Theoretical study of a multi-dimensional pressureless model with unilateral constraint

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Abstract

The aim of this paper is to extend to multi-dimension the study of a pressureless model of gas system with unilateral constraint. Several difficulties are added with respect to the one-dimensional case. Indeed the geometry of the dynamics of blocks cannot be conserved and to solve this problem, we approximate the motion of each block by discrete jumps in all the directions separately in consecutive time steps. This leads to approximations of solutions for special initial data. Then the stability of these approximations have to be adapted to this new situation. We finally get the existence and the stability of solutions.

Key-words: conservation laws with constraint – pressureless gas – sticky blocks – splitting dynamics

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1 Introduction

1.1 Context

We consider the system of pressureless gases

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) = 0, \end{cases}$$

which was studied in [8], [11], [13], [15], [10]. It is known that this system gives Dirac distributions on ρ in finite time, even for smooth initial data. It is clearly incompatible with a constraint for the density. In [9], a system arises in the modeling of two-phase flows as

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \pi) = 0, \end{cases} \tag{1.1}$$

with constraint and pressure Lagrange multiplier

$$0 \le \rho \le 1, \qquad \pi \ge 0, \tag{1.2}$$

and extremality relation

$$(1-\rho)\pi = 0. \tag{1.3}$$

With respect to pressureless gases, the pressure term π allows to get the constraint $\rho \leq 1$ and when $\rho \neq 1$, $\pi = 0$ and we recover the pressureless system. Let us explain how the model (1.1)-(1.3) arises. We consider a liquid and gas two-phase flow. We denote ρ the volume fraction of the liquid, n its density and u its velocity. We denote $\rho_g = 1-\rho$ the volume fraction of the gas, n_g its density and u_g its velocity. Following [16], we can write two mass conservation equations and two momentum conservation equations,

$$\begin{cases} \partial_t(\rho n) + \partial_x(\rho n u) = 0, \\ \partial_t(\rho n u) + \partial_x(\rho n u^2) + \rho \partial_x p + \tau_l = M_l^D, \\ \partial_t(\rho_g n_g) + \partial_x(\rho_g n_g u_g) = 0, \\ \partial_t(\rho_g n_g u_g) + \partial_x(\rho_g n_g u_g^2) + \rho_g \partial_x p + \tau_g = M_g^D, \end{cases}$$

where the right-hand sides M_l^D and M_g^D are source terms reflecting interphase drag and τ_l and τ_g are phase pressure fluctuations around p the common pressure. As explained in [9], a simplified model can be derived for an incompressible liquid, and an infinitely light gas. We choose the standard simplified closure laws

$$M_l^D = -M_g^D = \mu\rho(1-\rho)\rho_l(u_g - u),$$

$$\tau_l = C_p n (u_g - u)^2 \partial_x \rho, \qquad \tau_g = 0,$$

and we assume the liquid to be incompressible, $n = \textit{constant}$, and the pressure to be governed by the law $p = p(n_g) = \kappa n_g^\gamma$. We then introduce a scaling of the gas density $\tilde{n}_g = n_g/n_0$, where n_0 is the average of n_g. We also set $\tilde{p} = p/n_0^\gamma$. In the new variable $\tilde{n_g}$ (we shall drop the tilde for convenience), we perform the Chapman-Enskog expansion in $\varepsilon = n_0/n_l << 1$, supposing that $u_g - u = O(\varepsilon)$, and we get the model

$$\partial_t \begin{pmatrix} (1-\rho)n_g \\ \rho \\ \rho u \end{pmatrix} + \partial_x \begin{pmatrix} (1-\rho)n_g u \\ \rho u \\ \rho u^2 + \varepsilon n_0^{\gamma-1} p(n_g) \end{pmatrix} = \varepsilon \begin{pmatrix} D(\rho, u, n_g) \\ 0 \\ 0 \end{pmatrix},$$

where $D(\rho, u, n_g)$ is a diffusive second-order term.
If we let $\varepsilon \to 0$ formally, we get the system of pressureless gases

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) = 0. \end{cases}$$

As we noticed previously, it is clearly incompatible with the constraint $\rho \leq 1$. Thus, we cannot neglect the pressure term and we make the more realistic assumption $\varepsilon n_0^{\gamma-1} p(n_g) \rightharpoonup \pi$ as $\varepsilon \to 0$, supposing that this term appears only when $\rho = 1$.
We finally get the model (1.1)-(1.3). The pressure-type term π is a Lagrange multiplier, it represents the residual pressure of the gas which has infinite density and volume fraction $1 - \rho = 0$. For more details, we refer to [9].

The model (1.1)-(1.3) is an hyperbolic constraint model which corresponds to gas dynamics when $\pi = 0$ and gives a bound for the density. Existence and weak stability of suitable weak solutions is obtained in [2].

There are now a lot of domains in which constraints models take place. For example, it allowed to get better models in traffic jams since paper [6]. Indeed, we start from the Aw-Rascle model,

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho(u + p(\rho))) + \partial_x(\rho u(u + p(\rho))) = 0, \end{cases}$$

which is a very well accepted model for traffic flow. We observe that, in this model, upper bounds on the density are not necessarily preserved through the time evolution of the solution. In practice, the density of cars is bounded from above by a maximal density ρ^* corresponding to a bumper to bumper situation. However, the Aw-Rascle model does not exclude cases where, depending on the smallest invariant region which contains the initial data, solutions satisfy the maximal density constraint $\rho \leq \rho^*$ initially but evolve in finite time to a state, still uniformly bounded, but which violates this constraint. Then paper [6] presents a model which improve the Aw-Rascle model and preserves the constraints. In order to obtain this, we take in the Aw-Rascle model, the pressure

$$p_\varepsilon(\rho) = \varepsilon \left(\frac{1}{\rho} - \frac{1}{\rho^*} \right)^{-\gamma} \mathbb{1}_{\rho \leq \rho^*}$$

and assuming that this term have a limit π when $\varepsilon \to 0$, which acts only when $\rho = \rho^*$, it leads to the system

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho(u + \pi)) + \partial_x(\rho u(u + \pi)) = 0, \end{cases}$$

with the constraints

$$0 \leq \rho \leq \rho^*, \quad \pi \geq 0, \quad (\rho^* - \rho)\pi = 0.$$

After this paper, some improvements of the model have been completed in [7] for ρ^* depending on u (case where le maximum density of cars depends on the cars velocities) and in [5] for ρ^* depending on x (multi-lines case).

We could find other hyperbolic problems with constraints in [1], [17] and [18]. In [4], the isentropic case of the problem (1.1)-(1.3) was studied with other constraints. See also [3] for a numerical version of this kind of problems. The case with viscosity was studied in [19]. In that direction, the limit of barotropic compressible Navier-Stokes to constraint Navier-Stokes was proved in [12] for the one-dimensional case and in [20] for the multi-dimensional case.

1.2 Main result

In the present paper, we are focusing on extending the existence and stability result of [2] for this system (1.1)-(1.3) in multi-dimension.

An important tool for this result is the sticky block dynamics. In dimension one, the density and momentum of blocks are sum of terms of the form

$$(\rho(t), \rho(t)u(t)) = (1, u_i(t)) \mathbb{1}_{a_i^l(t) \leq x \leq a_i^r(t)}$$

with a density equals to 1 and a velocity $u_i(t)$ constant on the block $a_i^l(t) \leq x \leq a_i^r(t)$. The time evolution is defined as follows. The number of blocks n indeed depends on t, but is piecewise constant. As long as the blocks do not meet, they move at constant velocity $u_i(t)$. When two or more blocks collide, they get stuck, building a new block. Then, in dimension one, the dynamics of blocks is easy because after a collision, we still have a single block.

In multi-dimension, we extend the notion of blocks as

$$(\rho(t), \rho(t)u(t)\rho(t)v(t)) = (1, u_i(t), v_i(t))\mathbb{1}_{a_i^l(t)\leq x\leq a_i^r(t)}\mathbb{1}_{b_i^l(t)\leq y\leq b_i^r(t)}$$

with density equals to 1 and velocity $(u_i, v_i)(t)$ constant on the block $a_i^l(t) \leq x \leq a_i^r(t)$, $b_i^l(t) \leq y \leq b_i^r(t)$. Then a geometric problem appears since when two rectangular parallelepipeds collide, they do not form a rectangular parallelepiped. One idea of this paper to pass over this difficulty is to approximate the motion of each block by discrete jumps in all the directions separately in consecutive time steps. In other words, we make a splitting with respect to the various directions of space on consecutive time steps and then, on each time interval, we do vary only one direction then, on the next interval, another direction and so on to keep the geometry at each collision. Then by letting the time step going to 0 and thereby forcing the splitting to be more rapid, we hope to find the limit of the speed on any directions. The purpose of this paper is to achieve this approach and prove that it works.

Furthermore for block initial data in the one-dimensional case, we get explicit solutions. Here, in the multi-dimensional case, we will only get approximations of solutions for these special initial data. Then, the stability and existence of solutions will require additional steps to work.

In order to simplify the presentation, we will detail the two-dimensional case, knowing that the ideas and proofs are the same in any dimension. We will consider the following model with constraint in two dimensions which is the natural extension of (1.1)-(1.3):

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) + \partial_y(\rho v) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \pi_1) + \partial_y(\rho uv) = 0, \\ \partial_t(\rho v) + \partial_x(\rho uv) + \partial_y(\rho v^2 + \pi_2) = 0, \end{cases} \tag{1.4}$$

with the constraints

$$0 \leq \rho \leq 1, \quad \pi_1 \geq 0, \quad \pi_2 \geq 0, \tag{1.5}$$

and the exclusion relations

$$\rho\pi_1 = \pi_1, \quad \rho\pi_2 = \pi_2. \tag{1.6}$$

Let us also consider initial data

$$\begin{cases} \rho(0, x, y) = \rho^0(x, y), \\ \rho(0, x, y)u(0, x, y) = \rho^0(x, y)u^0(x, y), \\ \rho(0, x, y)v(0, x, y) = \rho^0(x, y)v^0(x, y), \end{cases} \tag{1.7}$$

with the regularities

$$\rho^0 \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2), \quad u^0, v^0 \in L^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2). \tag{1.8}$$

Let us define precisely the weak solutions we shall consider. We are looking for solutions satisfying

$$\rho \in L^\infty_t(]0,+\infty[, L^\infty_{xy}(\mathbb{R}^2) \cap L^1_{xy}(\mathbb{R}^2)) \cap C_t([0,+\infty[, L^\infty_{w*}(\mathbb{R}^2)), \tag{1.9}$$

$$u, v \in L^\infty_t(]0,+\infty[, L^\infty_{xy}(\mathbb{R}^2)), \tag{1.10}$$

$$\pi_1, \pi_2 \in \mathcal{M}_{loc}([0,+\infty[\times\mathbb{R}^2). \tag{1.11}$$

Hence, (1.4), (1.7) must be satisfied in the sense of distributions:
for all $\varphi \in C^\infty_c([0,+\infty[\times\mathbb{R}^2)$,

$$\int_{[0,+\infty[} \iint_{\mathbb{R}^2} (\rho\partial_t\varphi + \rho u\partial_x\varphi + \rho v\partial_y\varphi)\, dx\, dy\, dt + \iint_{\mathbb{R}^2} \rho^0(x,y)\varphi(0,x,y)\, dx\, dy = 0, \tag{1.12}$$

$$\int_{[0,+\infty[} \iint_{\mathbb{R}^2} (\rho u\partial_t\varphi + \rho u^2\partial_x\varphi + \rho uv\partial_y\varphi)\, dx\, dy\, dt + \int_{[0,+\infty[} \iint_{\mathbb{R}^2} \partial_x\varphi\ \pi_1 + \iint_{\mathbb{R}^2} (\rho^0 u^0)(x,y)\varphi(0,x,y)\, dx\, dy = 0, \tag{1.13}$$

and

$$\int_{[0,+\infty[} \iint_{\mathbb{R}^2} (\rho v\partial_t\varphi + \rho uv\partial_x\varphi + \rho v^2\partial_y\varphi)\, dx\, dy\, dt + \int_{[0,+\infty[} \iint_{\mathbb{R}^2} \partial_y\varphi\ \pi_2 + \iint_{\mathbb{R}^2} (\rho^0 v^0)(x,y)\varphi(0,x,y)\, dx\, dy = 0. \tag{1.14}$$

Notice that the constraint (1.6) cannot be obtained for every solutions because the product is not necessarily defined in any case. It was already the case in dimension one. It is of course still the case in multi-dimension. We will have a discussion about this in section 5 to deal with cases where the product is well defined. We get the constraint (1.6) for blocks (definition in section 2) and for limits of blocks whose convergence of the densities is in $C_t([0,+\infty[, L^1_{xy}(\mathbb{R}^2))$ with measures $\pi_1, \pi_2 \in \mathcal{M}_t([0,+\infty[, L^\infty_{xy}(\mathbb{R}^2))$.

The main result we get in this paper is the following.

Theorem 1.1 (Existence of solutions) *Let us consider initial data (ρ^0, u^0, v^0) with regularities (1.8). Then there exists $(\rho, u, v, \pi_1, \pi_2)$, with regularities (1.9)-(1.11), which are solutions of (1.4) with the constraint (1.5) and satisfy the bounds*

$$0 \le \rho \le 1, \quad \iint_{\mathbb{R}^2} \rho(t,x,y)\, dx\, dy \le \iint_{\mathbb{R}^2} \rho^0(x,y)\, dx\, dy, \tag{1.15}$$

$$essinf\ u^0 \leq u \leq esssup\ u^0, \quad essinf\ v^0 \leq v \leq esssup\ v^0, \tag{1.16}$$

$$\iint\limits_{\mathbb{R}\times[-a,a]} |\partial_x u(t,x,y)| \leq (1+a_t^2) \iint\limits_{\mathbb{R}^2} \left|\partial_x u^0(x,y)\right|, \tag{1.17}$$

$$\iint\limits_{[-a,a]\times\mathbb{R}} |\partial_y u(t,x,y)| \leq (1+a_t^2) \iint\limits_{\mathbb{R}^2} \left|\partial_y u^0(x,y)\right|, \tag{1.18}$$

$$\iint\limits_{\mathbb{R}\times[-a,a]} |\partial_x v(t,x,y)| \leq (1+a_t^2) \iint\limits_{\mathbb{R}^2} \left|\partial_x v^0(x,y)\right|, \tag{1.19}$$

$$\iint\limits_{[-a,a]\times\mathbb{R}} |\partial_y v(t,x,y)| \leq (1+a_t^2) \iint\limits_{\mathbb{R}^2} \left|\partial_y v^0(x,y)\right|, \tag{1.20}$$

for any $a > 0$ and with $a_t = a + t\max(\|u^0\|_\infty, \|v^0\|_\infty)$. The solution $(\rho, \rho u, , \rho v)$ is the limit in the distributional sense of blocks $(\rho_k, \rho_k u_k, \rho_k v_k)$. In the case where $\pi_1, \pi_2 \in \mathcal{M}_t([0,+\infty[, L^\infty_{xy}(\mathbb{R}^2))$ and $\rho_k \to \rho$ in $C_t([0,+\infty[, L^1_{xy}(\mathbb{R}^2))$, then the products $\rho\pi_1$ and $\rho\pi_2$ exist and we also get the constraint (1.6).

