\nabla_X f + F(X,v)\cdot\nabla_v f = g, \qquad (1.8)$$

where $X \in \mathbb{R}^{N+1}$, $v \in \mathbb{R}^M$.

The paper is organized by the following way. In Section 2, we prove Theorem 1. In Section 3, we prove Theorem 2. In Section 4, we compare the two results (Theorem 3) and finally in Section 5, we prove the extension to L^p spaces (Theorem 4 and 5).

2 First Theorem in the L^2 framework

We first recall the following classical averaging lemma (see [12], [5]).

Proposition 1 (Golse, Lions, Perthame, Sentis) *Let $a \in L^\infty_{loc}(\mathbb{R}^M, \mathbb{R}^N)$, $f, g \in L^2(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}^M_v)$, such that*

$$\partial_t f + a(v) \cdot \nabla_x f = g. \tag{2.1}$$

Let $\psi \in L^\infty(\mathbb{R}^M_v)$, with compact support in some $[-A, A]^M$, such that there exists $0 < \alpha \leq 1$ and $C > 0$ such that

$$meas \left(\{ v \in [-A, A]^M \,;\, u - \varepsilon < a(v) \cdot \sigma < u + \varepsilon \} \right) \leq C \varepsilon^\alpha \tag{2.2}$$

for any $(u, \sigma) \in S^N$ and $\varepsilon > 0$. Then the averaging

$$\rho_\psi(t, x) = \int_{\mathbb{R}^M} f(t, x, v) \psi(v) \, dv$$

is in $H^{\alpha/2}(\mathbb{R}_t \times \mathbb{R}^N_x)$ with the estimate

$$\|\rho_\psi\|_{H^{\alpha/2}} \leq \tilde{C}(N) \left(\|\psi\|_{L^2} + \sqrt{K} \|\psi\|_{L^\infty} \right) \left(\|f\|_{L^2} + \|g\|_{L^2} \right).$$

We use this averaging lemma to prove an other result, which deals with test function depending on (t, x, v).

Proposition 2 *Let $a \in L^\infty_{loc}(\mathbb{R}^M_v, \mathbb{R}^N_x)$, $f, g \in L^2(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}^M_v)$, such that*

$$\partial_t f + a(v) \cdot \nabla_x f = g. \tag{2.3}$$

Let $\psi \in L^\infty_c(\mathbb{R}^M_v, W^{N+2,\infty}(\mathbb{R}^{N+1}_{tx}))$ with compact support with respect to v in some $[-A, A]^M$. We assume that there exists $0 < \alpha \leq 1$ and $C > 0$ such that

$$meas \left(\{ v \in [-A, A]^M \,;\, u - \varepsilon < a(v) \cdot \sigma < u + \varepsilon \} \right) \leq C \varepsilon^\alpha \tag{2.4}$$

for any $(u, \sigma) \in S^N$ and $\varepsilon > 0$.
Then the averaging

$$\rho_\psi(t, x) = \int_{\mathbb{R}} f(t, x, v) \psi(t, x, v) \, dv$$

is in $H^{\alpha/2}_{loc}(\mathbb{R}_t \times \mathbb{R}^N_x)$ with the bound

$$\|\rho_\psi\|_{H^{\alpha/2}_K} \leq C(N, K) \left(\|f\|_{L^2} + \|g\|_{L^2} \right) \|\psi\|_{(L^2 \cap L^\infty)_v (W^{N+2,\infty}_{tx})}$$

for any compact K.

Proof. We fix a compact K on X. We take $\tilde{K} = [-S, S]^{N+1}$ such that $K \subset \tilde{K}$ and χ a C^∞ function such that $\chi = 1$ on K and 0 outside $\tilde{K}$. Finally, we set $\tilde{\psi} = \psi\chi$.
Since $\tilde{\psi}$ has a compact support with respect to X, we can extend it by periodicity in these variables. Then Fourier gives

$$\tilde{\psi}(X, v) = \sum_{\beta \in \mathbb{Z}^{N+1}} c_\beta(v) e^{iS\beta \cdot X}.$$

We write this formula through

$$\tilde{\psi}(X, v) = \sum_{\beta \in \mathbb{Z}^{N+1}} \Big((1 + |\beta|^r) c_\beta(v) \Big) \cdot \frac{e^{iS\beta \cdot X}}{1 + |\beta|^r},$$

with $r = N/2 + 1$. We set

$$\phi_\beta(X) = \frac{e^{iS\beta \cdot X}}{1 + |\beta|^r}, \quad \text{and} \quad \psi_\beta(v) = (1 + |\beta|^r) c_\beta(v).$$

We use the decreasing of Fourier coefficients for $W^{N+2,\infty}(\mathbb{R}_X^{N+1})$ function, that is to say that

$$|c_\beta(v)| \leq \frac{C_1}{(S|\beta|)^{N+2}} \|\tilde{\psi}(\cdot, v)\|_{W_X^{N+2,\infty}}.$$

Thus we have

$$\begin{aligned}
&\int_{\mathbb{R}^M} \sum_{\beta \in (\mathbb{Z}^{N+1})^*} |\psi_\beta(v)|^2 \, dv \\
\leq \;& \int_{\mathbb{R}^M} \sum_{\beta \in (\mathbb{Z}^{N+1})^*} (1 + |\beta|^r)^2 |c_\beta(v)|^2 \, dv \\
\leq \;& \frac{C_2}{S^{2N+4}} \int_{\mathbb{R}^M} \sum_{\beta \in (\mathbb{Z}^{N+1})^*} \frac{4|\beta|^{2r}}{|\beta|^{2N+4}} \|\tilde{\psi}(\cdot, v)\|^2_{W_X^{N+2,\infty}} \, dv \\
\leq \;& \frac{4C_2}{S^{2N+4}} \sum_{\beta \in (\mathbb{Z}^{N+1})^*} \frac{1}{|\beta|^{N+2}} \|\psi\|^2_{L^2_v(W_X^{N+2,\infty})} < +\infty.
\end{aligned} \tag{2.5}$$

On K, we notice that

$$\begin{aligned}
\rho_\psi(X) &= \int_{\mathbb{R}} f(X, v) \psi(X, v) \, dv \, \chi(X) \\
&= \int_{\mathbb{R}} f(X, v) \tilde{\psi}(X, v) \, dv, \\
&= \int_{\mathbb{R}^M} f(X, v) \sum_{\beta \in \mathbb{Z}^{N+1}} \phi_\beta(X) \psi_\beta(v) \, dv.
\end{aligned}$$

To apply Fubini's Theorem, we want that, for *a.e.* X,

$$\int_{\mathbb{R}^M} \sum_{\beta \in (\mathbb{Z}^{N+1})^*} |f(X, v) \phi_\beta(X) \psi_\beta(v)| \, dv < +\infty.$$

It comes from

$$\begin{aligned}
& \int_{\mathbb{R}^M} \sum_{\beta\in(\mathbb{Z}^{N+1})^*} |f(X,v)\phi_\beta(X)\psi_\beta(v)|\,dv \\
\leq\ & \int_{\mathbb{R}^M} |f(X,v)| \sum_{\beta\in(\mathbb{Z}^{N+1})^*} |\phi_\beta(X)\psi_\beta(v)|\,dv \\
\leq\ & \sqrt{\int_{\mathbb{R}^M} |f(X,v)|^2\,dv}\sqrt{\int_{\mathbb{R}^M}\left(\sum_{\beta\in(\mathbb{Z}^{N+1})^*} |\phi_\beta(X)\psi_\beta(v)|\right)^2 dv} \\
\leq\ & \|f(X,\cdot)\|_{L^2_v}\sqrt{\sum_{\beta\in(\mathbb{Z}^{N+1})^*} |\phi_\beta(X)|^2 \int_{\mathbb{R}^M} \sum_{\beta\in(\mathbb{Z}^{N+1})^*} |\psi_\beta(v)|^2\,dv} \\
\leq\ & \|f(X,.\cdot)\|_{L^2_v}\sqrt{\sum_{\beta\in(\mathbb{Z}^{N+1})^*} \frac{1}{(1+|\beta|^r)^2} \int_{\mathbb{R}^M} \sum_{\beta\in(\mathbb{Z}^{N+1})^*} |\psi_\beta(v)|^2\,dv} < +\infty
\end{aligned}$$

since $2r > N+1$ and from (2.5). Thus we can write, on K,

$$\rho_\psi(X) = \sum_{\beta\in\mathbb{Z}^{N+1}} \phi_\beta(X)\rho_{\psi_\beta}(X),$$

with

$$\rho_{\psi_\beta}(X) = \int_{\mathbb{R}} f(X,v)\psi_\beta(v)\,dv.$$

The classical averaging lemma (Proposition 1) gives that

$$\|\rho_{\psi_\beta}\|_{H^{\alpha/2}_K} \leq \tilde{C}(N)\left(\|\psi_\beta\|_{L^2} + \sqrt{C}\|\psi_\beta\|_{L^\infty}\right)(\|f\|_{L^2} + \|g\|_{L^2}).$$

We now use the following property: For $u_1 \in C^s(\Omega)$, $u_2 \in H^s(\Omega)$, with $s \in]0,1[$, with Ω a bounded open set of $\mathbb{R}^{N+1}$, we have $u_1u_2 \in H^s(\Omega)$ with

$$\|u_1u_2\|_{H^s} \leq C_3\|u_1\|_{C^s}\|u_2\|_{H^s}.$$

This result gives, for $s = \alpha/2$,

$$\begin{aligned}
& \|\rho_\psi\|_{H^{\alpha/2}_K} \\
\leq\ & C_3 \sum_{\beta\in\mathbb{Z}^{N+1}} \|\phi_\beta\|_{C^{\alpha/2}_K}\|\rho_{\psi_\beta}\|_{H^{\alpha/2}_K} \\
\leq\ & C_4 \sum_{\beta\in(\mathbb{Z}^{N+1})^*} \|\phi_\beta\|_{C^{\alpha/2}_K}(\|\psi_\beta\|_{L^2} + \|\psi_\beta\|_{L^\infty})(\|f\|_{L^2} + \|g\|_{L^2}) + C_3\|\rho_{\psi_0}\|_{H^{\alpha/2}_K} \\
\leq\ & C_5\left(\sum_{\beta\in(\mathbb{Z}^{N+1})^*} \frac{1}{|\beta|^{r-\alpha/2}} \frac{\|\psi\|_{(L^2\cap L^\infty)_v(W^{N+2,\infty}_X)}}{|\beta|^{N+2-r}}(\|f\|_{L^2} + \|g\|_{L^2}) + \|\tilde{\psi}\|_{C^1_K}\right) \\
\leq\ & C_5\left(\sum_{\beta\in(\mathbb{Z}^{N+1})^*} \frac{1}{|\beta|^{N+2-\alpha/2}}(\|f\|_{L^2} + \|g\|_{L^2})\|\psi\|_{(L^2\cap L^\infty)_v(W^{N+2,\infty}_X)} + \|\psi\|_{C^{N+2}_c}\right).
\end{aligned}$$

Since $N+2-\alpha/2 > N+1$, the proof is completed. $\square$

With this Proposition, we turn now to the proof of the first Theorem.

Proof of Theorem 1. Let K be a compact in $\mathbb{R}_X^{N+1}$. We set $\mathcal{K} = K \times [-A,A]^M$. We perform locally a change of variables in order to rewrite the equation (1.8) without the term $\nabla_v f$ and to apply the previous result. For any (X,v), using the characteristics since $b(v) = (1, a(v)) \neq 0$, there exists a neighborhood $\mathcal{B}_{Xv} \subset \mathcal{K}$ and a C^{N+3} function $w \mapsto V(X,w)$ such that

$$b(V(X,w)) \cdot \nabla_X V(X,w) = F(X, V(X,w)).$$

Denoting by

$$\tilde{f}(X,w) = f(X,V(X,w)), \quad \tilde{g}(X,w) = g(X,V(X,w)), \quad \tilde{b}(w) = b(V(X,w)),$$

the equation (1.8) rewrites

$$\tilde{b}(w) \cdot \nabla_X \tilde{f} = \tilde{g}. \tag{2.6}$$

There exists a finite number of $\mathcal{B}_{xv}$ to recover this compact: there exists $\{(x_l,v_l)\}_{l=1,\cdots,L}$ such that $\mathcal{K} \subset \bigcup\limits_{l=1,\cdots,L} \mathcal{B}_{x_l v_l}$. For this recovering, we use a partition of unity: on $X \in K$, we have

$$\psi = \psi \mathbb{1}_{\mathcal{K}} = \sum_{l=1}^{L} \chi_l(X,v)\psi,$$

where the function χ_l are C^∞ and have a compact support in $\mathcal{B}_{x_l v_l}$. Now, for $X \in K$,

$$\begin{aligned} \rho_\psi(X) &= \sum_{l=1}^{L} \int_{\mathbb{R}^M} f(X,v)\chi_l(X,v)\psi(v)\,dv \\ &= \sum_{l=1}^{L} \int_{\mathcal{B}_{x_l v_l}} f(X,v)\chi_l(X,v)\psi(v)\,dv \\ &= \sum_{l=1}^{L} \int_{V(\mathcal{B}_{x_l v_l})} \tilde{f}(X,w)\chi_l(X,V(X,w))\psi(V(X,w))J_l(X,w)\,dw \end{aligned}$$

performing the variable change $v \mapsto w = V(X,v)$ on every neighborhood corresponding to l and denoting by $J_l(X,w)$ the associated jacobian. We set

$$\overline{\psi_l}(X,w) = \chi_l(X,V(X,w))\psi(V(X,w))J_l(X,w).$$

Since a and F have C^{N+3} regularity, J_l has C^{N+2} one. Furthermore $\psi \in C_c^{N+2}$, thus $\overline{\psi_l} \in (L^2 \cap L^\infty)_c(\mathbb{R}_v^M, W^{N+2,\infty}(\mathbb{R}_X^{N+1}))$. We apply the previous result: the averaging

$$\rho_{\overline{\psi_l}}(X) = \int_{\mathbb{R}^M} \tilde{f}(X,w)\overline{\psi_l}(X,w)\,dw$$

is in $H_{loc}^{\alpha/2}(\mathbb{R}_X^{N+1})$. Now $\|\rho_\psi\|_{H_K^{\alpha/2}} \leq \sum\limits_{l=1}^{L} \|\rho_{\overline{\psi_l}}\|_{H_K^{\alpha/2}}$ conclude the proof. $\square$

3 Case with a constant force field

When F is a non zero constant vector, we can obtained a different result. The way to get it is quiet different and we have to restrict to the case of a constant force field. A key tool is a generalized uniform version of the classical method of the station nary phase. We work on equation (1.8) with F constant, $F \in \mathbb{R}^M$, $F \neq 0$. Let us denote a directional $v-$derivative along vector F by

$$D \quad = \quad F \cdot \nabla_v. \tag{3.1}$$

The smoothing effect depends on the (γND) assumption of Theorem 2. Indeed, it is exactly the following non-degeneracy condition about D-derivatives of $b(.)$:

$$\forall (v, \sigma) \in \mathbb{R}^M \times S^N, \qquad \sum_{k=0}^{\gamma-1} \left| D^k b(v) \cdot \sigma \right| > 0. \qquad (\gamma ND)$$

Before proving the Theorem 2 we gives some useful results about oscillatory integrals following the Stein's book [22].

Proposition 3 ([22]) *Suppose $\phi \in C^{k+1}(\mathbb{R}, \mathbb{R})$ so that, for some $k \geq 1$,*

$$\frac{d^k \phi}{dv^k}(v) \geq 1, \qquad \forall v \in]\alpha, \beta[. \tag{3.2}$$

Then

$$\left| \int_\alpha^\beta e^{i\lambda\phi(v)} dv \right| \leq c_k \cdot \frac{1}{|\lambda|^{1/k}}$$

holds when

1. *$k \geq 2$ or*
2. *$k = 1$ and ϕ' is monotonous.*

Furthermore, the bound c_k is independent of λ and ϕ.