The scheme of the proof we have to keep in mind to read the paper is the following. In order to get the existence of solutions, we approximate the initial data by blocks. For these blocks, we prove the existence of approximations of solutions. We obtain the limit of these approximations to get the existence of solutions for initial data with a block form. Finally, by a stability result, we find a solution for the starting initial data. We could draw the scheme of the proof in figure 1.

(ρ^0, u^0, v^0) initial data

approximation by blocks $\big\downarrow$ Proposition 2.3

(ρ^0_k, u^0_k, v^0_k) approximations by blocks initial data

dynamics of blocks $\big\downarrow$ Proposition 3.9

$(\rho_{kl}, u_{kl}, v_{kl}, (\pi_1)_{kl}, (\pi_2)_{kl})$ approximations of solutions

k fixed, $l \to +\infty$ $\big\downarrow$ Theorem 4.2

$(\rho_k, u_k, v_k, (\pi_1)_k, (\pi_2)_k)$ solutions with initial data (ρ^0_k, u^0_k, v^0_k)

$k \to +\infty$ $\big\downarrow$ Theorem 4.1

$(\rho, u, v, \pi_1, \pi_2)$ solution with initial data (ρ^0, u^0, v^0)

Figure 1.

The paper is organized as follows. In section 2, we define the notion of blocks and we prove a result of discretization of general initial data by blocks (first arrow of figure 1). In section 3, we study particular solutions and approximations of solutions in the class of blocks functions (second arrow of figure 1). In section 4, we obtain stability results for solutions and for approximations of solutions (third and fourth arrows of figure 1). Finally, in section 5, we study a space in which the constraint (1.6) can be taken and conclude to the existence result.

2 Discrete blocks

The first step of our proof is to define the blocks we are going to use and to give an approximation result of any initial data by block initial data.

2.1 Definition of blocks

Let us define the blocks we are going to use in the coming steps of the paper.

Definition 2.1 *We call block initial data a function* $(\rho^0, \rho^0 u^0, \rho^0 v^0)$ *depending on* (x, y) *of the form*

$$\rho^0(x,y)(1,u^0(x,y),v^0(x,y)) = \sum_{i=-I}^{I'} \sum_{j=-J}^{J'} \rho_{ij}(1,u_{ij},v_{ij}) \mathbb{1}_{(x,y)\in P_{ij}}, \tag{2.1}$$

where

$$\mathbb{1}_{(x,y)\in P_{ij}} = \mathbb{1}_{a_{ij}\le x\le b_{ij}} \mathbb{1}_{c_{ij}\le y\le d_{ij}}, \tag{2.2}$$

with $I, I', J, J' \in \mathbb{N}$ *and, for* $-I \le i \le I'$, $-J \le j \le J'$,

$$\rho_{ij} \in \{0,1\},$$

$a_{ij}, b_{ij}, c_{ij}, d_{ij}, u_{ij}, v_{ij} \in \mathbb{R}$ *such that* $b_{ij} \le a_{i+1,j}$ *and* $d_{ij} \le c_{i,j+1}$.

Definition 2.2 *Let* $\Delta t, \Delta x, \Delta y > 0$. *We call discrete block a function* $(\rho, \rho u, \rho v)$ *depending on* (t, x, y) *of the form*

$$\rho(t,x,y)(1,u(t,x,y),v(t,x,y)) = \sum_{i=-I}^{I'} \sum_{j=-J}^{J'} \sum_{l=0}^{+\infty} \rho_{ijl}(1,u_{ijl},v_{ijl}) \mathbb{1}_{(t,x,y)\in P_{ijl}}, \tag{2.3}$$

at level of discretization $(\Delta t, \Delta x, \Delta y)$, *where*

$$\mathbb{1}_{(t,x,y)\in P_{ijl}} = \mathbb{1}_{l\Delta t\le t<(l+1)\Delta t} \mathbb{1}_{a_{ijl}+i\Delta x\le x<a_{ijl}+(i+1)\Delta x} \mathbb{1}_{b_{ijl}+j\Delta y\le y<b_{ijl}+(j+1)\Delta y}, \tag{2.4}$$

with $I, I', J, J' \in \mathbb{N}$ *and, for* $-I \le i \le I'$, $-J \le j \le J'$, $l \in \mathbb{N}$,

$$\rho_{ijl} \in \{0,1\},$$

$a_{ijl}, b_{ijl}, c_{ijl}, d_{ijl}, u_{ijl}, v_{ijl} \in \mathbb{R}$ *such that* $a_{ijl}+\Delta x \le a_{i+1,jl}$ *and* $b_{ijl}+\Delta y \le b_{i,j+1,l}$.

Remark 2.1 It looks like the standard numerical discretization, taking a function piecewise constant on a square grid. But the density takes only values 0 and 1 and we have to take into account the time's evolution.

Remark 2.2 To simplify the presentation, we can assume that $I = J = 0$ which is just a translation of indices and $I' = J'$ by adding zero terms to have the same number of terms (adding some $\rho_{ij}(1, u_{ij}, v_{ij})\mathbb{1}_{(x,y)\in P_{ij}}$ with $\rho_{ij} = 0$). In the following, we may sometimes use this change of notations by setting $N := I' + 1 = J' + 1$.

Remark 2.3 The two definitions above are consistent together because for $t = 0$, the relation (2.3) has only the term for $l = 0$ remaining and we get

$$\rho(0, x, y)(1, u(0, x, y), v(0, x, y)) = \sum_{i=-I}^{I'} \sum_{j=-J}^{J'} \rho_{ij0}(1, u_{ij0}, v_{ij0})\mathbb{1}_{(0,x,y)\in P_{ij0}},$$

with

$$\mathbb{1}_{(0,x,y)\in P_{ij0}} = \mathbb{1}_{a_{ij0}+i\Delta x\le x<a_{ij0}+(i+1)\Delta x}\mathbb{1}_{b_{ij0}+j\Delta y\le y<b_{ij0}+(j+1)\Delta y}.$$

2.2 Discretization with blocks

We will improve here a result of [2] and [7] in order to get the following approximation lemma of initial data. We approximate any initial data by blocks initial data. A technical difficulty is here to deal with multi-variable functions instead of real-variable functions. In particular, the arguments with BV functions have to be changed. There are also additional difficulties in the definition of the blocks which approximate the initial data. It leads to more complicated BV estimates.

Remark 2.4 We first start by recalling the one dimensional context. To pass to the limit in multi-dimension situations, instead of TV estimate of W, we need to consider $|\partial_x W|$, ...

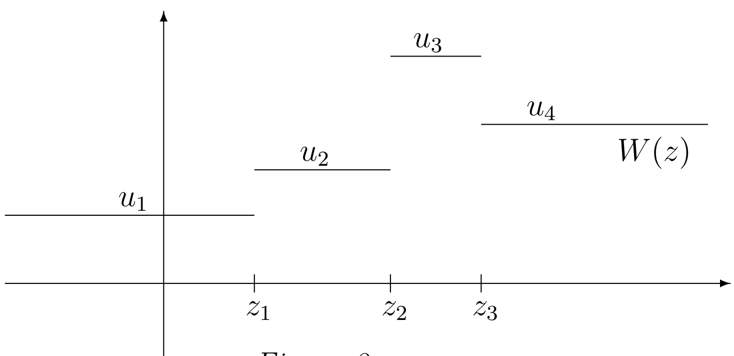

Figure 2.

The function W has for derivative a measure in the distributional sense given by

$$W'(z) = (u_2 - u_1)\delta_{z_1}(z) + (u_3 - u_2)\delta_{z_2}(z) + (u_4 - u_3)\delta_{z_3}(z)$$

and

$$\int_{\mathbb{R}} |W'(z)| = |u_2 - u_1| + |u_3 - u_2| + |u_4 - u_3|,$$

where δ is the Dirac distribution.

This result can be extended to functions $W(x,y)$ varying with respect to x at y fixed or varying with respect to y at x fixed. In the case of variations on $x(=z)$, those variations being piecewise constants, with y being fixed, that is to say for example

$$W(x,y) = u_1 \mathbb{1}_{x\in]-\infty,z_1[} + u_2 \mathbb{1}_{x\in]z_2,z_3[} + u_3 \mathbb{1}_{x\in]z_3,z_4[} + u_4 \mathbb{1}_{x\in]z_4,+\infty[}, \tag{2.5}$$

then we have

$$\int_{\mathbb{R}} |\partial_x W(x,y)| = |u_2 - u_1| + |u_3 - u_2| + |u_4 - u_3|.$$

Similarly, for the case of variations on $y(=z)$, those variations being piecewise constants, with x being fixed, that is to say for example

$$W(x,y) = u_1 \mathbb{1}_{y\in]-\infty,z_1[} + u_2 \mathbb{1}_{y\in]z_2,z_3[} + u_3 \mathbb{1}_{y\in]z_3,z_4[} + u_4 \mathbb{1}_{y\in]z_4,+\infty[},$$

then we have

$$\int_{\mathbb{R}} |\partial_y W(x,y)| = |u_2 - u_1| + |u_3 - u_2| + |u_4 - u_3|.$$

The approximation by blocks result for initial data is the following.

Proposition 2.3 *Let $\rho^0 \in L^1(\mathbb{R}^2)$, $u^0, v^0 \in L^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$ such that $0 \leq \rho^0 \leq 1$. Then, there exists a sequence of block initial data $(\rho_k^0, u_k^0, v_k^0)_{k\geq 1}$ such that, for any $k \in \mathbb{N}^*$,*

$$\rho_k^0 \in L^1(\mathbb{R}^2), \quad u_k^0, v_k^0 \in L^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2) \tag{2.6}$$

with the bounds

$$0 \leq \rho_k^0 \leq 1, \quad \iint\limits_{\mathbb{R}^2} \rho_k^0(x,y)\,dx\,dy \leq \iint\limits_{\mathbb{R}^2} \rho^0(x,y)\,dx\,dy, \tag{2.7}$$

$$essinf\ u^0 \leq u_k^0 \leq esssup\ u^0, \quad essinf\ v^0 \leq v_k^0 \leq esssup\ v^0, \tag{2.8}$$

$$\iint\limits_{\mathbb{R}\times[-a,a]} \left|\partial_x u_k^0(x,y)\right| \leq (1+a^2) \iint\limits_{\mathbb{R}^2} \left|\partial_x u^0(x,y)\right|, \tag{2.9}$$

$$\iint\limits_{[-a,a]\times\mathbb{R}} \left|\partial_y u_k^0(x,y)\right| \leq (1+a^2) \iint\limits_{\mathbb{R}^2} \left|\partial_y u^0(x,y)\right|, \tag{2.10}$$

$$\iint\limits_{\mathbb{R}\times[-a,a]} \left|\partial_x v_k^0(x,y)\right| \leq (1+a^2) \iint\limits_{\mathbb{R}^2} \left|\partial_x v^0(x,y)\right|, \tag{2.11}$$

$$\iint\limits_{[-a,a]\times\mathbb{R}} \left|\partial_y v_k^0(x,y)\right| \leq (1+a^2) \iint\limits_{\mathbb{R}^2} \left|\partial_y v^0(x,y)\right|, \tag{2.12}$$

for any $a > 0$, and for which the convergences $\rho_k^0 \rightharpoonup \rho^0$, $\rho_k^0 u_k^0 \rightharpoonup \rho^0 u^0$ and $\rho_k^0 v_k^0 \rightharpoonup \rho^0 v^0$ hold in the distributional sense.