This Proposition can be found in [22] p 332. Elias M. Stein obtain $c_k \leq 5 \cdot 2^{k-1} - 2$ in his proof. Notice that c_k is independent of the length of the interval $]\alpha, \beta[$. For $|\lambda| < 1$, the bound for the oscillatory integral blows up. Indeed, for $k = 1$, we can relax the monotonous assumption on ϕ by the following bounds

$$|\phi'(v)| \geq \delta > 0, \quad \forall v \in]\alpha, \beta[, \qquad \widetilde{c}_1 = 2 + \delta^{-1} \int_\alpha^\beta |\phi"(v)| dv,$$

Indeed, integrating by parts and using the inequality $\min(a, \beta b) \leq \min(1, \beta) \max(a, b)$ for all non negative a, b, β, we get

$$\left| \int_\alpha^\beta e^{i\lambda\phi(v)} dv \right| \leq \max(|\beta - \alpha|, \widetilde{c}_1) \cdot \max(1, \frac{1}{\delta}) \cdot \min(1, \frac{1}{|\lambda|}).$$

Furthermore, the bound given in Proposition 3 blow up for small λ, so we replace it by the length of the interval and get the following Corollary.

Corollary 1 *Let be $\delta > 0$. Suppose $\phi \in C^{k+1}(\mathbb{R},\mathbb{R})$ so that, for some $k \geq 1$,*

$$\left|\frac{d^k\phi}{dv^k}(v)\right| \geq \delta, \qquad \forall v \in]\alpha,\beta[. \tag{3.3}$$

Then $\left|\int_\alpha^\beta e^{i\lambda\phi(v)}dv\right| \leq \max(|\beta-\alpha|,\widetilde{c}_k)\cdot\max(1,\frac{1}{\delta^{1/k}})\min(1,\frac{1}{|\lambda|^{1/k}})$,
where $\widetilde{c}_k$ is independent of λ, ϕ and $]\alpha,\beta[$ for $k \geq 2$ and $\widetilde{c}_1 = 2+\delta^{-1}\int_\alpha^\beta |\phi"(v)|dv$.

Notice, that, for $k \geq 2$, $\widetilde{c}_k = c_k$ given in Proposition 3.
Following Stein's book (Corollary p 334), we obtain the following Proposition.

Proposition 4 ([22]) *Let $\psi \in W^{1,1}(]\alpha,\beta[)$, $\phi \in C^{k+1}(\mathbb{R},\mathbb{R})$ such that, for some $\delta > 0$ and $k \geq 1$,*

$$\left|\frac{d^k\phi}{dv^k}(v)\right| \geq \delta, \quad \forall v \in]\alpha,\beta[.$$

Then

$$\left|\int_\alpha^\beta \psi(v)e^{i\lambda\phi(v)}dv\right| \leq \frac{\max(|\beta-\alpha|,\widetilde{c}_k)}{\min(1,\delta^{1/k})\max(1,|\lambda|^{1/k}))}\left(\|\psi\|_{L^\infty(]\alpha,\beta[)} + \|\psi'\|_{L^1(]\alpha,\beta[)}\right),$$

where $\widetilde{c}_k$ is independent of λ, ϕ, ψ and $]\alpha,\beta[$ for $k \geq 2$, and $\widetilde{c}_1 = 2 + \delta^{-1}\int_\alpha^\beta |\phi"(v)|dv$.

Proof. This is classically proved in writing the integral $\int_\alpha^\beta \psi(v)e^{i\lambda\phi(v)}dv$ as $\int_\alpha^\beta \psi(v)I'(v)dv$, with $I(v) = \int_\alpha^v e^{i\lambda\phi(u)}du$, integrating by parts and using the uniform estimate for $|I(v)|$ from previous Corollary. $\square$

Now we generalize the Proposition 4 in the case with parameters and a like (γND) assumption.

Proposition 5 *Suppose P is a compact set of parameter p, $A > 0$, $\psi(u;p)$ belongs in $L_p^\infty(P, W_u^{1,1}(]-A,A[))$ and $\phi(u;p) \in C^{\gamma+1}(\mathbb{R}_u \times P_p,\mathbb{R})$, such that, for all (u,p) in $K = [-A,A]\times P$,*

$$\sum_{k=1}^{\gamma}\left|\frac{\partial^k\phi}{\partial u^k}\right|(u;p) \;>\; 0. \tag{3.4}$$

Then, for any $]\alpha,\beta[\subset]-A,A[$,

$$\begin{aligned}&\left|\int_\alpha^\beta \psi(u;p)e^{i\lambda\phi(u;p)}du\right|\\ \leq\;& d_\gamma\cdot\min\left(1,\frac{1}{|\lambda|^{1/\gamma}}\right)\cdot\left(\|\psi\|_{L^\infty(K)} + \left\|\frac{\partial\psi}{\partial u}\right\|_{L^\infty(P,L^1(]-A,A[))}\right),\end{aligned}$$

where the constant d_γ is independent of λ and only depends on A, $\sup_K \left|\frac{\partial^2\phi}{\partial u^2}\right|$, $\inf_K \frac{1}{\gamma}\sum_{k=1}^{\gamma}\left|\frac{\partial^k\phi}{\partial u^k}\right|$.

Proof. Since K is a compact set, we can choose $0<\delta\leq 1$ such that, everywhere on K:

$$0<\delta<\frac{1}{\gamma}\sum_{k=1}^{\gamma}\left|\frac{\partial^k\phi}{\partial u^k}\right|(u;p).$$

Let us define the open set $Z_k=\{(u;p),\ |\partial_u^k\phi(u;p)|>\delta\}$, for $k=1,\cdots,\gamma$. Necessarily $K\subset\bigcup_{k=1}^{\gamma}Z_k$, and then there exists a partition of unity such that $\sum_{k=1}^{\gamma}\rho_k\equiv 1$ on K and such that the support of ρ_k is included in Z_k. Let us define $\psi_k=\ \rho_k\psi$ and $I=I_1+\cdots+I_\gamma$ where $I_k(p)=\int_a^b\psi_k(u;p)e^{i\lambda\phi(u;p)}du$. We apply the Proposition 4 on each I_k where the exponent " $'$ " denotes ∂_u:

$$|I_k|\leq\frac{\max(2A,\tilde{c}_k)}{\delta^{1/k}\max(1,|\lambda|^{1/k})}\sup_P\left(\|\psi_k(.,p)\|_{L^\infty(]-A,A[)}+\|\psi_k'(.,p)\|_{L^1(]-A,A[)}\right).$$

Since for any fixed p and $J=]-A,A[$, we have

$$\begin{aligned}&\left(\|\psi_k(.,p)\|_{L^\infty(J)}+\|\psi_k'(.,p)\|_{L^1(J)}\right)\\ \leq\ &\left(\|\rho_k\|_{L^\infty(J)}+\|\rho_k'\|_{L^1(J)}\right)\left(\|\psi(.,p)\|_{L^\infty(J)}+\|\psi'(.,p)\|_{L^1(J)}\right),\end{aligned}$$

it is enough to take

$$d_\gamma=\sum_k\frac{\max(2A,\tilde{c}_k)}{\delta^{1/k}}\left(\|\rho_k\|_{L^\infty(K)}+\|\partial_u\rho_k\|_{L^\infty(P,L^1(J))}\right)$$

to conclude the proof. □

We are now able to prove the second Theorem.

Proof of the Theorem 2.
The proof follows three steps. First, we choose a suitable variable associated to D. Secondly, we use Fourier transform with respect to X and solve a linear ordinary differential equation with respect to v_1. Third, we obtain Sobolev estimates for ρ_ψ with Proposition 5.

Step 1, change of coordinates: With a suitable choice of orthonormal coordinates, we assume, without loss of generality that

$$D=F\cdot\nabla_v=|F|\frac{\partial}{\partial v_1}$$

where $|F|$ is the euclidean norm of vector F and $v = (v_1, v_2, \cdots, v_M) \equiv (v_1; w)$. Notice that the jacobian for an orthonormal change of variables is one, thus the estimates on ρ_ψ are invariant through such choice for v_1. With such notations, equation (1.8) becomes

$$b(v) \cdot \nabla_X f + |F| \frac{\partial f}{\partial v_1} = g. \tag{3.5}$$

Step 2, linear o.d.e.: Denoting by $\mathcal{F}(f)$ the Fourier transform of f with respect to X, and by Y the dual variable of X, equation (3.5) becomes

$$|F| \frac{\partial}{\partial v_1} \mathcal{F}(f) + i(b(v) \cdot Y)\mathcal{F}(f) \;=\; \mathcal{F}(g). \tag{3.6}$$

For almost all fixed Y, we solve an ordinary differential equation with respect to v_1. For this purpose, we choose the initial v_1, namely $v_1^0 \in]0,1[$, such that

$$\begin{aligned} &\int_{\mathbb{R}_Y^{N+1} \times \mathbb{R}_w^{M-1}} |\mathcal{F}(f)|^2(Y; v_1^0; w) dY dw \\ \leq \; &\int_{\mathbb{R}_{v_1}} \int_{\mathbb{R}_Y^{N+1} \times \mathbb{R}_w^{M-1}} |\mathcal{F}(f)|^2(Y; v_1; w) dY dw dv_1. \end{aligned} \tag{3.7}$$

Existence of such v_1^0 is a consequence of the Fubini's Theorem. Indeed, let $h(v_1) = \int_{\mathbb{R}^{N+1}} \int_{\mathbb{R}_w^{M-1}} |\mathcal{F}(f)|^2(Y; v_1; w) dY dw \geq 0$. The function h is defined almost everywhere, belongs in $L^1(\mathbb{R}_{v_1})$ and satisfies $\|h\|_{L^1(\mathbb{R}_{v_1})} = \|f\|^2_{L^2_{X,v}}$. Since the h function cannot be everywhere greater than its mean value on $]0,1[$, there exists $v_1^0 \in]0,1[$ such that $h(v_1^0) \leq \int_0^1 h(v_1) dv_1$, which confirms (3.7).
We finally write an explicit formula for $\mathcal{F}(f)$ with $B(v)$ a primitive with respect to v_1 of $-b/|F|$:

$$\begin{aligned} B(v) &= B(v_1; w) = -\int_{v_1^0}^{v_1} \frac{b(u; w)}{|F|} du \\ \mathcal{F}(f)(Y, v_1; w) &= \mathcal{F}(f)(Y, v_1^0; w) e^{iB(v) \cdot Y} \\ &\quad + \frac{1}{|F|} \int_{v_1^0}^{v_1} \mathcal{F}(g)(Y, u; w) e^{i(B(v_1; w) - B(u; w)) \cdot Y} du. \end{aligned}$$

Step 3, $H^{1/\gamma}$ estimates with oscillatory integrals: We decompose $\rho_\psi(t, x) = \int_{\mathbb{R}^M} f(t, x, v)\psi(v)\, dv$ in two parts from the explicit expression of $\mathcal{F}(f)$ in the step 2: $\mathcal{F}(\rho_\psi) = \widehat{\rho}_f + \widehat{\rho}_g$. The first term is

$$\widehat{\rho}_f(Y) \;=\; \int_{\mathbb{R}_w^{M-1}} \mathcal{F}(f)(Y, v_1^0; w) \int_{\mathbb{R}_u} \psi(u; w) e^{iB(u; w) \cdot Y} du dw.$$

In this integral, there is an oscillatory integral which is parametrized by w and $Y = \lambda\sigma$ with $\lambda = |Y|$ and $\sigma \in S^N$; it is

$$Osc(Y, w) \;=\; \int_{\mathbb{R}_u} \psi(u; w) e^{i\lambda B(u; w) \cdot \sigma} du. \tag{3.8}$$

To use the Proposition 5, we set $p = (\sigma, w)$ which belongs in the compact set $P = S^N \times [-A, A]^{M-1}$ with $A > 1 > v_1^0 > 0$ such that $supp\,\psi \subset [-A, A]^M$. The condition (3.4) of the Proposition 5 for oscillatory integral (3.8) is

$$\sum_{k=1}^{\gamma} \left| \frac{\partial^k B(u; w)}{\partial u^k} \cdot \sigma \right| \quad > 0$$

which is exactly the (γND) assumption for $b(.)$. Thanks to the (γND) assumption and the Proposition 5, there exists a constant L such that for all $(Y, w) \in \mathbb{R}^d \times [-A, A]^{M-1}$, and for all α, β such that $-A < \alpha < \beta < A$, we have

$$\max(1, |Y|^{1/\gamma}) \left| \int_\alpha^\beta \psi(u; w) e^{i\lambda B(u;w)\cdot\sigma} du \right| \ \leq \ L. \tag{3.9}$$

Using the constant L and the compact support of ψ we have

$$\max(1, |Y|^{1/\gamma}) |\widehat{\rho}_f(Y)| \ \leq \ L \int_{[-A,A]^{M-1}} |\mathcal{F}(f)(Y, v_1^0; w)| dw.$$

By Cauchy-Schwarz inequality, we get

$$\max(1, |Y|^{2/\gamma}) |\widehat{\rho}_f(Y)|^2 \ \leq \ (2A)^{M-1} L^2 \int_{[-A,A]^{M-1}} |\mathcal{F}(f)(Y, v_1^0; w)|^2 dw.$$

Finally, since v_1^0 satisfies (3.7), we obtain

$$\int_{\mathbb{R}^{N+1}} \max(1, |Y|^{2/\gamma}) |\widehat{\rho}_f(Y)|^2 dY \ \leq \ (2A)^{M-1} L^2 \int_{\mathbb{R}^{N+1}\times\mathbb{R}^M} |\mathcal{F}(f)(Y, v)|^2 dv dY,$$

which gives $\widehat{\rho}_f \in H^{1/\gamma}$.

The second term $\widehat{\rho}_g$ is bounded in the same way. More precisely, we set

$$\widehat{\rho}_g(Y) \ = \ \int_{\mathbb{R}^{M-1}} H(Y, w) dw$$

with

$$H(Y, w) \ = \ \frac{1}{|F|} \int_{-A}^{A} \int_{v_1^0}^{v_1} \mathcal{F}(g)(Y, u; w) e^{i(B(v_1;w) - B(u;w))\cdot Y} du dv_1.$$

Using the Fubini's Theorem and the notation

$$\Psi(Y, u; w) \ = \ \psi(u; w) e^{iB(u;w)\cdot Y},$$

we have another expression for $H(Y, w)$:

$$\begin{aligned} H(Y, w) \ = \ & \frac{1}{|F|} \int_{v_1^0}^{A} \mathcal{F}(g)(Y, u; w) e^{-iB(u;w)\cdot Y} \left(\int_u^A \Psi(Y, v_1; w) dv_1 \right) du \\ & + \frac{1}{|F|} \int_{-A}^{v_1^0} \mathcal{F}(g)(Y, u; w) e^{-iB(u;w)\cdot Y} \left(\int_{-A}^{u} \Psi(Y, v_1; w) dv_1 \right) du, \end{aligned}$$

where there are two oscillatory integrals $\int_u^A \Psi(Y,v_1;w)dv_1$ and $\int_{-A}^u \Psi(Y,v_1;w)dv_1$ which are uniformly bounded thanks to inequality (3.9). Then we have

$$\max(1,|Y|^{1/\gamma})|H(Y;w)| \leq \frac{L}{|F|}\int_{-A}^{A}|\mathcal{F}(g)(Y,u;w)|\,du.$$

With the Cauchy-Schwarz inequality, we obtain

$$\max(1,|Y|^{2/\gamma})|H(Y;w)|^2 \leq \frac{2AL^2}{|F|^2}\int_{-A}^{A}|\mathcal{F}(g)(Y,u;w)|^2\,du$$

and finally

$$\max(1,|Y|^{2/\gamma})|\widehat{\rho}_g(Y)|^2 \leq (2A)^M\frac{L^2}{|F|^2}\int_{\mathbb{R}^M}|\mathcal{F}(g)(Y,v)|^2 dv.$$

Then $\rho_g \in H^{1/\gamma}$, thus finally ρ_ψ is also in this space, which concludes the proof of the Theorem. $\square$

4 About non degeneracy conditions

Theorem 1 and Theorem 2 assume two different non degeneracy conditions on the vector field $a(v) \in \mathbb{R}^N$, $v \in$ supp $\psi \subset \mathbb{R}^M$. These conditions involve two parameters, namely $\alpha = \alpha_{a(.)} \in]0,1]$ in (1.5) and $\gamma = \gamma_{a(.),F} \in \mathbb{N}^*$ in (1.6), directly linked to the smoothing effect for the averaging in $H^{\alpha/2}_{loc}$ or $H^{1/\gamma}$. In this section, we give some optimal upper bounds for α and $1/\gamma$ to compare the two results obtained by different ways. Indeed, for $M = 1$ and $N \geq 2$, Theorem 2 gives a better smoothing effect than Theorem 1. Conversely, when $N = M$, Theorem 1 is stronger than Theorem 2. In this part, we study these various properties and in particular, we prove the Theorem 3.
More precisely, let A be positive, we obtain the optimal α and γ, namely

$$\begin{aligned}
\alpha_{opt}(N,M) &= \sup_{a(.)\in C^\infty([-A,A]_v^M,\mathbb{R}_x^N)} \alpha,\\
\gamma_{opt}(N,M) &= \min_{a(.)\in C^\infty(\mathbb{R}_v^M,\mathbb{R}_x^N),\,F\in\mathbb{R}^N\setminus\{0\}} \gamma.
\end{aligned}$$

We start by obtaining the easiest estimate which is a lower bound for γ.