Proof. Let $k \in \mathbb{N}^*$ and set for any $i, j \in \mathbb{Z}$

$$m_{ijk} = \int_{\frac{i}{k}}^{\frac{i+1}{k} - \frac{1}{k^2}} \int_{\frac{j}{k}}^{\frac{j+1}{k} - \frac{1}{k^2}} \rho^0(x, y)\, dx\, dy,$$

$$u^0_{ijk} = k^2 \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} u^0(x, y)\, dx\, dy, \quad v^0_{ijk} = k^2 \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} v^0(x, y)\, dx\, dy.$$

Now we set, for any $(x, y) \in \mathbb{R}^2$,

$$\rho^0_k(x, y) = \sum_{i,j=-k^2}^{k^2} \mathbb{1}_{]\frac{i}{k}, \frac{i}{k} + \sqrt{m_{ijk}}[}(x) \mathbb{1}_{]\frac{j}{k}, \frac{j}{k} + \sqrt{m_{ijk}}[}(y), \tag{2.13}$$

$$\begin{aligned} u^0_k(x, y) \;\; = \;\; & \sum_{i,j=-k^2}^{k^2} u^0_{ijk} \mathbb{1}_{]\frac{i}{k}, \frac{i+1}{k}[}(x) \mathbb{1}_{]\frac{j}{k}, \frac{j+1}{k}[}(y) & (2.14)\\ & + \sum_{j=-k^2}^{k^2} \left(u^0_{-k^2,j,k} \mathbb{1}_{]-\infty,-k[}(x) + u^0_{k^2,j,k} \mathbb{1}_{]k+1/k,+\infty[}(x) \right) \mathbb{1}_{]\frac{j}{k}, \frac{j+1}{k}[}(y) \\ & + \sum_{i=-k^2}^{k^2} \left(u^0_{i,-k^2,k} \mathbb{1}_{]-\infty,-k[}(y) + u^0_{i,k^2,k} \mathbb{1}_{]k+1/k,+\infty[}(y) \right) \mathbb{1}_{]\frac{i}{k}, \frac{i+1}{k}[}(x) \\ & + \left(u^0_{k^2,k^2,k} \mathbb{1}_{]k+1/k,+\infty[}(y) + u^0_{k^2,-k^2,k} \mathbb{1}_{]-\infty,-k[}(y) \right) \mathbb{1}_{]k+1/k,+\infty[}(x) \\ & + \left(u^0_{-k^2,k^2,k} \mathbb{1}_{]k+1/k,+\infty[}(y) + u^0_{-k^2,-k^2,k} \mathbb{1}_{]-\infty,-k[}(y) \right) \mathbb{1}_{]-\infty,-k[}(x) \end{aligned}$$

and

$$\begin{aligned} v^0_k(x, y) \;\; = \;\; & \sum_{i,j=-k^2}^{k^2} v^0_{ijk} \mathbb{1}_{]\frac{i}{k}, \frac{i+1}{k}[}(x) \mathbb{1}_{]\frac{j}{k}, \frac{j+1}{k}[}(y) & (2.15)\\ & + \sum_{j=-k^2}^{k^2} \left(v^0_{-k^2,j,k} \mathbb{1}_{]-\infty,-k[}(x) + v^0_{k^2,j,k} \mathbb{1}_{]k+1/k,+\infty[}(x) \right) \mathbb{1}_{]\frac{j}{k}, \frac{j+1}{k}[}(y) \\ & + \sum_{i=-k^2}^{k^2} \left(v^0_{i,-k^2,k} \mathbb{1}_{]-\infty,-k[}(y) + v^0_{i,k^2,k} \mathbb{1}_{]k+1/k,+\infty[}(y) \right) \mathbb{1}_{]\frac{i}{k}, \frac{i+1}{k}[}(x) \\ & + \left(v^0_{k^2,k^2,k} \mathbb{1}_{]k+1/k,+\infty[}(y) + v^0_{k^2,-k^2,k} \mathbb{1}_{]-\infty,-k[}(y) \right) \mathbb{1}_{]k+1/k,+\infty[}(x) \\ & + \left(v^0_{-k^2,k^2,k} \mathbb{1}_{]k+1/k,+\infty[}(y) + v^0_{-k^2,-k^2,k} \mathbb{1}_{]-\infty,-k[}(y) \right) \mathbb{1}_{]-\infty,-k[}(x). \end{aligned}$$

Figure 3 allows to visualise how we approximate.

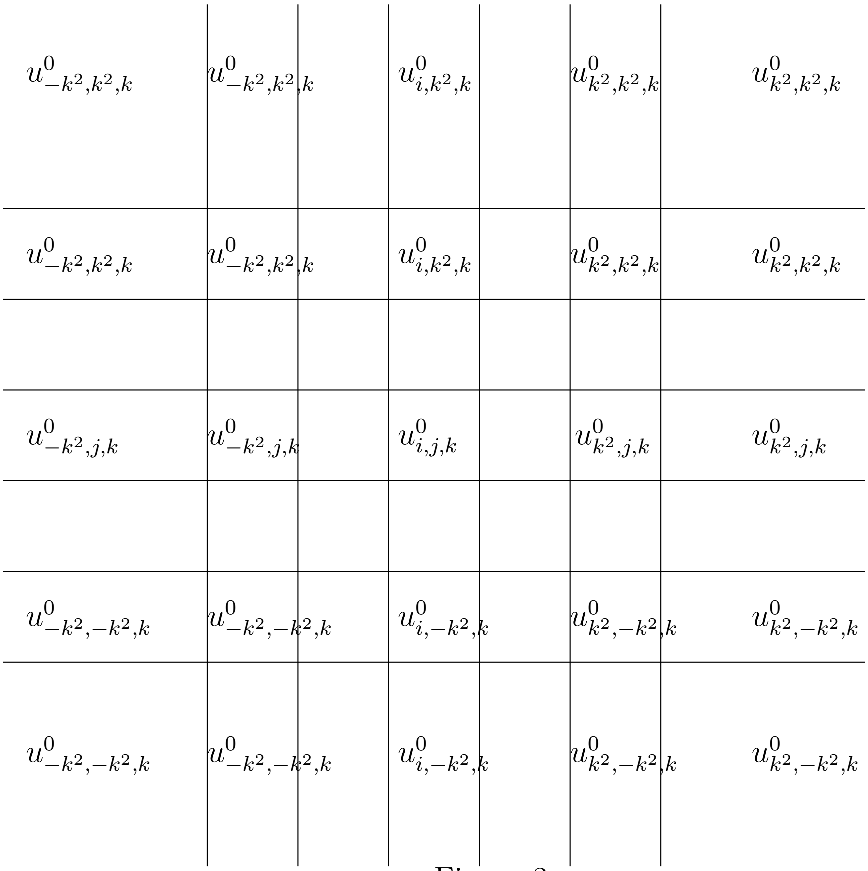

Figure 3.

Notice that this point is very different from the one dimensional case because we need to have a definition of u^0 and v^0 on every $\mathbb{R}^2$, even in the vacuum, in order to define u^0_{ijk} and v^0_{ijk} everywhere. Because of the geometry of the blocks, a linear extension between the blocks is not possible as in the one dimensional case. Furthermore, for a fixed k, the problem with the infinity have to be treated with the sum $\sum_{j=-k^2}^{k^2}$ and $\sum_{i=-k^2}^{k^2}$ and with the terms in $\mathbb{1}_{]-\infty,-k[}$ and $\mathbb{1}_{]k+1/k,+\infty[}$ which extend by different constants at infinity the function u and v without increasing the variations in the two directions. Notice that we have

$$\rho^0_k(x,y)u^0_k(x,y) = \sum_{i,j=-k^2}^{k^2} u^0_{ijk}\mathbb{1}_{]\frac{i}{k},\frac{i}{k}+\sqrt{m_{ijk}}[}(x)\mathbb{1}_{]\frac{j}{k},\frac{j}{k}+\sqrt{m_{ijk}}[}(y) \tag{2.16}$$

and

$$\rho^0_k(x,y)v^0_k(x,y) = \sum_{i,j=-k^2}^{k^2} v^0_{ijk}\mathbb{1}_{]\frac{i}{k},\frac{i}{k}+\sqrt{m_{ijk}}[}(x)\mathbb{1}_{]\frac{j}{k},\frac{j}{k}+\sqrt{m_{ijk}}[}(y). \tag{2.17}$$

Notice also that $\sqrt{m_{ijk}} \le \dfrac{1}{k} - \dfrac{1}{k^2} < \dfrac{1}{k}$. This point is important in order that blocks are disjoints. We have (2.7), in particular since

$$\begin{aligned}
\iint\limits_{\mathbb{R}^2} \rho_k^0(x,y)\,dx\,dy &= \sum_{i,j=-k^2}^{k^2} m_{ijk} \\
&= \sum_{i,j=-k^2}^{k^2} \int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} \int_{\frac{j}{k}}^{\frac{j+1}{k}-\frac{1}{k^2}} \rho^0(x,y)\,dx\,dy \\
&\le \iint\limits_{\mathbb{R}^2} \rho^0(x,y)\,dx\,dy.
\end{aligned}$$

We have clearly (2.8). We turn to the proof of (2.9). For $y \in]\frac{j}{k}, \frac{j+1}{k}[$, the function u_k^0 takes the value

$$u_k^0(x,y) = \sum_{i=-k^2}^{k^2} u_{ijk}^0 \mathbb{1}_{]\frac{i}{k},\frac{i+1}{k}[}(x) + \left(u^0_{-k^2,j,k}\mathbb{1}_{]-\infty,-k[}(x) + u^0_{k^2,j,k}\mathbb{1}_{]k,+\infty[}(x)\right)$$

which is a form similar to (2.5), then we get, for such a y,

$$\int_{\mathbb{R}_x} \left|\partial_x u_k^0(x,y)\right| = \sum_{i=-k^2}^{k^2-1} |u^0_{i+1,jk} - u^0_{ijk}|.$$

For $y \in]-\infty, -k[$, the function u_k^0 takes the value

$$u_k^0(x,y) = \sum_{i=-k^2}^{k^2} u^0_{i,-k^2,k}\mathbb{1}_{]\frac{i}{k},\frac{i+1}{k}[}(x) + u^0_{k^2,-k^2,k}\mathbb{1}_{]k+1/k,+\infty[}(x) + u^0_{-k^2,-k^2,k}\mathbb{1}_{]-\infty,-k[}(x),$$

which gives, for such a y,

$$\int_{\mathbb{R}_x} \left|\partial_x u_k^0(x,y)\right| = \sum_{i=-k^2}^{k^2-1} |u^0_{i+1,-k^2,k} - u^0_{i,-k^2,k}|.$$

Similarly, for $y \in]k+1/k, +\infty[$, we have

$$\int_{\mathbb{R}_x} \left|\partial_x u_k^0(x,y)\right| = \sum_{i=-k^2}^{k^2-1} |u^0_{i+1,k^2,k} - u^0_{i,k^2,k}|.$$

Then, for any $y \in \mathbb{R}$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}_x} \left|\partial_x u_k^0(x,y)\right| &= \sum_{i,j=-k^2}^{k^2-1} |u^0_{ijk} - u^0_{i-1,jk}|\,\mathbb{1}_{]\frac{j}{k},\frac{j+1}{k}[}(y) \\
&\quad + \sum_{i=-k^2}^{k^2-1} |u^0_{i+1,-k^2,k} - u^0_{i,-k^2,k}|\,\mathbb{1}_{]-\infty,-k[}(y) \\
&\quad + \sum_{i=-k^2}^{k^2-1} |u^0_{i+1,k^2,k} - u^0_{i,k^2,k}|\,\mathbb{1}_{]k+1/k,+\infty[}(y),
\end{aligned}$$

and we get

$$\iint_{\mathbb{R}\times[-a,a]} \left|\partial_x u_k^0(x,y)\right| \leq \sum_{i,j=-k^2}^{k^2} |u_{ijk}^0 - u_{i-1,jk}^0| \frac{1}{k} \tag{2.18}$$
$$+ \sum_{i=-k^2}^{k^2-1} |u_{i+1,-k^2,k}^0 - u_{i,-k^2,k}^0| \, (a-k)\mathbb{1}_{a>k}$$
$$+ \sum_{i=-k^2}^{k^2-1} |u_{i+1,k^2,k}^0 - u_{i,k^2,k}^0| \, (a-k)\mathbb{1}_{a>k}.$$

Now

$$\begin{aligned}
|u_{ijk}^0 - u_{i-1,jk}^0| &= k^2 \left| \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} u^0(x,y)\,dx\,dy - \int_{\frac{i-1}{k}}^{\frac{i}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} u^0(x,y)\,dx\,dy \right| \\
&= k^2 \left| \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} u^0(x,y) - u^0(x-\frac{1}{k},y)\,dx\,dy \right| \\
&\leq k^2 \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} \left| u^0(x,y) - u^0(x-\frac{1}{k},y) \right| dx\,dy,
\end{aligned}$$

therefore

$$\begin{aligned}
\sum_{i,j=-k^2}^{k^2-1} |u_{ijk}^0 - u_{i-1,jk}^0| \frac{1}{k} &\leq \sum_{i,j=-k^2}^{k^2} k \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} \left| u^0(x,y) - u^0(x-\frac{1}{k},y) \right| dx\,dy \\
&\leq \iint_{\mathbb{R}^2} \left| \frac{u^0(x,y) - u^0(x-\frac{1}{k},y)}{1/k} \right| dx\,dy \\
&\leq \iint_{\mathbb{R}^2} \left| \partial_x u^0(x,y) \right|.
\end{aligned}$$

Furthermore

$$\begin{aligned}
&\sum_{i=-k^2}^{k^2-1} |u_{i+1,-k^2,k}^0 - u_{i,-k^2,k}^0| \, (a-k)\mathbb{1}_{a>k} \\
\leq \; & a\mathbb{1}_{a>k} \sum_{i=-k^2}^{k^2-1} k^2 \int_{\frac{i+1}{k}}^{\frac{i+2}{k}} \int_{-k}^{-k+1/k} \left| u^0(x,y) - u^0(x-\frac{1}{k},y) \right| dx\,dy \\
\leq \; & a^2 \int_{-k+\frac{1}{k}}^{k+\frac{1}{k}} \int_{-k}^{-k+1/k} \left| \frac{u^0(x,y) - u^0(x-\frac{1}{k},y)}{1/k} \right| dx\,dy
\end{aligned}$$

and then

$$\sum_{i=-k^2}^{k^2-1} |u_{i+1,-k^2,k}^0 - u_{i,-k^2,k}^0| \, (a-k)\mathbb{1}_{a>k} + \sum_{i=-k^2}^{k^2-1} |u_{i+1,k^2,k}^0 - u_{i,k^2,k}^0| \, (a-k)\mathbb{1}_{a>k}$$

$$\leq a^2 \iint\limits_{\mathbb{R}^2} \left| \frac{u^0(x,y) - u^0(x-\frac{1}{k},y)}{1/k} \right| dx\, dy$$
$$\leq a^2 \iint\limits_{\mathbb{R}^2} \left| \partial_x u^0(x,y) \right|.$$

Finally we get

$$\iint\limits_{\mathbb{R}\times[-a,a]} \left| \partial_x u_k^0(x,y) \right| \leq (1+a^2) \iint\limits_{\mathbb{R}^2} \left| \partial_x u^0(x,y) \right|.$$

We proceed similarly to get the other inequalities (2.10)-(2.12).
We refer to appendix A for the convergences of ρ_k^0, $\rho_k^0 u_k^0$ and $\rho_k^0 v_k^0$ in the distributional sense. □

3 Approximations of solutions

In this section, we obtain approximations of solutions for any block initial data. In order to do this, we first present some particular solutions for the system. Then, we are able to give approximations of solutions for the case with a dynamics without constraints and for the case of a shock between blocks during the evolution. Finally, we obtain a merging result of these two cases to get the general case.