Proposition 6 *For all N, M, we have $\gamma \geq \gamma_{opt}(N,M) = N+1$.*

Proof. We use notations from Section 3. Following this section, the (γND) condition can be rewritten and means that we cannot find $\sigma \in S^N$ such that, $\sigma \perp b(v)$, $\sigma \perp Db(v)$, ..., $\sigma \perp D^{\gamma-1}b(v)$. There are γ conditions to satisfy. Since $b(v)$ belongs in $\mathbb{R}^{N+1}$, we necessarily have $\gamma \geq N+1$. Indeed $N+1$ is the minimal possible value for γ. For instance, if $D = \dfrac{\partial}{\partial v_1}$, $b(v) = (1, v_1, v_1^2, \cdots, v_1^N)$, with $v = (v_1, v_2, \cdots, v_M)$, we have $\gamma_{opt} = N+1$. $\square$

The optimal α is more difficult to get and it is obtained in the following subsections, see also [15]. The evaluation of exponent α also implies new asymptotic expansions involving piecewise smooth functions in [16].

4.1 $M = 1$, one dimensional velocity

Proposition 7 *For $M = 1$, we have $\alpha \leq \alpha_{opt}(N,1) = \dfrac{1}{N}$.*

To obtain this optimal α for $M = 1$, we need some other notations and following results. The proof of Proposition 7 is reached at the end of this subsection 4.1.

Let $\varphi \in C^\infty([a,b],\mathbb{R})$ and $v \in [a,b]$, the multiplicity of φ on v is defined by

$$m_\varphi[v] \;=\; \inf\{k \in \mathbb{N},\, \varphi^{(k)}(v) \neq 0\} \qquad \in \overline{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}.$$

It means that if $k = m_\varphi$ then $\varphi^{(k)}(v) \neq 0$ and $\varphi^{(j)}(v) = 0$ for $j = 0, 1, \cdots, k-1$. For instance $m_\varphi[v] = 0$ means $\varphi(v) \neq 0$; $m_\varphi[v] = 1$ means $\varphi(v) = 0$, $\varphi'(v) \neq 0$ and $m_\varphi[v] = +\infty$ means $\varphi^{(j)}(v) = 0$ for all $j \in \mathbb{N}$.
Set the multiplicity of φ on $[a,b]$ by

$$m_\varphi \;=\; \sup_{v \in [a,b]} m_\varphi[v] \quad \in \overline{\mathbb{N}}.$$

Notice that the case where φ only belongs in C^k, m_φ is well defined only if $m_\varphi[v] \leq k$ for all $v \in [a,b]$.

Lemma 1 *Let $\varphi \in C^k([a,b],\mathbb{R})$ with $a < b$, and*

$$Z(\varphi,\varepsilon) \;=\; \{v \in [a,b],\; |\varphi(v)| \leq \varepsilon\}.$$

If m_φ is well defined ($m_\varphi \leq k$) then there exists $C > 0$ such that, for all $\varepsilon > 0$,

$$\operatorname{meas}(Z(\varphi,\varepsilon)) \leq C\varepsilon^\alpha \quad \textit{with} \quad \alpha = \frac{1}{m_\varphi}. \tag{4.1}$$

Furthermore, if m_φ is positive, for all $\beta > \alpha$, we have $\displaystyle\lim_{\varepsilon \to 0} \frac{\operatorname{meas}(Z(\varphi,\varepsilon))}{\varepsilon^\beta} = +\infty$ (Optimality).

Proof. The case $m_\varphi = 0$ is clear since there is no zero in this situation. The quantity m_φ is positive simply means that the set $Z(\varphi,0)$ of roots of φ is not empty. Since any root of φ has a finite multiplicity, the compact set $Z(\varphi,0)$ is discrete and then finite: $Z(\varphi,0) = \{z_1, \cdots, z_\nu\}$. For each z_i and $h > 0$, let $V_i(h)$ be $]z_i - h, z_i + h[\cap [a,b]$. For any $0 < h < |b-a|$, we have

$$h \leq \operatorname{meas}(V_i(h)) \leq 2h.$$

For any root z_i, there exists $h_i \in]0, |b-a|[$, $A_i > 0$ and $\delta_i > 0$ such that

$$\delta_i |h|^{k_i} \;\leq |\varphi(z_i + h)| \leq\; A_i |h|^{k_i} \quad \text{ for all } h \in V_i(h_i), \tag{4.2}$$

with $k_i = m_\varphi[z_i]$. This is a direct consequence of the Taylor-Lagrange formula. Let V be $\displaystyle\bigcup_i V_i(h_i)$ and $\varepsilon_0 = \min\left(1, \min_{v \in [a,b]\setminus V} |\varphi(v)|\right)$. By the continuity of φ

on the compact set $[a,b]\setminus V$, ε_0 is positive. Then for all $0<\varepsilon<\varepsilon_0$, we have $Z(\varphi,\varepsilon)\subset V$. If $\varepsilon\geq|\varphi(z_i+h)|$ for $|h|<h_i$, then from (4.2), we have $(\varepsilon/\delta_i)^{1/k_i}\geq|h|$. This last inequality implies for $0<\varepsilon<\varepsilon_0\leq 1$ that $Z(\varphi,\varepsilon)$ is a subset of $\bigcup_i V_i((\varepsilon/\delta_i)^{1/k_i})$ and then

$$\text{meas}(Z(\varphi,\varepsilon))\leq 2\sum_{i=1}^{\nu}(\varepsilon/\delta_i)^{1/k_i}\leq\left(2\sum_{i=1}^{\nu}\delta_i^{-1/k_i}\right)\varepsilon^{1/m_\varphi}.$$

It gives inequality (4.1). To obtain the optimality of α, let z_j be a root of φ with maximal multiplicity i.e. $m_\varphi[z_j]=m_\varphi=k$. Again from (4.2), $V_j((\varepsilon/A_j)^{1/k})$ is a subset of $Z(\varphi,\varepsilon)$ for all $\varepsilon\in]0,\varepsilon_0[$. Then we have $(\varepsilon/A_j)^{1/k}\leq\text{meas}(Z(\varphi,\varepsilon))$, which is enough to get the optimality of $\alpha=1/k$ and concludes the proof. □

An upper bound of $\alpha_{opt}(N,1)$ is a consequence of the previous Lemma.

Lemma 2 *For all N, we have $\alpha_{opt}(N,1)\leq 1/N$.*

Proof. For any $a(.)\in C^\infty(\mathbb{R}_v,\mathbb{R}_x^N)$ and $A>0$, we set

$$\varphi(v;u,\sigma)\;=\;a(v)\cdot\sigma-u=b(v)\cdot(-u,\sigma),$$

defined for $v\in[-A,A]$, with $u\in\mathbb{R}$, $(-u,\sigma)\in S^N$, $b(v)=(1,a(v))\in\mathbb{R}^{N+1}$ and $m=\sup\limits_{(-u,\sigma)\in S^N} m_{\varphi(.;u,\sigma)}$.

Let v be fixed, we choose $(-u,\sigma)$ such that $m_\varphi[v]\geq N$ in order to obtain a lower bound for m.

Since $rank\{b(v),b'(v),\cdots,b^{(N-1)}(v)\}\leq N$, there exists $(-u,\sigma)$ such that $u^2+|\sigma|^2=1$ and $(-u,\sigma)\perp\{b(v),b'(v),\cdots,b^{(N-1)}(v)\}$. Then with such u and σ, $m_{\varphi(.;u,\sigma)}[v]\geq N$ which implies $m\geq N$ and consequently, from the optimality obtained in Lemma 1, we get $\alpha\leq\alpha_{opt}(N,1)\leq\dfrac{1}{N}$. □

When the function $v\to\varphi(v;p)$ depends on a parameter p, some results are obtained in the two following Lemma to bound the quantity C of Lemma 1 independently of the parameter p.

Lemma 3 *Let $k\geq 1$, I an interval of $\mathbb{R}$, $\phi\in C^k(I,\mathbb{R})$ and $\delta>0$.*
If $|\phi^{(k)}(v)|\geq\delta>0$ for all $x\in I$ then there exists a constant $\overline{c}_k$ independent of ϕ,I,δ such that

$$meas(Z(\phi,\varepsilon))\leq\overline{c}_k(\varepsilon/\delta)^{1/k},\quad \textit{where } Z(\phi,\varepsilon)=\{v\in I,\ |\phi(v)|\leq\varepsilon\}.$$

Proof. Since the result is independent of the interval I and of the $\phi^{(k-1)}(0)$ sign, let us suppose that $I=\mathbb{R}$ with $|\phi^{(k)}(v)|\geq\delta>0$ on $\mathbb{R}$, and $\phi^{(k-1)}(0)\leq 0$.

We first treat the case $k=1$. If $\phi'(v)$ stays positive, we have $\phi(0)+\delta v\leq\phi(v)$ for $0\leq v$ and since $\phi(0)\leq 0$, there exists a unique $c\geq 0$ such that $\phi(c)=0$. In the other case, $\phi'(v)$ stays negative, and we find a unique $c\leq 0$ such that $\phi(c)=0$. Then $|\phi(v)|\geq\delta|v-c|$ for all v, and $|\phi(v)|\leq\varepsilon$ implies $|v-c|\leq\varepsilon/\delta$ i.e. $Z(\phi,\varepsilon)\subset[c-\varepsilon/\delta,c+\varepsilon/\delta]$. So the lemma is proved for $k=1$ with $\overline{c}_1=2$.

We now prove the Lemma by induction on k. Let us suppose that the case k is known. As for $k = 1$, there exists an unique c such that $\phi^{(k)}(c) = 0$. Thus for all v we have $|\phi^{(k)}(v)| \geq \delta|v - c|$. Let $\eta > 0$ and set $W = Z(\phi, \varepsilon) \cap [c - \eta, c + \eta]$, $U = Z(\phi, \varepsilon) \cap (] - \infty, c - \eta[\cup]c + \eta, +\infty[)$. We have $\text{meas}(W) \leq 2\eta$ and by our inductive hypothesis, since $|\phi^{(k)}(v)| \geq \delta|v - c| \geq \delta\eta$ on U, $\text{meas}(U) \leq \overline{c}_k(\varepsilon/(\delta\eta))^{1/k}$. Now the relation $Z(\phi, \varepsilon) = W \cup U$ gives $\text{meas}(Z(\phi, \varepsilon)) \leq \inf_{\eta>0} \left(2\eta + \overline{c}_k(\varepsilon/(\delta\eta))^{1/k}\right)$ which implies by a simple computation of the minimum that $\text{meas}(Z(\phi, \varepsilon)) \leq \overline{c}_{k+1}(\varepsilon/\delta)^{1/(k+1)}$, where $\overline{c}_{k+1} = 2^{1/(k+1)}(k + 1)k^{1/(k+1)-1}\overline{c}_k^{1-1/(k+1)}$ which concludes the proof. $\square$

Lemma 4 *Let P be a compact set of parameters, k a positive integer, $A > 0$, $V = [-A, A]$, $K = V \times P$, $\phi(v; p) \in C^0(P, C^k(V, \mathbb{R}))$, such that, for all (v, p) in the compact K, we have*

$$\sum_{j=1}^{k} \left|\frac{\partial^j \phi}{\partial v^j}\right|(v; p) \quad > \quad 0.$$

Let $Z(\phi(.; p), \varepsilon) = \{v \in V,\ |\phi(v; p)| \leq \varepsilon\}$, then there exists a constant C such that

$$\sup_{p\in P} meas(Z(\phi(.; p), \varepsilon)) \quad \leq \quad C\varepsilon^{1/k}.$$

Proof. Since K is a compact set, we can choose $0 < \delta \leq 1$ such that, everywhere on K, we have $0 < 2\delta < \frac{1}{k}\sum_{i=1}^{k} \left|\frac{\partial^i \phi}{\partial v^i}\right|(v; p)$.

For each $(v; p) \in K$, there exists an integer $i \in \{1, \cdots, k\}$, a number $r > 0$ and an open set O_p with $p \in O_p \subset P$ such that $|\partial^i_v \phi| > \delta$ on $U(v, p) =]v - r, v + r[\times O_p$. Therefore, we have

$$\text{meas}(Z(\phi(.; p), \varepsilon)\cap]v - r, v + r[) \leq \overline{c}_i(\varepsilon/\delta)^{1/i} \leq \overline{c}\,\varepsilon^{1/k}/\delta$$

using Lemma 3, where $\overline{c} = \max_{i=1,\cdots,k} \overline{c}_i$.

By compactness of K, there exists a finite number of such set $U_j = U(v_j, p_j)$ such that $K \subset \bigcup_{j=1}^{\nu} U_j$. Thus, for each p, $Z(\phi(.; p), \varepsilon)$ intersects at most ν intervals $]v_j - r_j, v_j + r_j[$ where Lemma 3 is applied. This allows to write $\text{meas}(Z(\phi(.; p), \varepsilon)) \leq \nu c\,\varepsilon^{1/k}/\delta$ for all p and to conclude the proof. $\square$

Lemma 5 *Let $a(v)$ be the field $(v^1, v^2, \cdots, v^N)$ then $\alpha_{a(.)} = 1/N$.*

Proof. From Lemma 2, we have yet $\alpha_{a(.)} \leq 1/N$. So, it suffices to prove that $\alpha = 1/N$ satisfies (1.5) to conclude.

For all v, $rank\{a'(v), \cdots, a^{(N)}(v)\} = N$, thus it is impossible to find $\sigma \in S^{N-1}$ such that $\sigma \perp \{a'(v), \cdots, a^{(N)}(v)\}$. Let $\varphi(v; u, \sigma)$ be $a(v) \cdot \sigma - u$. Since

$\partial_v^j \varphi(v;u,\sigma) = a^{(j)}(v)\cdot\sigma$ for $j \geq 1$, we have everywhere $\sum_{j=1}^{N} |\partial_v^j \varphi(v;u,\sigma)| > 0$.

Furthermore, for $|u| > 1 + a_{max}$, where $a_{max} = \sup_{|v|\leq A} |a(v)|$, we have $|\varphi(v;u,\sigma)| > 1$ for any $v \in [-A,A]$ and $\sigma \in S^{N-1}$. Thus we can apply Lemma 4 for $0 < \varepsilon \leq 1$ on the compact set $[-A,A]_v \times [-a_{max}-1, a_{max}+1]_u \times S_\sigma^{N-1}$ which concludes the proof with $\alpha_{a(.)} = 1/N$. □

Proof of Proposition 7. With Lemma 2, we have $\alpha_{opt}(N,1) \leq 1/N$. From Lemma 5, necessarily $\alpha_{opt}(N,1) = 1/N$ which concludes the proof. □

4.2 $M = N$

The case when the space dimension is equal to the velocity dimension is the most physical one and then is very important. In this case, we can get the best smoothing effect with $\alpha = 1$.