3.1 Some particular solutions

We start first by studying the dynamics when constraints don't act. It leads to the study of pressureless dynamics equations in dimension two, which are given by

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) + \partial_y(\rho v) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_y(\rho u v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho u v) + \partial_y(\rho v^2) = 0. \end{cases} \tag{3.1}$$

We will prove now that some particular functions are solution of this system.

Proposition 3.1 *Let $u, v, a_0, b_0 \in \mathbb{R}$ and $c, d > 0$. The functions*

$$\tilde{\rho}(1, \tilde{u}, \tilde{v})(t,x,y) = (1,u,v)\mathbb{1}_{0\leq t}\mathbb{1}_{a(t)\leq x\leq a(t)+c}\mathbb{1}_{b(t)\leq y\leq b(t)+d}, \tag{3.2}$$

where $a(t) = a_0 + ut$ and $b(t) = b_0 + vt$, are solution of (3.1) in the distributional sense with the initial data

$$(1,u,v)\mathbb{1}_{a_0\leq x\leq a_0+c}\mathbb{1}_{b_0\leq y\leq b_0+d}.$$

Proof. Let $\varphi \in C_c^\infty([0,+\infty[, \mathbb{R}^2)$, we would like to compute

$$\int_0^{+\infty} \int_{a(t)}^{a(t)+c} \int_{b(t)}^{b(t)+d} S(\tilde{u},\tilde{v})(\partial_t \varphi + \tilde{u}\partial_x \varphi + \tilde{v}\partial_y \varphi)(t,x,y)\, dy\, dx\, dt$$

for $S(\overline{u},\overline{v})=1$, $S(\overline{u},\overline{v})=\overline{u}$ and $S(\overline{u},\overline{v})=\overline{v}$. With the functions (3.2), it turns to the computations of

$$\int_0^{+\infty}\int_{a(t)}^{a(t)+c}\int_{b(t)}^{b(t)+d} S(u,v)(\partial_t\varphi+u\partial_x\varphi+v\partial_y\varphi)(t,x,y)\,dy\,dx\,dt$$

where u,v are given constants and then $S(u,v)$ also. First

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{a(t)}^{a(t)+c}\int_{b(t)}^{b(t)+d}\varphi(t,x,y)\,dy\,dx\right) \qquad (3.3)\\
= &\int_{a(t)}^{a(t)+c}\int_{b(t)}^{b(t)+d}\partial_t\varphi(t,x,y)\,dy\,dx\\
&+a'(t)\left(\int_{b(t)}^{b(t)+d}\varphi(t,a(t)+c,y)\,dy-\int_{b(t)}^{b(t)+d}\varphi(t,a(t),y)\,dy\right)\\
&+b'(t)\left(\int_{a(t)}^{a(t)+c}\varphi(t,x,b(t)+d)\,dx-\int_{a(t)}^{a(t)+c}\varphi(t,x,b(t))\,dx\right),
\end{aligned}$$

furthermore

$$\begin{aligned}
&\int_0^{+\infty}\int_{a(t)}^{a(t)+c}\int_{b(t)}^{b(t)+d}\partial_x\varphi(t,x,y)\,dy\,dx\,dt\\
= &\int_0^{+\infty}\int_{b(t)}^{b(t)+d}\varphi(t,a(t)+c,y)-\varphi(t,a(t),y)\,dy\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{+\infty}\int_{a(t)}^{a(t)+c}\int_{b(t)}^{b(t)+d}\partial_y\varphi(t,x,y)\,dy\,dx\,dt\\
= &\int_0^{+\infty}\int_{a(t)}^{a(t)+c}\varphi(t,x,b(t)+d)-\varphi(t,x,b(t))\,dx\,dt.
\end{aligned}$$

Integrating with respect to t the relation (3.3) and using that $a'(t)=u$ and $b'(t)=v$, we get that

$$\begin{aligned}
&\int_0^{+\infty}\int_{a(t)}^{a(t)+c}\int_{b(t)}^{b(t)+d} S(u,v)(\partial_t\varphi+u\partial_x\varphi+v\partial_y\varphi)(t,x,y)\,dy\,dx\,dt\\
= &-\int_{a_0}^{a_0+c}\int_{b_0}^{b_0+d} S(u,v)\varphi(0,x,y)\,dy\,dx.
\end{aligned}$$

Applying this to $S(u,v)=1$, $S(u,v)=u$ and $S(u,v)=v$, we get the three expected equations with the corresponding initial data. □

Remark 3.1 Notice that these functions are not discrete blocks because of the continuous form with respect to t. Here the discretization is just with respect to (x,y) but not in t. In our definition of discrete blocks, the discretization is with respect to all the variables (t,x,y). It is necessary because this continuity with respect to t cannot be conserved all along this paper when we are doing splitting in time with respect to the various directions.

The previous dynamics concern some particular evolutions as long as there is no collision. Now we consider the case with a collision in the x direction at some time t^*. Then the two (or more) blocks collide, they get stuck, building a new block, with volume the sum of the volumes, and with momentum the sum of momenta. The new velocity u_f is chosen such as to preserve total momentum. This is then the classical one-dimension dynamics in the x direction taking into account the other directions.

Proposition 3.2 *Let $t^*, \mu > 0$, $x^*, u_l, u_r, c, d, v \in \mathbb{R}$ with $u_l > u_r$. The functions*

$$\begin{aligned}\hat{\rho}(1,\hat{u},\hat{v})(t,x,y) &= \mathbb{1}_{0\le t<t^*}\Big((1,u_l,v)\mathbb{1}_{a_l(t)-c\le x\le a_l(t)}\mathbb{1}_{b(t)\le y\le b(t)+\mu}\\ &\qquad +(1,u_r,v)\mathbb{1}_{a_r(t)\le x\le a_r(t)+d}\mathbb{1}_{b(t)\le y\le b(t)+\mu}\Big)\\ &\quad +(1,u_f,v)\mathbb{1}_{t^*\le t}\,\mathbb{1}_{a_f(t)-c\le x\le a_f(t)+d}\mathbb{1}_{b(t)\le y\le b(t)+\mu},\end{aligned}$$

and the measures

$$\pi_1(t,x,y) = \begin{cases} \delta(t-t^*)(u_l-u_f)(x-(x^*-c)) & \text{if } x^*-c\le x\le x^*,\\ \delta(t-t^*)(u_f-u_r)((x^*+d)-x) & \text{if } x^*\le x\le x^*+d,\\ 0 & \text{otherwise,}\end{cases} \tag{3.4}$$

and $\pi_2 = 0$, where δ is the Dirac mass, $a_l(t) = x^* + u_l(t-t^*)$, $a_r(t) = x^* + u_r(t-t^*)$ and $a_f(t) = x^* + u_f(t-t^*)$ (the point x^* being the point of collision) with $cu_l + du_r = (c+d)u_f$, are solution of (1.4), (1.5) and (1.6) in the distributional sense.

Proof. Let φ be a test function and $S : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. We have

$$\begin{aligned} &< \partial_t(\hat{\rho}S(\hat{u},\hat{v})) + \partial_x(\hat{\rho}S(\hat{u},\hat{v})\hat{u}) + \partial_y(\hat{\rho}S(\hat{u},\hat{v})\hat{v}), \varphi >\\ =\; & -\int_0^{t^*}\int_{a_l(t)-c}^{a_l(t)}\int_{b(t)}^{b(t)+\mu} S(u_l,v)(\partial_t\varphi + u_l\partial_x\varphi + v\partial_y\varphi)\,dy\,dx\,dt \qquad (3.5)\\ & -\int_0^{t^*}\int_{a_r(t)}^{a_r(t)+d}\int_{b(t)}^{b(t)+\mu} S(u_r,v)(\partial_t\varphi + u_r\partial_x\varphi + v\partial_y\varphi)\,dy\,dx\,dt \qquad (3.6)\\ & -\int_{t^*}^{+\infty}\int_{a_f(t)-c}^{a_f(t)+d}\int_{b(t)}^{b(t)+\mu} S(u_f,v)(\partial_t\varphi + u_f\partial_x\varphi + v\partial_y\varphi)\,dy\,dx\,dt. \qquad (3.7)\end{aligned}$$

Notice that

$$\begin{aligned} &\frac{d}{dt}\left(\int_{a_l(t)-c}^{a_l(t)}\int_{b(t)}^{b(t)+\mu}\varphi(t,x,y)\,dy\,dx\right)\\ =\; & \int_{a_l(t)-c}^{a_l(t)}\int_{b(t)}^{b(t)+\mu}\partial_t\varphi(t,x,y)\,dy\,dx\\ & + \int_{b(t)}^{b(t)+\mu}(\varphi(t,a_l(t),y) - \varphi(t,a_l(t)-c,y))u_1\,dy\\ & + \int_{a_l(t)-c}^{a_l(t)}(\varphi(t,x,b(t)+\mu) - \varphi(t,x,b(t)))v\,dx,\end{aligned}$$

then we get

$$\begin{aligned}
&\int_0^{t^*}\int_{a_l(t)-c}^{a_l(t)}\int_{b(t)}^{b(t)+\mu} S(u_l,v)\,\partial_t\varphi(t,x,y)\,dy\,dx\,dt\\
=&\int_{a_l(t^*)-c}^{a_l(t^*)}\int_{b(t^*)}^{b(t^*)+\mu} S(u_l,v)\,\varphi(t^*,x,y)\,dy\,dx\\
&-\int_0^{t^*}\int_{b(t)}^{b(t)+\mu} S(u_l,v)u_l\left(\varphi(t,a_l(t),y)-\varphi(t,a_l(t)-c,y)\right)dy\,dt\\
&-\int_0^{t^*}\int_{a_l(t)-c}^{a_l(t)} S(u_l,v)v\left(\varphi(t,x,b(t)+\mu)-\varphi(t,x,b(t))\right)dx\,dt\\
=&\int_{a_l(t^*)-c}^{a_l(t^*)}\int_{b(t^*)}^{b(t^*)+\mu} S(u_l,v)\,\varphi(t^*,x,y)\,dy\,dx\\
&-\int_0^{t^*}\int_{a_l(t)-c}^{a_l(t)}\int_{b(t)}^{b(t)+\mu} S(u_l,v)u_l\,\partial_x\varphi(t,x,y)\,dy\,dx\,dt\\
&-\int_0^{t^*}\int_{a_l(t)-c}^{a_l(t)}\int_{b(t)}^{b(t)+\mu} S(u_l,v)v\,\partial_y\varphi(t,x,y)\,dy\,dx\,dt.
\end{aligned}$$

We have similar equations for both terms (3.6) and (3.7) and we get

$$\begin{aligned}
&< \partial_t(\hat\rho S(\hat u,\hat v))+\partial_x(\hat\rho S(\hat u,\hat v)\hat u)+\partial_y(\hat\rho S(\hat u,\hat v)\hat v),\varphi >\\
=&\ -\int_{b(t^*)}^{b(t^*)+\mu}\left(\int_{x^*-c}^{x^*}(S(u_l)-S(u_f))\varphi(t^*,x,y)\,dx\right.\\
&\left.\qquad+\int_{x^*}^{x^*+d}(S(u_r)-S(u_f))\varphi(t^*,x,y)\,dx\right)dy.
\end{aligned}$$

For $S(u)=1$, it gives $\partial_t\hat\rho+\partial_x(\hat\rho\hat u)+\partial_y(\hat\rho\hat v)=0$, for $S(u,v)=v$, it gives $\partial_t(\hat\rho\hat v)+\partial_x(\hat\rho\hat u\hat v)+\partial_y(\hat\rho\hat v^2)=0$ and for $S(u)=u$, we get $\partial_t(\hat\rho\hat u)+\partial_x(\hat\rho\hat u^2+\pi_1)+\partial_y(\hat\rho\hat u\hat v)=0$ where π_1 is defined by (3.4). Notice that $\pi_1\geq 0$ since $u_l>u_r$ and $u_f=(cu_l+du_r)/(c+d)$, and that the constraint relations are satisfied. □

Remark 3.2 If we do the same with a shock in the y direction, it gives a term $\pi_2\neq 0$.

3.2 Discrete approximations in the case of dynamics without constraints

Let Δt, Δx and $\Delta y>0$. We prove here that we can approximate the solution of proposition 3.1 with discrete blocks. We first define the dynamics of blocks we are going to use in this case. The key idea is to perform discrete jumps successively in both directions in consecutive time steps. In other words, during a time Δt, we allow only the x direction movement to act, then during the following Δt time, we allow only the y direction movement to act and so on with alternatively a movement on x direction and on y direction.

Definition 3.3 *Let $u, v, a_0, b_0 \in \mathbb{R}$ and $c, d > 0$. Let $N \in \mathbb{N}^*$. We take $\Delta x = c/N$, $\Delta y = d/N$ and $\Delta t = 1/N$. We consider the approximations given by the following sum of blocks:*

$$(\rho_N, \rho_N u_N, \rho_N v_N)(t,x,y) = \sum_{i,j=0}^{N-1} \sum_{l=0}^{+\infty} (1,u,v) \mathbb{1}_{(t,x,y)\in P_{ijl}} \tag{3.8}$$

where

$$\mathbb{1}_{(t,x,y)\in P_{ijl}} = \mathbb{1}_{l\Delta t \le t < (l+1)\Delta t} \mathbb{1}_{a_l + i\Delta x \le x < a_l + (i+1)\Delta x} \mathbb{1}_{b_l + j\Delta y \le y < b_l + (j+1)\Delta y} \tag{3.9}$$

with the sequences $(a_n)_n$ and $(b_n)_n$ defined as follows. Starting from a_0 and b_0, we construct the sequences as

$$a_{2k+1} = a_0 + \left[\frac{2(k+1)u\Delta t}{\Delta x}\right] \Delta x, \qquad b_{2k+1} = b_{2k},$$

and

$$b_{2k+2} = b_0 + \left[\frac{2(k+1)v\Delta t}{\Delta y}\right] \Delta y, \qquad a_{2k+2} = a_{2k+1},$$

where the big square brackets denote the integer part.

Remark 3.3 At time $t = (2k+1)\Delta t$, we make a jump for the block in the x direction, and at time $t = (2k+2)\Delta t$, we make a jump for the block in the y direction, staying on the fixed grid at level N and taking an approximation of the movement.

Remark 3.4 For the extension in three dimensions, we also consider Δz and a sequence $(c_n)_n$ in this third direction.

We first start by proving the two following technical lemmas.

Lemma 3.4 *We use the discrete blocks which were constructed in definition 3.3 and the associated notations. We set*

$$a_\Delta(t) = \sum_{l=0}^{+\infty} a_l \mathbb{1}_{l\Delta t \le t < (l+1)\Delta t}.$$

Then we have

$$|a(t) - a_\Delta(t)| \le |u|\Delta t + \Delta x,$$

and

$$|b(t) - b_\Delta(t)| \le |v|\Delta t + \Delta y. \tag{3.10}$$

Proof. Using that $2(k+1)u\Delta t - \Delta x < \left[\frac{2(k+1)u\Delta t}{\Delta x}\right] \Delta x \le 2(k+1)u\Delta t$, for $t \in [(2k+1)\Delta t, (2k+3)\Delta t[$, we have

$$\begin{aligned} |a(t) - a_\Delta(t)| &= |a_{2k+1} - a_0 - ut| = \left| \left[\frac{2(k+1)u\Delta t}{\Delta x}\right] \Delta x - ut \right| \\ &\le |u||2(k+1)\Delta t - t| + \Delta x \\ &\le |u|\Delta t + \Delta x. \end{aligned}$$

Then, for any $t \geq 0$, we get

$$|a(t) - a_\Delta(t)| \leq |u|\Delta t + \Delta x.$$

Similarly, we have $|b(t) - b_\Delta(t)| \leq |v|\Delta t + \Delta y$. $\square$

Lemma 3.5 *We use the constructed discrete blocks and the associated notations of definition 3.3 and lemma 3.4. Setting, for any test function* $\varphi \in C_c^\infty([0, +\infty[, \mathbb{R}^2)$,

$$A(\varphi) = \sum_{l=0}^{+\infty} \int_{l\Delta t}^{(l+1)\Delta t} \int_{a(t)}^{a(t)+c} \int_{b(t)}^{b(t)+d} \varphi(t, x, y)\, dy\, dx\, dt \tag{3.11}$$

and

$$A_N(\varphi) = \sum_{i,j=0}^{N-1} \sum_{l=0}^{+\infty} \int_{l\Delta t}^{(l+1)\Delta t} \int_{a_l+i\Delta x}^{a_l+(i+1)\Delta x} \int_{b_l+j\Delta y}^{b_l+(j+1)\Delta y} \varphi(t, x, y)\, dy\, dx\, dt. \tag{3.12}$$

Then we have $A_N(\varphi) \to A(\varphi)$ *when* $N \to +\infty$.

Proof. Since $c = N\Delta x$ and $d = N\Delta y$, notice that

$$A(\varphi) = \sum_{i,j=0}^{N-1} \sum_{l=0}^{+\infty} \int_{l\Delta t}^{(l+1)\Delta t} \int_{a(t)+i\Delta x}^{a(t)+(i+1)\Delta x} \int_{b(t)+j\Delta y}^{b(t)+(j+1)\Delta y} \varphi(t, x, y)\, dy\, dx\, dt. \tag{3.13}$$

Let us denote by T a real such that the support in time of φ is in $[0, T]$. Denote by L_N an integer such that $L_N \Delta t \geq T$. We have

$$\begin{aligned}
A_N(\varphi) - A(\varphi) &= \sum_{i,j=0}^{N-1} \sum_{l=0}^{+\infty} \int_{l\Delta t}^{(l+1)\Delta t} \left(\int_{a_l+i\Delta x}^{a_l+(i+1)\Delta x} \int_{b_l+j\Delta y}^{b_l+(j+1)\Delta y} \varphi(t, x, y)\, dy\, dx \right. \\
&\qquad \left. - \int_{a(t)+i\Delta x}^{a(t)+(i+1)\Delta x} \int_{b(t)+j\Delta y}^{b(t)+(j+1)\Delta y} \varphi(t, x, y)\, dy\, dx \right) dt \\
&= \sum_{i,j=0}^{N-1} \sum_{l=0}^{L_N} \int_{l\Delta t}^{(l+1)\Delta t} \int_{a_l+i\Delta x}^{a_l+(i+1)\Delta x} \int_{b_l+j\Delta y}^{b_l+(j+1)\Delta y} \\
&\qquad \Big(\varphi(t, x, y) - \varphi(t, x + a(t) - a_l, y + b(t) - b_l) \Big)\, dy\, dx\, dt.
\end{aligned}$$

Let $\varepsilon > 0$. Since φ is continuous and has a compact support, there exists $\eta > 0$ such that for any (t, x_1, y_1) and (t, x_2, y_2) in the support of φ, if $|x_1 - x_2| \leq \eta$ and $|y_1 - y_2| \leq \eta$, then $|\varphi(t, x_1, y_1) - \varphi(t, x_2, y_2)| \leq \varepsilon$. Let $N_0 \in \mathbb{N}^*$ be such that N_0 is greater than $(|u| + c)/\eta$ and $(|v| + d)/\eta$. Let $N \in N^*$ be greater than N_0. Now

$$|a(t) - a_\Delta(t)| \leq |u|\Delta t + \Delta x = |u|\frac{1}{N} + \frac{c}{N} \leq \eta$$

and $|b(t)-b_\Delta(t)| \le |v|\dfrac{1}{N}+\dfrac{d}{N} \le \eta$, therefore

$$\begin{aligned}
|A_N(\varphi)-A(\varphi)| &\le \sum_{i,j=0}^{N-1}\sum_{l=0}^{L_N}\int_{l\Delta t}^{(l+1)\Delta t}\int_{a_l+i\Delta x}^{a_l+(i+1)\Delta x}\int_{b_l+j\Delta y}^{b_l+(j+1)\Delta y}\varepsilon\, dy\, dx\, dt \\
&\le \sum_{i,j=0}^{N-1}\sum_{l=0}^{L_N}\int_{l\Delta t}^{(l+1)\Delta t}\Delta x\Delta y\,\varepsilon\, dt \\
&\le N\Delta x N\Delta y L_N\Delta\,\varepsilon \\
&\le cdT\,\varepsilon. \qquad (3.14)
\end{aligned}$$

It gives that $A_N(\varphi)\to A(\varphi)$ when $N\to+\infty$. $\square$

Proposition 3.6 *Let $u, v, a_0, b_0 \in \mathbb{R}$ and $c, d > 0$. Then there exists discrete blocks $(\rho_N, \rho_N u_N, \rho_N v_N)$ with initial data*

$$\mathbb{1}_{a_0\le x\le a_0+c}\mathbb{1}_{b_0\le y\le b_0+d}(1,u,v)$$

such that

$$\begin{cases} \partial_t\rho_N+\partial_x(\rho_N u_N)+\partial_y(\rho_N v_N)\to 0, \\ \partial_t(\rho_N u_N)+\partial_x(\rho_N u_N^2)+\partial_y(\rho_N u_N v_N)\to 0, \\ \partial_t(\rho_N v_N)+\partial_x(\rho_N u_N v_N)+\partial_y(\rho_N v_N^2)\to 0, \end{cases} \qquad (3.15)$$

when $N\to+\infty$, in the distributional sense.

Proof. Let $\varphi\in C_c^\infty([0,+\infty[,\mathbb{R}^2)$. The solution $(\tilde\rho,\tilde\rho\tilde u,\tilde\rho\tilde u)$ of proposition 3.1 satisfies

$$\begin{aligned}
0 &= \int_0^{+\infty}\iint_{\mathbb{R}^2}(\tilde\rho\partial_t\varphi+\tilde\rho\tilde u\partial_x\varphi+\tilde\rho\tilde v\partial_y\varphi)\,dy\,dx\,dt \\
&= \sum_{l=0}^{+\infty}\int_{l\Delta t}^{(l+1)\Delta t}\int_{a(t)}^{a(t)+\Delta x}\int_{b(t)}^{b(t)+\Delta y}(\partial_t\varphi+u\partial_x\varphi+v\partial_y\varphi)\,dy\,dx\,dt \\
&= A(\partial_t\varphi)+uA(\partial_x\varphi)+vA(\partial_y\varphi).
\end{aligned}$$

We also have

$$\begin{aligned}
&\int_0^{+\infty}\iint_{\mathbb{R}^2}(\rho_N\partial_t\varphi+\rho_N u_N\partial_x\varphi+\rho_N v_N\partial_y\varphi)\,dy\,dx\,dt \\
&= \sum_{i,j=0}^{N-1}\sum_{l=0}^{+\infty}\int_{l\Delta t}^{(l+1)\Delta t}\int_{a_l+i\Delta x}^{a_l+(i+1)\Delta x}\int_{b_l+j\Delta y}^{b_l+(j+1)\Delta y}(\partial_t\varphi+u\partial_x\varphi+v\partial_y\varphi)\,dy\,dx\,dt \\
&= A_N(\partial_t\varphi)+uA_N(\partial_x\varphi)+vA_N(\partial_y\varphi).
\end{aligned}$$

Since $A(\partial_t\varphi)+uA(\partial_x\varphi)+vA(\partial_y\varphi)=0$, then we get that

$$A_N(\partial_t\varphi)+uA_N(\partial_x\varphi)+vA_N(\partial_y\varphi)\underset{N\to+\infty}{\to}0$$

applying the lemma 3.5 to $\partial_t\varphi$, $\partial_x\varphi$ and $\partial_y\varphi$. That is to say

$$\int_0^{+\infty}\iint\limits_{\mathbb{R}^2}\left(\rho_N\partial_t\varphi+\rho_N u_N\partial_x\varphi+\rho_N v_N\partial_y\varphi\right)\,dy\,dx\,dt\underset{Nt\to+\infty}{\to}0$$

for any test function φ.
Since the speeds u and v are constant, they can be factorized on every terms and then we get also that

$$\int_0^{+\infty}\iint\limits_{\mathbb{R}^2}\left(\rho_N u_N\partial_t\varphi+\rho_N u_N^2\partial_x\varphi+\rho_N v_N u_N\partial_y\varphi\right)\,dy\,dx\,dt\underset{N\to+\infty}{\to}0$$

and

$$\int_0^{+\infty}\iint\limits_{\mathbb{R}^2}\left(\rho_N v_N\partial_t\varphi+\rho_N u_N v_N\partial_x\varphi+\rho_N v_N^2\partial_y\varphi\right)\,dy\,dx\,dt\underset{N\to+\infty}{\to}0$$

for any test function φ. $\square$

3.3 Discrete approximations in the constraint case

We define now the dynamics to approximate the solution of proposition 3.2 by discrete blocks. We first consider that the shock happens during the x direction movement in the splitting.

Definition 3.7 *We take, for $N\in\mathbb{N}^*$, $\Delta x=c/N$, $\Delta y=d/N$ and $\Delta t=1/N$. We start at $t=0$ from a situation with two disjoint blocks:*

$$(\rho,\rho u,\rho v)(0,x,y)=(1,u_l,v)\mathbb{1}_{x\in[\alpha^l-P\Delta x,\alpha^l[}+(1,u_r,v)\mathbb{1}_{x\in[\alpha^r,\alpha^r+Q\Delta x[}.$$

If $0<\dfrac{\alpha^r-\alpha^l}{u_l-u_r}\le\Delta t$, then a collision has to happen in time $t^*=\dfrac{\alpha^r-\alpha^l}{u_l-u_r}$. In order to have the conservation of the mass and a good approximation of the conservation of the momentum, at time t^*, we replace the dynamics by the following situation:

$$(\rho,\rho u,\rho v)(t^*,x,y)=(1,u_f,v)\mathbb{1}_{x\in[\alpha^f-P\Delta x,\alpha^f+Q\Delta x[}$$

with the velocity

$$u_f=\frac{Pu_r+Qu_r}{P+Q}$$

and where

$$\alpha^f=\left[\frac{u_lt^*+\alpha^l+u_f(\Delta t-t^*)}{\Delta x}\right]\Delta x,$$

with the big square brackets denoting the integer part.

Remark 3.5 *We have similar formulas for a shock in the y direction substituting Δx by Δy and u by v.*
If the shock is not between 0 and Δt, but let say between $L\Delta t$ and $(L+1)\Delta t$, we just have to make a translation of these formulas.

We prove now that the discrete blocks defined previously are approximations of the solution of proposition 3.2.

Proposition 3.8 *We denote by* $(\hat\rho_N, \hat\rho_N\hat u_N, \hat\rho_N\hat v_N)$ *the discrete blocks constructed in definition 3.7 (see also formula (3.16) for the part of this function which is located at the collision). Then the functions* $(\hat\rho_N, \hat\rho_N\hat u_N, \hat\rho_N\hat v_N)$ *have functions* $(\hat\rho, \hat\rho\hat u, \hat\rho\hat v)$ *of proposition 3.2 for limit in the distributional sense when* $N \to +\infty$.