Proposition 8 *For $N = M$, we have $\alpha_{opt}(N,N) = 1$.*

Proof. Since $\alpha \leq 1$, it suffices to find $a(.)$ such that $\alpha = 1$.
Let $a(.) : \mathbb{R}_v^N \to \mathbb{R}_x^N$ be a global diffeomorphism, $A > 0$, $(u,\sigma) \in S^N$ and $\varphi(v) = a(v)\cdot\sigma - u$. Let $Z(\varphi,\varepsilon) = \{|v| \leq A, |\varphi(v)| \leq \varepsilon\}$. Since $Da(v) \in GL_N(\mathbb{R})$ and $\sigma \neq 0$, then $\nabla_v\varphi \neq 0$ and the set $Z(\varphi,0)$ is empty or a manifold of dimension $N-1$.
Notice that for any v, there exists $(u,\sigma) \in S^N$ such that $a(v)\cdot\sigma - u = 0$, i.e. $Z(\varphi,0) \neq \phi$. For instance, let $\widetilde{\sigma}$ belongs in S^{N-1} and set $\widetilde{u} = a(v)\cdot\widetilde{\sigma}$, then $(u,\sigma) = \dfrac{1}{\sqrt{\widetilde{u}^2+1}}(\widetilde{u},\widetilde{\sigma})$ satisfied the conditions.
We thus consider that $Z(\varphi,0)$ is not empty.
There exists δ such that $0 < \delta < |\nabla_v\varphi(v)| < 1/\delta$ for all $|v| \leq A, u^2 + |\sigma|^2 = 1$.
Using the mean inequality, we obtain $\delta|v-v'| \leq |\varphi(v) - \varphi(v')| \leq \dfrac{|v-v'|}{\delta}$, which implies for all $\varepsilon < 1$, with $B(x,r) = \{y, |x-y| \leq r\} \subset \mathbb{R}^N$, that

$$\bigcup_{z\in Z(\varphi,0)} B(z,\delta\varepsilon) \quad \subset Z(\varphi,\varepsilon) \subset \bigcup_{z\in Z(\varphi,0)} B(z,\varepsilon/\delta)$$

and $Z(\varphi,0)$ is diffeomorph to a hyperplane, so $meas(Z(\varphi,\varepsilon))$ is of order ε. More precisely, there exists a constant $C > 0$, only dependent of A, δ and $||Da(.)||_{B(0,A)}$ such that $0 < C < \dfrac{meas(Z(\varphi,\varepsilon))}{\varepsilon} < C^{-1}$.
Notice that if $a(.)$ is a local diffeomorphism, α is still 1. □

Incidentally, we also have $\alpha_{opt}(N,M) = 1$ for all $M \geq N$.

5 Theorem in the L^p framework

We now turn to the L^p case. It will be an interpolation result of the L^2 obtained bound and an estimate in L^1 using some operators in Hardy spaces. We note

$\mathcal{H}^1(\mathbb{R}^{N+1})$ the Hardy space and $\mathcal{H}^1(\mathbb{R}^N\times\mathbb{R})$ the product Hardy space as done in [2] (see [22] for more details about such spaces).

We will use the two following Proposition. The first one is an interpolation result (see [18], [2] and [5]) and the second is about multiplier ([2]).

Proposition 9 (Bézard, Interpolation) *Let T be a $\mathbb{C}$-linear operator, bounded in*

$$L^2(\mathbb{R}_t\times{\mathbb{R}^N}_x\times{\mathbb{R}^M}_v)\to W^{\beta,2}(\mathbb{R}_t\times\mathbb{R}^N_x),$$

and in

$$L^1(\mathbb{R}^M_v,\mathcal{H}^1(\mathbb{R}^N\times\mathbb{R}))\to\mathcal{H}^1(\mathbb{R}^{N+1}_{t,x}),$$

for some $\gamma\geq 0$. Then T is bounded

$$L^p(\mathbb{R}_t\times\mathbb{R}^N_x\times\mathbb{R}^N_v)\to W^{s,p}(\mathbb{R}_t\times\mathbb{R}^N_x),$$

for $1<p\leq 2$, with $s=2\beta/p'$.

Proposition 10 (Bézard, Multiplier on $\mathcal{H}^1$) *Let $m(y,y_{N+1})$ be a function of $(y,y_{n+1})\in\mathbb{R}^N\times\mathbb{R}$ which is C^∞ out of $[y=0$ or $y_{N+1}=0]$, and verifying for all α, β,*

$$|\partial_y^\alpha\partial_{y_{N+1}}^\beta m(y,y_{N+1})|\leq\frac{C_{\alpha\beta}}{|y|^\alpha|y_{N+1}|^\beta},$$

then m defines a bounded Fourier multiplier on $\mathcal{H}^1(\mathbb{R}^N\times\mathbb{R})$.

Proof of Theorems 4 and 5.
For Theorem 4 (respectively Theorem 5), we use the averaging lemma of Theorem 1 (respectively Theorem 2) which gives that $T(f,g)=\rho_\psi$ is bounded from L^2 to $H^{\alpha/2}_{loc}$ (respectively $H^{1/\gamma}$).

We turn to the estimate in L^1. We denote by $\mathcal{F}$ the Fourier transform with respect to X. Taking this Fourier transform in $b(v)\cdot\nabla_X f+F(X)\cdot\nabla_v f=g$, we have

$$\mathcal{F}(f)=\frac{\mathcal{F}(g)-\mathcal{F}(F\cdot\nabla_v f)}{i(b(v)\cdot Y)}.$$

Let $\chi\in C_c^\infty(\mathbb{R})$, $\chi(0)=1$, $\chi'(0)=0$ and $\chi''(0)\neq 0$ be an even, non increasing function in $[0,+\infty[$. We set L such that $\mathrm{supp}\chi\subset[-L,L]$. We have

$$\begin{aligned} f(Y,v) &= \mathcal{F}^{-1}\Big[\chi(b(v)\cdot Y)\mathcal{F}(f)(Y,v)+(1-\chi(b(v)\cdot Y))\mathcal{F}(f)(Y,v)\Big]\\ &= \mathcal{F}^{-1}\Big[\chi(b(v)\cdot Y)\mathcal{F}(f)(Y,v)\Big]\\ &\quad+\mathcal{F}^{-1}\Big[(1-\chi(b(v)\cdot Y))\frac{\mathcal{F}(g)-\mathcal{F}(F\cdot\nabla_v f)}{i(b(v)\cdot Y)}\Big],\end{aligned}$$

and then, in order to bound the operator $f\mapsto\displaystyle\int_{\mathbb{R}^M}f(Y,v)\psi(v)\,dv$, we have to bound the following operators

$$Q:f\mapsto\int_{\mathbb{R}^M}\mathcal{F}^{-1}\Big[\chi(b(v)\cdot Y)\mathcal{F}(f)(Y,v)\Big]\psi(v)\,dv,\tag{5.1}$$

$$W : g \mapsto \int_{\mathbb{R}^M} \mathcal{F}^{-1} \left[\frac{1-\chi(b(v)\cdot Y)}{i(b(v)\cdot Y)} \mathcal{F}(g)(Y,v) \right] \psi(v)\, dv \tag{5.2}$$

and

$$R : f \mapsto -\int_{\mathbb{R}^M} \mathcal{F}^{-1} \left[\frac{1-\chi(b(v)\cdot Y)}{i(b(v)\cdot Y)} \mathcal{F}(F\cdot\nabla_v f)(Y,v) \right] \psi(v)\, dv. \tag{5.3}$$

As in the classical case (by this we refer to [2], [5]), we transform the operators in order they involve only one direction in X. Indeed, the manipulation of product structure for Hardy space which depends of a moving direction is difficult to deal with. Thus, for any v, we take R_v an orthogonal transform in $\mathbb{R}^{N+1}$ such that

$$R_v \left(\frac{b(v)}{|b(v)|} \right) = e_{N+1},$$

where e_{N+1} is the last vector of the canonical base, and we set

$$f_*(X,v) = f(R_v^{-1}(X), v)$$

and

$$Q_* f_* = Qf.$$

Since $f \mapsto f_*$ is an isometry on L^p_{Xv}, we have now to study Q_* instead of Q. We perform similar transformations for the two other operators and we get W_* and R_*.
For the two first operators, as in the classical proof, we have

$$\|Qf\|_{\mathcal{H}^1(\mathbb{R}^{N+1})} \le C \|f\|_{L^1(\mathbb{R}^M_v, \mathcal{H}^1(\mathbb{R}^N\times\mathbb{R}))},$$

and

$$\|Wg\|_{\mathcal{H}^1(\mathbb{R}^{N+1})} \le C \|g\|_{L^1(\mathbb{R}^M_v, \mathcal{H}^1(\mathbb{R}^N\times\mathbb{R}))}.$$