Proof. We consider the case of a shock in the x direction with the previous notations. Denote by L (which changes with Δt, that is to say with N) the integer such that $t^* \in [L\Delta t, (L+1)\Delta t[$, and we notice that the part of the functions located near the collision can be written as

$$(\hat\rho_N, \hat\rho_N\hat u_N, \hat\rho_N\hat v_N)(t,x,y) = (1, u_f, v)\mathbb{1}_{L\Delta t \le t < t^*}\mathbb{1}_{(x,y)\in\mathcal{P}} \tag{3.16}$$

where

$$\mathbb{1}_{(x,y)\in\mathcal{P}} = \mathbb{1}_{\alpha^f - P\Delta x \le x < \alpha^f + Q\Delta x}\mathbb{1}_{b_l \le y < b_l + \mu}. \tag{3.17}$$

Notice that before $L\Delta t$ and after $(L+1)\Delta t$, the movement is without constraints and we have studied it already. Notice also that after the shock, the positions of the blocks move as in the case without constraints starting with the new defined positions at the instant of shock.
We consider a test function $\varphi \in C_c^\infty([0,+\infty[, \mathbb{R}^2)$. We have

$$\begin{aligned}
&\int_0^{+\infty}\iint_{\mathbb{R}^2} (\hat\rho_N\partial_t\varphi + \hat\rho_N\hat u_N\partial_x\varphi + \hat\rho_N\hat v_N\partial_y\varphi)\, dy\, dx\, dt \\
= \;&\int_{L\Delta t}^{(L+1)\Delta t}\int_{\alpha^f - P\Delta x}^{\alpha^f + Q\Delta x}\int_{b_l}^{b_l+\mu} (\partial_t\varphi + u\partial_x\varphi + v\partial_y\varphi)\, dy\, dx\, dt \\
&+ R_N(\varphi),
\end{aligned}$$

where $R_N(\varphi) \to 0$ corresponding to the part of $\hat\rho_N$ which follows a movement without constraints and has already been studied. We will consider the difference with the corresponding terms for $(\hat\rho, \hat\rho\hat u, \hat v)$. We have then to consider the quantity

$$\begin{aligned}
B_N(\varphi) = &\int_{L\Delta t}^{t^*}\int_{\alpha^f - c}^{\alpha^f}\int_{b_l}^{b_l+\mu} \Big(\varphi(t,x,y) - \varphi(t, x - \alpha^f + a_l(t), y)\Big)\, dy\, dx\, dt \\
&+ \int_{L\Delta t}^{t^*}\int_{\alpha^f}^{\alpha^f + d}\int_{b_l}^{b_l+\mu} \Big(\varphi(t,x,y) - \varphi(t, x - \alpha^f + a_r(t), y)\Big)\, dy\, dx\, dt \\
&+ \int_{t^*}^{(L+1)\Delta t}\int_{\alpha^f - c}^{\alpha^f + d}\int_{b_l}^{b_l+\mu} \Big(\varphi(t,x,y) - \varphi(t, x - \alpha^f + a^f(t), y)\Big)\, dy\, dx\, dt.
\end{aligned}$$

We have $a_l(t) = \alpha^l + u_l(t - L\Delta)$, $a_r(t) = \alpha^r + u_r(t - L\Delta t)$ and $x^* = \alpha^l + u_l(t^* - L\Delta t)$, then for $t \in [L\Delta t, (L+1)\Delta t[$,

$$|\alpha^f - a^f(t)| \le |u_f(t - L\Delta t)| + \Delta x \le |u_f|\Delta t + \Delta x,$$

$$|\alpha^f - a_l(t)| \le |(u_l - u_f)(t^* - L\Delta t)| + |u_f(\Delta t - t)| + \Delta x \le (|u_l - u_f| + |u_f|)\Delta t + \Delta x,$$

and

$$|\alpha^f - a_r(t)| \leq |(u_r - u_f)(t^* - L\Delta t)| + |u_f(\Delta t - t)| + \Delta x \leq (|u_r - u_f| + |u_f|)\Delta t + \Delta x.$$

Then we do as in the case without constraints (for the terms $A_N(\varphi)$) to get that $B_N(\varphi) \to 0$ when $N \to +\infty$. □

3.4 General case of approximations of solutions and BV estimates

We want now to get approximations of solutions for any block initial data of the form of our approximation processus, that is to say with the following form (2.13)-(2.15). These blocks satisfy (2.16)-(2.17) and then have the form

$$(\rho^0(x,y), \rho^0(x,y)u^0(x,y), \rho^0(x,y)v^0(x,y)) \\ = \sum_{i=-I}^{I'} \sum_{j=-J}^{J'} (1, u_{ij}, v_{ij}) \mathbb{1}_{a_{ij} \leq x \leq b_{ij}} \mathbb{1}_{c_{ij} \leq y \leq d_{ij}} \tag{3.18}$$

which is a linear sum of terms as the ones considered in previous subsections. Then we have the following merging result.

Proposition 3.9 *Let $\rho^0 \in L^1(\mathbb{R}^2)$, $u^0, v^0 \in L^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$ such that $0 \leq \rho^0 \leq 1$. We consider the sequence of block initial data $(\rho_k^0, u_k^0, v_k^0)_{k\geq 1}$ defined by (2.13)-(2.15). Then, for any $k \in \mathbb{N}^*$, there exists $(\rho_{kl}, \rho_{kl}u_{kl}, \rho_{kl}v_{kl})_l$ discrete blocks associated to the initial data $(\rho_k^0, \rho_k^0 u_k^0, \rho_k^0 v_k^0)$ and $(\pi_1)_{kl}, (\pi_2)_{kl} \in \mathcal{M}_{loc}([0,+\infty[\times\mathbb{R}^2)$ such that*

$$\begin{cases} \partial_t \rho_{kl} + \partial_x(\rho_{kl}u_{kl}) + \partial_y(\rho_{kl}v_{kl}) = R_{kl} \rightharpoonup 0, \\ \partial_t(\rho_{kl}u_{kl}) + \partial_x(\rho_{kl}u_{kl}^2 + (\pi_1)_{kl}) + \partial_y(\rho_{kl}u_{kl}v_{kl}) = S_{kl} \rightharpoonup 0, \\ \partial_t(\rho_{kl}v_{kl}) + \partial_x(\rho_{kl}u_{kl}v_{kl}) + \partial_y(\rho_{kl}v_{kl}^2 + (\pi_2)_{kl}) = T_{kl} \rightharpoonup 0, \end{cases} \tag{3.19}$$

when $l \to +\infty$, in the distributional sense.

Proof. As long as there is no collision, each block moves freely and then proposition 3.6 gives approximations of the solution by discrete blocks. Until the first collision between two blocks, $(\rho_{kl}, \rho_{kl}u_{kl}, \rho_{kl}v_{kl})$ is thus defined by the sum of functions like defined in definition 3.3. Every time a collision between two blocks happens, let us say during a movement in direction x (it is similar in the y direction), proposition 3.8 gives an approximation of the solution by discrete blocks, thus at this time, the corresponding part of $(\rho_{kl}, \rho_{kl}u_{kl}, \rho_{kl}v_{kl})$ is modified according to definition 3.7. Then, it moves freely as in proposition 3.6 until the next collision. This way, it defined approximations of solutions as expected. □

Remark 3.6 If we take initial data such that

$$b_{i+1,j} < a_{ij} \text{ and } d_{i,j+1} < c_{ij} \text{ for any } i,j, \tag{3.20}$$

then collision doesn't appear at time $t = 0$ and then we have $\pi_1(0, x, y) = 0$ and $\pi_2(0, x, y) = 0$.
For the block discretization of our processus in proposition 2.3, we have $\sqrt{m_{ijk}} \leq \frac{1}{k} - \frac{1}{k^2} < \frac{1}{k}$ and then we are in the situation of (3.20) and thus this discretization by blocks will lead to solutions with no initial measure.

We turn now to the proof of L^∞ and BV estimates for these functions.

Proposition 3.10 *The blocks of proposition 3.9 satisfy, for any $t \geq 0$,*

$$0 \leq \rho_{kl} \leq 1, \tag{3.21}$$

$$essinf\ u^0 \leq u_{kl} \leq esssup\ u^0, \quad essinf\ v^0 \leq v_{kl} \leq esssup\ v^0, \tag{3.22}$$

$$\iint_{\mathbb{R}\times[-a,a]} |\partial_x u_{kl}(t, x, y)| \leq \iint_{\mathbb{R}\times[-a_t,a_t]} |\partial_x u_k^0(x, y)|, \tag{3.23}$$

$$\iint_{[-a,a]\times\mathbb{R}} |\partial_y u_{kl}(t, x, y)| \leq \iint_{[-a_t,a_t]\times\mathbb{R}} |\partial_y u_k^0(x, y)|, \tag{3.24}$$

$$\iint_{\mathbb{R}\times[-a,a]} |\partial_x v_{kl}(t, x, y)| \leq \iint_{\mathbb{R}\times[-a_t,a_t]} |\partial_x v_k^0(x, y)|, \tag{3.25}$$

and

$$\iint_{[-a,a]\times\mathbb{R}} |\partial_y v_{kl}(t, x, y)| \leq \iint_{[-a_t,a_t]\times\mathbb{R}} |\partial_y v_k^0(x, y)|, \tag{3.26}$$

for any $a > 0$ and setting

$$a_t = a + t \max(\|u^0\|_\infty, \|v^0\|_\infty).$$

Furthermore, the sequences of measures $((\pi_1)_{kl})_{l\geq1}$ and $((\pi_2)_{kl})_{l\geq1}$ are bounded in $\mathcal{M}_{loc}([0, +\infty[\times\mathbb{R}^2)$.

Proof. The L^∞ bounds are obvious from construction. For simplicity, we skip the indice k and l and denote by u a function u_{kl}. We have a relation like (2.18). It allows to consider the evolution across shocks of a quantity like

$$\sum_{i=2}^{n^0} |u_i^0 - u_{i-1}^0|,$$

with u_i^0 the velocities of the successive blocks, When a collision happens, for example at time t^* between blocks k and $k+1$ to simplify the presentation, we have after the collision a speed of the form $u^* = u_k^0\, c/(c + d) + u_{k+1}^0\, d/(c + d)$ due to proposition 3.2. First, we have

$$\begin{aligned}
|u_{k+2}^0 - u^*| &\leq |u_{k+2}^0 - u_{k+1}^0| + |u_{k+1}^0 - u^*| \\
&\leq |u_{k+2}^0 - u_{k+1}^0| + \left|u_{k+1}^0 \frac{c+d}{c+d} - u_k^0 \frac{c}{c+d} - u_{k+1}^0 \frac{d}{c+d}\right| \\
&\leq |u_{k+2}^0 - u_{k+1}^0| + \frac{c}{c+d} |u_{k+1}^0 - u_k^0|.
\end{aligned}$$

Similarly, we get $|u^* - u^0_{k-1}| \le |u^0_k - u^0_{k-1}| + \frac{d}{c+d}|u^0_{k+1} - u^0_k|$, and by adding these two last inequalities, we get

$$|u^* - u^0_{k-1}| + |u^0_{k+2} - u^*| \le |u^0_k - u^0_{k-1}| + |u^0_{k+1} - u^0_k| + |u^0_{k+2} - u^0_{k+1}|. \quad (3.27)$$

Since

$$\iint_{\mathbb{R}\times[-a,a]} |\partial_x u(t^*, x, y)| = \sum_{i=2}^{k-1} |u^0_i - u^0_{i-1}| + |u^* - u^0_{k-1}| + |u^0_{k+2} - u^*| + \sum_{i=k+3}^{n^0} |u^0_i - u^0_{i-1}|,$$

and since the blocks moved at maximum of $t\max(\|u^0\|_\infty, \|v^0\|_\infty)$ during a time t, the blocks between $-a$ end a at time t are between those initially between $-a_t$ and a_t. Then we get (3.23) until t^*. Finally, collision after collision, we get (3.23). We obtain similarly (3.24)-(3.26).
We turn now to the bounds of the measures. Since $(\rho_{kl})_{kl}$, $(u_{kl})_{kl}$ and $(v_{kl})_{kl}$ are L^∞ bounded and $S_{kl}, T_{kl} \rightharpoonup 0$, we get that $((\pi_1)_{kl})_l$ and $((\pi_2)_{kl})_l$ are bounded in the distributional sense. Since they are non-negative measures, we conclude. □

Remark 3.7 The combination of the bounds (2.9)-(2.12) and (3.23)-(3.26) gives

$$\iint_{\mathbb{R}\times[-a,a]} |\partial_x u_{kl}(t, x, y)| \le \iint_{\mathbb{R}\times[-a_t,a_t]} \left|\partial_x u^0_k(x, y)\right| \le (1+a_t^2) \iint_{\mathbb{R}^2} \left|\partial_x u^0(x, y)\right|, \quad (3.28)$$

$$\iint_{[-a,a]\times\mathbb{R}} |\partial_y u_{kl}(t, x, y)| \le \iint_{[-a_t,a_t]\times\mathbb{R}} \left|\partial_y u^0_k(x, y)\right| \le (1+a_t^2) \iint_{\mathbb{R}^2} \left|\partial_y u^0(x, y)\right|, \quad (3.29)$$

$$\iint_{\mathbb{R}\times[-a,a]} |\partial_x v_{kl}(t, x, y)| \le \iint_{\mathbb{R}\times[-a_t,a_t]} \left|\partial_x v^0_k(x, y)\right| \le (1+a_t^2) \iint_{\mathbb{R}^2} \left|\partial_x v^0(x, y)\right|, \quad (3.30)$$

$$\iint_{[-a,a]\times\mathbb{R}} |\partial_y v_{kl}(t, x, y)| \le \iint_{[-a_t,a_t]\times\mathbb{R}} \left|\partial_y v^0_k(x, y)\right| \le (1+a_t^2) \iint_{\mathbb{R}^2} \left|\partial_y v^0(x, y)\right|, \quad (3.31)$$

for any $a > 0$ and with $a_t = a + t\max(\|u^0\|_\infty, \|v^0\|_\infty)$.

We have now to get stability results in order to get solution for the system with constraint for a large class of initial data.

4 Stability

The results we prove in this section have two specific purposes. First, we will get a result of stability of solutions. Then, by an adaptation of this proof, we will obtain a result for the limit of approximations of solutions.

4.1 Stability theorem

We first prove a stability property of a sequence of solutions using the technical results of the appendix B.

Theorem 4.1 (Stability of solutions) *Let us consider a sequence of solutions* $(\rho_k, u_k, v_k, (\pi_1)_k, (\pi_2)_k)_{k\geq 1}$*, with regularities (1.9)-(1.11), satisfying (1.4) with the constraint (1.5) and initial data* (ρ_k^0, u_k^0, v_k^0)*. We assume the following bounds for initial data:*

$$(\rho_k^0)_{k\geq 1} \text{ is bounded in } L^\infty(\mathbb{R}^2) \text{ and in } L^1(\mathbb{R}^2), \tag{4.1}$$

$$(u_k^0)_{k\geq 1}, (v_k^0)_{k\geq 1} \text{ are bounded in } L^\infty(\mathbb{R}^2) \text{ and in } BV_{loc}(\mathbb{R}^2). \tag{4.2}$$

The solutions are supposed to satisfy

$$0 \leq \rho_k \leq 1, \quad \iint\limits_{\mathbb{R}^2} \rho_k(t,x,y)\,dx\,dy \leq \iint\limits_{\mathbb{R}^2} \rho_k^0(x,y)\,dx\,dy, \tag{4.3}$$

$$essinf\ u_k^0 \leq u_k \leq esssup\ u_k^0, \quad essinf\ v_k^0 \leq v_k \leq esssup\ v_k^0, \tag{4.4}$$

$$\iint\limits_{\mathbb{R}\times[-a,a]} |\partial_x u_k(t,x,y)| \leq (1+a_t^2)K_1, \qquad \iint\limits_{[-a,a]\times\mathbb{R}} |\partial_y u_k(t,x,y)| \leq (1+a_t^2)K_2, \tag{4.5}$$

$$\iint\limits_{\mathbb{R}\times[-a,a]} |\partial_x v_k(t,x,y)| \leq (1+a_t^2)K_3, \qquad \iint\limits_{[-a,a]\times\mathbb{R}} |\partial_y v_k(t,x,y)| \leq (1+a_t^2)K_4, \tag{4.6}$$

for any $a > 0$ *and with* $a_t = a + tC$*, with* K_1, K_2, K_3, K_4 *and* C *constants. Finally we assume that*

$$((\pi_1)_k)_{k\geq 1} \text{ and } ((\pi_2)_k)_{k\geq 1} \text{ are bounded in } \mathcal{M}_{loc}([0,+\infty[\times\mathbb{R}^2). \tag{4.7}$$

Then, extracting a subsequence if necessary, as $k \to \infty$*, we have in the distributional sense*

$$(\rho_k, u_k, v_k, (\pi_1)_k, (\pi_2)_k) \rightharpoonup (\rho, u, v, \pi_1, \pi_2),$$

where $(\rho, u, v, \pi_1, \pi_2)$*, with regularities (1.9)-(1.11), are solution of (1.4) with the constraint (1.5) and satisfy the bounds*

$$0 \leq \rho \leq 1, \quad \iint\limits_{\mathbb{R}^2} \rho(t,x,y)\,dx\,dy \leq \iint\limits_{\mathbb{R}^2} \rho^0(x,y)\,dx\,dy, \tag{4.8}$$

$$essinf\ u^0 \leq u \leq esssup\ u^0, \quad essinf\ v^0 \leq v \leq esssup\ v^0, \tag{4.9}$$

$$\iint\limits_{\mathbb{R}\times[-a,a]} |\partial_x u(t,x,y)| \leq (1+a_t^2)K_1, \qquad \iint\limits_{[-a,a]\times\mathbb{R}} |\partial_y u(t,x,y)| \leq (1+a_t^2)K_2, \tag{4.10}$$

$$\iint\limits_{\mathbb{R}\times[-a,a]} |\partial_x v(t,x,y)| \leq (1+a_t^2)K_3, \qquad \iint\limits_{[-a,a]\times\mathbb{R}} |\partial_y v(t,x,y)| \leq (1+a_t^2)K_4, \tag{4.11}$$

for any $a > 0$*.*

Proof. Since $(\rho_k, u_k, v_k)_{k\geq 1}$ are bounded in $L^\infty(]0,+\infty[\times\mathbb{R}^2)$, then there exists a subsequence such that

$$\rho_k \rightharpoonup \rho, \quad u_k \rightharpoonup u, \quad v_k \rightharpoonup v \quad \text{ in } L^\infty_{w*}(]0,+\infty[\times\mathbb{R}^2). \tag{4.12}$$

From (4.7), there exists a subsequence such that

$$(\pi_1)_k \rightharpoonup \pi_1, \quad (\pi_2)_k \rightharpoonup \pi_2 \quad \text{ in } \mathcal{M}_{loc}([0,+\infty[\times\mathbb{R}^2). \tag{4.13}$$

From the first equation of (1.4), the sequence $(\rho_k)_{k\geq 1}$ satisfies the estimate: $\forall T > 0$, $\forall \varphi \in C_c^\infty(\mathbb{R}^2)$, $\forall t, s \in [0,T]$, $\forall k \in \mathbb{N}^*$,

$$\left| \iint\limits_{\mathbb{R}^2} (\rho_k(t,x,y) - \rho_k(s,x,y))\varphi(x,y)\,dx\,dy \right| \leq C_\varphi |t-s|, \tag{4.14}$$

with

$$C_\varphi = \sup_{k\geq 1} \|u_k^0\|_{L^\infty} \left(\iint\limits_{\mathbb{R}^2} |\partial_x \varphi|\,dx\,dy \right) + \sup_{k\geq 1} \|v_k^0\|_{L^\infty} \left(\iint\limits_{\mathbb{R}^2} |\partial_y \varphi|\,dx\,dy \right).$$

Then, applying lemma 7.2, $\rho_k \to \rho$ in $C([0,T], L^\infty_{w*}(\mathbb{R}^2))$. Furthermore $(u_k)_{k\geq 1}$ is bounded in $BV_{loc}(\mathbb{R}^2)$ uniformly in time sur $[0,T]$. We can then apply lemma 7.1, with $C_a = (1 + a_T^2)K$, with $K = \max(K_1, K_2)$ and we get that $\rho_k u_k \rightharpoonup \rho u$ in $L^\infty_{w*}(]0,T[\times\mathbb{R}^2)$. Similarly, we have $\rho_k v_k \rightharpoonup \rho v$ in $L^\infty_{w*}(]0,T[\times\mathbb{R}^2)$.
Now the second equation of (1.4) gives that

$$\begin{aligned} &\frac{d}{dt} \iint\limits_{\mathbb{R}^2} (\rho_k u_k)(t,x,y)\varphi(x,y)\,dx\,dy \\ = &\iint\limits_{\mathbb{R}^2} (\rho_k u_k^2)(t,x,y)\partial_x\varphi(x,y)\,dx\,dy + \iint\limits_{\mathbb{R}^2} (\rho_k u_k v_k)(t,x,y)\partial_y\varphi(x,y)\,dx\,dy \\ &+ \iint\limits_{\mathbb{R}^2} \partial_x\varphi(x,y)(\pi_1)_k(t,x,y), \end{aligned}$$

thus the sequence $\iint\limits_{\mathbb{R}^2} (\rho_k u_k)(t,x,y)\varphi(x,y)\,dx\,dy$ is bounded in BV_t. Therefore, in the same pattern as the proof of lemma 7.2 (see also [5]), we can extract a subsequence such that

$$\iint\limits_{\mathbb{R}^2} (\rho_k u_k)(t,x,y)\varphi(x,y)\,dx\,dy \to \iint\limits_{\mathbb{R}^2} (\rho u)(t,x,y)\varphi(x,y)\,dx\,dy \text{ in } L^1(]0,T[),$$

for all $\varphi \in C_c^\infty(\mathbb{R}^2)$. We can then apply lemma 7.1 with $\gamma_k = \rho_k u_k$ this time and $\omega_k = u_k$ (and also with v_k) and we get that $\rho_k u_k^2 \rightharpoonup \rho u^2$ and $\rho_k u_k v_k \rightharpoonup \rho u v$ in $L^\infty_{w*}(]0,T[\times\mathbb{R}^2)$. Similarly, we also have $\rho_k v_k^2 \rightharpoonup \rho v^2$ in $L^\infty_{w*}(]0,T[\times\mathbb{R}^2)$. We can now pass to the limit in the weak formulation to get (1.12)-(1.14) with the initial data (ρ^0, u^0, v^0). $\square$

4.2 Limit of approximations of solutions

In dimension one, we have directly obtained explicit solutions for any block initial data. In the current two-dimension case, at this stage, we only have approximations of solutions for general block initial data. We need to improve the previous stability result in the case where we only have

$$\begin{cases} \partial_t \rho_l + \partial_x(\rho_l u_l) + \partial_y(\rho_l v_l) = R_l \rightharpoonup 0, \\ \partial_t(\rho_l u_l) + \partial_x(\rho_l u_l^2 + (\pi_1)_l) + \partial_y(\rho_l u_l v_l) = S_l \rightharpoonup 0, \\ \partial_t(\rho_l v_l) + \partial_x(\rho_l u_l v_l) + \partial_y(\rho_l v_l^2 + (\pi_2)_l) = T_l \rightharpoonup 0 \end{cases} \tag{4.15}$$

when $l \to +\infty$, with a limit in the distribution sense, instead of having $R_l = S_l = T_l = 0$. We prove now that in this situation, we can extract a subsequence whose limit is a solution.

Theorem 4.2 (Limit of approximations) *Let $\rho^0 \in L^1(\mathbb{R}^2)$ and $u^0, v^0 \in L^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$ such that $0 \le \rho^0 \le 1$. We consider the sequence of block initial data $(\rho_k^0, u_k^0, v_k^0)_{k\ge 1}$ defined by (2.13)-(2.15). For any k, we consider the sequence $(\rho_{kl}, \rho_{kl}u_{kl}, \rho_{kl}v_{kl})_l$ defined by proposition 3.9. Then, extracting a subsequence if necessary, as $l \to +\infty$, we have*

$$(\rho_{kl}, u_{kl}, v_{kl}, (\pi_1)_{kl}, (\pi_2)_{kl}) \rightharpoonup (\rho_k, u_k, v_k, (\pi_1)_k, (\pi_2)_k)$$

in the distributional sense, where $(\rho_k, u_k, v_k, (\pi_1)_k, (\pi_2)_k)$ with regularities (1.9)-(1.11), is solution of (1.4) with the constraint (1.5) and satisfy the bounds

$$0 \le \rho_k \le 1, \qquad \iint_{\mathbb{R}^2} \rho_k(t,x,y)\,dx\,dy \le \iint_{\mathbb{R}^2} \rho^0(x,y)\,dx\,dy, \tag{4.16}$$

$$essinf\ u^0 \le u_k \le esssup\ u^0, \quad essinf\ v^0 \le v_k \le esssup\ v^0, \tag{4.17}$$

$$\iint_{\mathbb{R}\times[-a,a]} |\partial_x u_k(t,x,y)| \le (1+a_t^2) \iint_{\mathbb{R}^2} \left|\partial_x u^0(x,y)\right|, \tag{4.18}$$

$$\iint_{[-a,a]\times\mathbb{R}} |\partial_y u_k(t,x,y)| \le (1+a_t^2) \iint_{\mathbb{R}^2} \left|\partial_y u^0(x,y)\right|, \tag{4.19}$$

$$\iint_{\mathbb{R}\times[-a,a]} |\partial_x v_k(t,x,y)| \le (1+a_t^2) \iint_{\mathbb{R}^2} \left|\partial_x v^0(x,y)\right|, \tag{4.20}$$

$$\iint_{[-a,a]\times\mathbb{R}} |\partial_y v_k(t,x,y)| \le (1+a_t^2) \iint_{\mathbb{R}^2} \left|\partial_y v^0(x,y)\right|, \tag{4.21}$$

for any $a > 0$ and with $a_t = a + t\max(\|u^0\|_\infty, \|v^0\|_\infty)$.

Proof. The proof is very similar to the one of theorem 4.1 except an important dissimilarity, which is the relation (4.14). Here we get a relation of the form

$$\left| \iint_{\mathbb{R}^2} (\rho_{kl}(t,x,y) - \rho_{kl}(s,x,y))\varphi(x,y)\,dx\,dy \right| \le C_\varphi |t-s| + \left| \int_s^t \iint_{\mathbb{R}^2} R_{kl}\varphi \right|. \tag{4.22}$$

Adapting the proof of (3.14) but on a time space of length $|t-s|$ instead of T, we similarly get a bound for $\displaystyle\int_s^t \iint\limits_{\mathbb{R}^2} R_{kl}\varphi$ of the form $|t-s|\varepsilon C$ (instead of $T\varepsilon C$). Then we get again a bound of the form

$$\left|\iint\limits_{\mathbb{R}^2} (\rho_{kl}(t,x,y) - \rho_{kl}(s,x,y))\varphi(x,y)\,dx\,dy\right| \leq \tilde{C}_\varphi |t-s|, \tag{4.23}$$

and we have again, when $l \to +\infty$, $\rho_{kl} \to \rho_k$ in $C([0,T], L^\infty_{w*}(\mathbb{R}^2))$ and the rest of the proof is quite similar. $\square$

The first consequence of this result is that we will obtain solutions for any block initial data (not explicit in every cases here contrary to the one-dimensional case). Then by approximation of any initial data by initial blocks and the stability result, we will get existence of solutions for any initial data.

5 Existence result

Prior to get the existence result, let's start with discussing the constraint relation (1.6) which leads to the difficulty of defining the product $\rho\pi$ with π a measure and ρ not necessarily continuous. We expose how it is possible to define this term in a special class of solutions. To do this, we adapt the analysis done in [2]. Then, we will prove the existence result for any initial data and then for functions with enough regularity, we prove that we get the product $\rho\pi = \pi$.

5.1 Definition of $\rho\pi$ for π in $\mathcal{M}([0,+\infty[, L^\infty(\mathbb{R}^2))$

If the measure $\pi \in \mathcal{M}_{loc}([0,+\infty[\times\mathbb{R}^2)$ is also in the space $\mathcal{M}_t([0,+\infty[, L^\infty_{xy}(\mathbb{R}^2))$, then there exists C such that

$$\left|\int_{[0,+\infty[}\int_{\mathbb{R}}\int_{\mathbb{R}} \phi(t,x,y)\pi(t,x,y)\right| \leq C\|\phi\|_{L^\infty_t(]0,+\infty[,L^1_{xy}(\mathbb{R}^2))}, \quad \forall\phi \in C_c([0,+\infty[\times\mathbb{R}^2) \tag{5.1}$$

and we can define $<\pi,\phi>$ for $\phi \in C_c([0,+\infty[, L^1(\mathbb{R}^2))$ with

$$|<\pi,\phi>| \leq C\|\phi\|_{L^\infty_t(]0,+\infty[,L^1_{xy}(\mathbb{R}^2))}, \quad \forall\phi \in C_c([0,+\infty[, L^1(\mathbb{R}^2)).$$

See [21] for the representation theorem of Riesz for bounded continuous vector-valued functions.