The new term is the third one. We use the following rewrite of $R(f)$ in order to bound it. This is

$$\begin{aligned}
(Rf)(Y) &= -\mathcal{F}^{-1} \int_{\mathbb{R}^M} \left[\frac{1-\chi(b(v)\cdot Y)}{i(b(v)\cdot Y)} F\cdot\nabla_v \mathcal{F}(f)(Y,v) \right] \psi(v)\, dv \\
&= \mathcal{F}^{-1} \left(F \cdot \int_{\mathbb{R}^M} \mathcal{F}(f)(Y,v)\, \nabla_v \left[\frac{1-\chi(b(v)\cdot Y)}{i(b(v)\cdot Y)} \psi(v) \right] dv \right) \\
&= \mathcal{F}^{-1} \left(F \cdot \int_{\mathbb{R}^M} \mathcal{F}(f)(Y,v)\, \frac{1-\chi(b(v)\cdot Y)}{i(b(v)\cdot Y)} \nabla_v \psi(v)\, dv \right) \\
&\quad + \mathcal{F}^{-1} \left(F \cdot \int_{\mathbb{R}^M} \mathcal{F}(f)(Y,v)\, m_0(b(v)\cdot Y) \nabla_v (b(v)\cdot Y)\, \psi(v)\, dv \right)
\end{aligned} \tag{5.4}$$

with

$$m_0(y) = \frac{-y\chi'(y) - 1 + \chi(y)}{iy^2}.$$

We denote by $\mathcal{F}(R_1 f)$ and $\mathcal{F}(R_2 f)$ the two terms of this decomposition. We perform as previously orthogonal transformations and we have to study the obtained $(R_1)_*$ and $(R_2)_*$.
The term $(R_1)_*$ is the same than W_* but with $\nabla_v \psi$ instead of ψ. Thus we have the same result thanks to the regularity assumption on ψ.
Now, setting $T = m_0 \nabla_v$, we have

$$\begin{aligned}
(R_2)_*(f_*)(Y) &= F \cdot \int_{\mathbb{R}^M} \mathcal{F}^{-1}\Big(\mathcal{F}(f_*)(R_v(Y), v)\, T\Big(b(v) \cdot Y\Big)\Big)\, \psi(v)\, dv \\
&= F \cdot \int_{\mathbb{R}^M} \mathcal{F}^{-1}\Big(\mathcal{F}(f_*)(R_v(Y), v)\, T\Big(R_v(b(v)) \cdot R_v(Y)\Big)\Big)\, \psi(v)\, dv \\
&= F \cdot \int_{\mathbb{R}^M} \mathcal{F}^{-1}\Big(\mathcal{F}(f_*)(R_v(Y), v)\, T\Big(|b(v)| e_{N+1} \cdot R_v(Y)\Big)\Big)\, \psi(v)\, dv,
\end{aligned}$$

thus, setting $T_j = m_0 \partial_{v_j}$, we get

$$\begin{aligned}
&\|(R_2)_*(f_*)\|_{\mathcal{H}^1(\mathbb{R}^{N+1})} \\
\leq\ & \sum_j |F_j| \int_{\mathbb{R}^M} \Big\| \mathcal{F}^{-1}\Big(\mathcal{F}(f_*)(R_v(Y), v)\, T_j\Big(|b(v)| e_{N+1} \cdot R_v(Y)\Big)\Big) \Big\|_{\mathcal{H}^1(\mathbb{R}^{N+1})} |\psi(v)|\, dv \\
\leq\ & \sum_j |F_j| \int_{\mathbb{R}^M} \Big\| \mathcal{F}^{-1}\Big(\mathcal{F}(f_*)(Y, v)\, T_j\Big(|b(v)| e_{N+1} \cdot Y\Big)\Big) \Big\|_{\mathcal{H}^1(\mathbb{R}^{N+1})} |\psi(v)|\, dv \\
\leq\ & C_1 \sum_j |F_j| \int_{\mathbb{R}^M} \Big\| \mathcal{F}^{-1}\Big(\mathcal{F}(f_*)(Y, v)\, T_j\Big(|b(v)| e_{N+1} \cdot Y\Big)\Big) \Big\|_{\mathcal{H}^1(\mathbb{R}^N \times \mathbb{R})} |\psi(v)|\, dv,
\end{aligned}$$

using the invariance under orthogonal transformation in $\mathcal{H}^1(\mathbb{R}^{N+1})$ and thanks to the continuous injection of $\mathcal{H}^1(\mathbb{R}^N \times \mathbb{R})$ in $\mathcal{H}^1(\mathbb{R}^{N+1})$.
We use now the Proposition 10 with the terms

$$m_j(y, y_{N+1}) = T_j(|b(v)| e_{N+1} \cdot Y) = m_0(|b(v)| y_{N+1}) \partial_{v_j}(|b(v)|) y_{N+1}, \quad \text{for } j = 1, \cdots, M.$$

This term rewrites

$$m_j(y, y_{N+1}) = m_0(|b(v)| y_{N+1}) \frac{a(v) \cdot \partial_{v_j} a(v)}{|b(v)|} y_{N+1}.$$

Now $m_0(z) \underset{z \to 0}{\longrightarrow} -\frac{1}{2i} \chi''(0)$, therefore m_0 is C^∞. The terms with χ have a compact support and the other term is $1/y^2$, then every derivatives of m_0 is bounded at infinity.
We differentiate m_j with respect to y_{N+1}, it gives

$$\begin{aligned}
\partial^k_{y_{N+1}} m_j(y, y_{N+1}) &= \frac{a(v) \cdot \partial_{v_j} a(v)}{|b(v)|} \Big(m_0^{(k)}(|b(v)| y_{N+1}) |b(v)|^k y_{N+1} \\
&\qquad + k m_0^{(k-1)}(|b(v)| y_{N+1}) |b(v)|^{k-1} \Big).
\end{aligned}$$

There exists some constants C and C_k such that

$$|b(v)| \leq C, \qquad |b(v)|^{k-2} |a(v) \cdot \partial_{v_j} a(v)| \leq C_k$$

for v in the compact support of ψ. Thus

$$\left|\partial^k_{y_{N+1}} m_j(y,y_{N+1})\right| |y_{N+1}|^k \leq C_k \left(Cm_0^{(k)}(|b(v)|y_{N+1})y_{N+1} + km_0^{(k-1)}(|b(v)|y_{N+1})\right).$$

For $|y_{N+1}| \geq (R+1)/C$, we have $m_0^{(j)}(|b(v)|y_{N+1}) = 0$ for any j, and then $m_0^{(k)}(|b(v)|y_{N+1})y_{N+1} + km_0^{(k-1)}(|b(v)|y_{N+1}) = 0$ for $|y_{N+1}| \geq (R+1)/C$. Furthermore $|m_0^{(k)}(|b(v)|y_{N+1})y_{N+1} + km_0^{(k-1)}(|b(v)|y_{N+1})| \leq \|m_0^{(k)}\|_\infty \dfrac{R+1}{C} + k\|m_0^{(k-1)}\|_\infty$ for $|y_{N+1}| < (R+1)/C$. Finally, for any (y, y_{N+1}), we get

$$\left|\partial^k_{y_{N+1}} m_j(y,y_{N+1})\right| |y_{N+1}|^k \leq C_k \left(\|m_0^{(k)}\|_\infty (R+1) + k\|m_0^{(k-1)}\|_\infty\right)$$

uniformly with respect to v in the support of ψ. Then, we can apply Proposition 10 to get the boundary of $(R_2)_*$.
The interpolation result conclude, since $\beta = \alpha/2$ (respectively $\beta = 1/\gamma$), that the obtained regularity is $s = \alpha/p'$ (respectively $s = 2/(\gamma p')$). $\square$

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