Definition 5.1 *Let $\rho \in C_t([0,+\infty[, L^1_{loc}(\mathbb{R}^2))$ and $\pi \in \mathcal{M}_t([0,+\infty[, L^\infty_{xy}(\mathbb{R}^2))$. Then the product $\rho\pi$ is defined as a measure by $<\rho\pi,\phi> = <\pi,\rho\phi>$ for $\phi \in C_c([0,+\infty[\times\mathbb{R}^2)$.*

We notice that if $\pi \in \mathcal{M}_{loc}([0,+\infty[\times\mathbb{R}^2)$ satisfies

$$\int_{[0,+\infty[}\int_{\mathbb{R}}\int_{\mathbb{R}} |\phi(t,x,y)\pi(t,x,y)| \leq C\|\phi\|_{L^\infty_t(]0,+\infty[,L^1_{xy}(\mathbb{R}^2))}, \ \ \forall\phi \in C_c([0,+\infty[,L^1(\mathbb{R}^2)), \tag{5.2}$$

then $\pi \in \mathcal{M}_t([0,+\infty[,L^\infty_{xy}(\mathbb{R}^2))$ and

$$< \pi,\phi >= \int_{[0,+\infty[}\int_{\mathbb{R}}\int_{\mathbb{R}} \phi(t,x,y)\pi(t,x,y), \quad \forall\phi \in C_c([0,+\infty[,L^1(\mathbb{R}^2)).$$

As in [2], we have easily the following proposition which proves that the blocks of section 2 have this regularity and satisfy clearly the constraint (1.6).

Proposition 5.2 *For the sticky blocks of section 2, the pressures π_1 and π_2 satisfy (5.2). For these blocks, we also have $\rho \in C([0,+\infty[,L^1(\mathbb{R}^2))$ and $\rho\pi = \pi$ in $\mathcal{M}_{loc}([0,+\infty[\times\mathbb{R}^2)$.*

5.2 The exclusion relation $\rho\pi = \pi$

We prove a result for the product $\rho\pi$ when $\pi \in \mathcal{M}_t([0,+\infty[,L^\infty_{xy}(\mathbb{R}^2))$.

Proposition 5.3 *If we assume that the limit (ρ,u,v,π_1,π_2) of theorem 4.1 satisfies furthermore that $\rho_k \to \rho \in C_t([0,+\infty[,L^1_{xy}(\mathbb{R}^2))$ and if we assume that $\pi_1,\pi_2 \in \mathcal{M}_t([0,+\infty[,L^\infty_{xy}(\mathbb{R}^2))$, then the exclusion relations $\rho\pi_1 = \pi_1$ and $\rho\pi_2 = \pi_2$ hold.*

Proof. We prove it for π_1, the proof is similar for π_2. We consider the sequence $(\rho_k,u_k,v_k,(\pi_1)_k,(\pi_2)_k)_{k\geq 1}$ of blocks which approximates (ρ,u,v,π_1,π_2) in weak sense. Let $\varphi \in C_c^\infty([0,+\infty[\times\mathbb{R}^2)$. We can write, since $\pi_1 \in \mathcal{M}_t([0,+\infty[,L^\infty_{xy}(\mathbb{R}^2))$ and $\rho \in C([0,+\infty[,L^1_{loc}(\mathbb{R}^2))$,

$$< (\pi_1)_k,\rho_k\varphi > - < \pi_1,\rho\varphi >=< (\pi_1)_k,(\rho_k-\rho)\varphi > + < (\pi_1)_k - \pi,\rho\varphi > .$$

On one hand, $\rho\varphi \in C_c([0,+\infty[,L^1(\mathbb{R}^2))$ hence $< (\pi_1)_k - \pi_1,\rho\varphi > \underset{k\to+\infty}{\to} 0$. On the other hand, since $(\pi_1)_k \in \mathcal{M}_t([0,+\infty[,L^\infty_{xy}(\mathbb{R}^2))$,

$$\begin{aligned} | < (\pi_1)_k,(\rho_k-\rho)\varphi > | &\leq C_k \sup_t \int_{\mathbb{R}} |(\rho_k-\rho)\varphi|\,dx \\ &\leq C_k\|\varphi\|_{L^\infty_{t,x,y}}\|\rho_k-\rho\|_{L^\infty_t(L^1_{loc,x,y})}. \end{aligned}$$

We can take for the constant C_k the smallest one, that is to say

$$C_k = \sup_{\varphi\in L^\infty_t(L^1_{xy}),\varphi\neq 0} \frac{|\int_0^T \int_{\mathbb{R}^2}\varphi(\pi_1)_k|}{\|\varphi\|_{L^\infty_t(L^1_{xy})}}.$$

We consider the linear continuous applications f_k defined, for any $\varphi \in L^\infty_t(L^1_{xy})$, by $f_k(\varphi) = \int_0^T \iint_{\mathbb{R}^2} \varphi(\pi_1)_k$. For any $\varphi \in L^\infty_t(L^1_{xy})$, we have $f_k(\varphi) \to \int_0^T \iint_{\mathbb{R}^2} \varphi\pi$

and then $(f_k(\varphi))_k$ is bounded. We apply the Banach-Steinhaus theorem to this family of applications and get that $\sup\limits_k C_k < +\infty$.

Therefore we get $\lim\limits_{k\to+\infty} < (\pi_1)_k, \rho_k\varphi > = < \rho, \pi_1\varphi > = < \rho\pi_1, \varphi >$. Now $(\pi_1)_k = \rho_k(\pi_1)_k \to \tilde{\pi}_1$ and then $\rho\pi_1 = \pi_1$. $\square$

5.3 Existence of solutions

We are now able to prove the existence theorem 1.1.

Proof of theorem 1.1. Let ρ_k^0, u_k^0, v_k^0 ($k \in \mathbb{N}^*$) be the block initial data defined by (2.13)-(2.15) associated to ρ^0, u^0, v^0 provided by proposition 2.3. Proposition 3.9 gives, for any k, a sequence $(\rho_{kl}, u_{kl}, v_{kl}, (\pi_1)_{kl}, (\pi_2)_{kl})_l$ such that

$$\begin{cases} \partial_t\rho_{kl} + \partial_x(\rho_{kl}u_{kl}) + \partial_y(\rho_{kl}v_{kl}) = R_{kl} \underset{l\to+\infty}{\rightharpoonup} 0, \\ \partial_t(\rho_{kl}u_{kl}) + \partial_x(\rho u_{kl}^2 + (\pi_1)_{kl}) + \partial_y(\rho_{kl}u_{kl}v_{kl}) = S_{kl} \underset{l\to+\infty}{\rightharpoonup} 0, \\ \partial_t(\rho_{kl}v_{kl}) + \partial_x(\rho u_{kl}v_{kl}) + \partial_y(\rho_{kl}v_{kl}^2 + (\pi_2)_{kl}) = T_{kl} \underset{l\to+\infty}{\rightharpoonup} 0 \end{cases}$$

in the distributional sense. At k fixed, these functions satisfy the bounds of theorem 4.2 and we can apply it to get that, up to subsequence, and making a diagonal Cantor process, the convergence in the distributional sense $(\rho_{kl}, u_{kl}, v_{kl}, (\pi_1)_{kl}, (\pi_2)_{kl}) \underset{l\to+\infty}{\rightharpoonup} (\rho_k, u_k, v_k, (\pi_1)_k, (\pi_2)_k)$ for any k. The previous obtained limit $(\rho_k, u_k, v_k, (\pi_1)_k, (\pi_2)_k)$, with regularities (1.9)-(1.11), is solution of (1.4) with the constraint (1.5) and satisfies the bounds (4.16)-(4.21). We can now apply the theorem 4.1 to this sequence, and get, up to a subsequence when $k \to \infty$, $(\rho_k, u_k, v_k, (\pi_1)_k, (\pi_2)_k) \rightharpoonup (\rho, u, v, \pi_1, \pi_2)$, where $(\rho, u, v, \pi_1, \pi_2)$, with regularities (1.9)-(1.11), is solution of (1.4) with the constraint (1.5) and satisfy the bounds (1.15)-(1.20). By proposition 5.3, we finally have the constraint (1.6) in the case where $\rho_k \to \rho \in C_t([0, +\infty[, L^1_{xy}(\mathbb{R}^2))$ and $\pi_1, \pi_2 \in \mathcal{M}_t([0, +\infty[, L^\infty_{xy}(\mathbb{R}^2))$. $\square$

Remark 5.1 We get the constraint in the case of blocks and for the limit of approximation by blocks when the limit is in $C_t([0, +\infty[, L^1_{xy}(\mathbb{R}^2))$. In the most general case, we only have a convergence in $C_t([0, +\infty[, L^\infty_{w*}(\mathbb{R}^2))$.

6 Appendix A: weak convergences of block approximations

In this appendix, we prove the technical part of the proof of proposition 2.3, that is to say the convergence of $\rho_k^0 \rightharpoonup \rho^0$, $\rho_k^0 u_k^0 \rightharpoonup \rho^0 u^0$ and $\rho_k^0 v_k^0 \rightharpoonup \rho^0 v^0$ in the distributional sense.

Let $\varphi \in C_c^\infty(\mathbb{R}^2)$ and let $k_0 \in \mathbb{N}$ such that *supp* $\varphi \subset [-k_0, k_0]^2$. Using Taylor formula, there exists $(x_{ijk}, y_{ijk}) \in \left]\frac{i}{k}, \frac{i}{k} + \sqrt{m_{ijk}}\right[\times \left]\frac{j}{k}, \frac{j}{k} + \sqrt{m_{ijk}}\right[$ such that

$$\int_{\frac{i}{k}}^{\frac{i}{k}+\sqrt{m_{ijk}}} \int_{\frac{j}{k}}^{\frac{j}{k}+\sqrt{m_{ijk}}} \varphi(x, y)\, dx\, dy$$

$$
\begin{aligned}
= & \;\varphi\left(\frac{i}{k},\frac{j}{k}\right) m_{ijk} + \frac{1}{6}\int_{j/k}^{y_{ijk}} \partial^2_{xx}\varphi(x_{ijk},v)\,dv\, m_{ijk}^{3/2} + \frac{1}{2}\partial_x\varphi(x_{ijk},y_{ijk}) m_{ijk}^{3/2} \\
& + \frac{1}{2}\partial_y\varphi(x_{ijk},y_{ijk}) m_{ijk}^{3/2} + \frac{1}{6}\int_{i/k}^{x_{ijk}} \partial^2_{yy}\varphi(u,y_{ijk})\,du\, m_{ijk}^{3/2}.
\end{aligned}
$$

Knowing that

$$
\iint\limits_{\mathbb{R}^2} \rho_k^0(x,y)\varphi(x,y)\,dx\,dy = \sum_{i,j=-k^2}^{k^2} \int_{\frac{i}{k}}^{\frac{i}{k}+\sqrt{m_{ijk}}} \int_{\frac{j}{k}}^{\frac{j}{k}+\sqrt{m_{ijk}}} \varphi(x,y)\,dx\,dy,
$$

and

$$
\varphi\left(\frac{i}{k},\frac{j}{k}\right) m_{ijk} = \int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} \int_{\frac{j}{k}}^{\frac{j+1}{k}-\frac{1}{k^2}} \rho^0(x,y)\varphi\left(\frac{i}{k},\frac{j}{k}\right)\,dx\,dy,
$$

therefore, for $k > k_0$, we have

$$
\begin{aligned}
& \left| \iint\limits_{\mathbb{R}^2} \rho_k^0(x,y)\varphi(x,y)\,dx\,dy - \iint\limits_{\mathbb{R}^2} \rho^0(x,y)\varphi(x,y)\,dx\,dy \right| \\
\leq & \sum_{i,j=-kk_0}^{kk_0-1} \int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} \int_{\frac{j}{k}}^{\frac{j+1}{k}-\frac{1}{k^2}} \left\|\rho^0\right\|_\infty \left|\varphi\left(\frac{i}{k},\frac{j}{k}\right) - \varphi(x,y)\right| dx\,dy \\
& + \sum_{i,j=-kk_0}^{kk_0-1} \left(\int_{\frac{i+1}{k}-\frac{1}{k^2}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} \left\|\rho^0\right\|_\infty |\varphi(x,y)|\, dx\,dy \right. \\
& \qquad\qquad \left. + \int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} \int_{\frac{j+1}{k}-\frac{1}{k^2}}^{\frac{j+1}{k}} \left\|\rho^0\right\|_\infty |\varphi(x,y)|\, dx\,dy \right) \\
& + \frac{1}{6} \sum_{i,j=-kk_0}^{kk_0-1} \left(\left\|\partial^2_{xx}\varphi\right\|_\infty (y_{ijk} - \frac{j}{k}) + 3\left\|\partial_x\varphi\right\|_\infty \right. \\
& \qquad\qquad \left. + 3\left\|\partial_y\varphi\right\|_\infty + \left\|\partial^2_{yy}\varphi\right\|_\infty (x_{ijk} - \frac{i}{k}) \right) m_{ijk}^{3/2} \\
\leq & \sum_{i,j=-kk_0}^{kk_0-1} \int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} \int_{\frac{j}{k}}^{\frac{j+1}{k}-\frac{1}{k^2}} \left(\left\|\partial_x\varphi\right\|_\infty \left(x - \frac{i}{k}\right) + \left\|\partial_x\varphi\right\|_\infty \left(y - \frac{j}{k}\right) \right) dx\,dy \\
& + \left\|\varphi\right\|_\infty \sum_{i=-kk_0}^{kk_0-1} \left(\int_{\frac{i+1}{k}-\frac{1}{k^2}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} dx\,dy + \int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} \int_{\frac{j+1}{k}-\frac{1}{k^2}}^{\frac{j+1}{k}} dx\,dy \right) \\
& + \frac{1}{6} \sum_{i,j=-kk_0}^{kk_0-1} \left((\left\|\partial^2_{xx}\varphi\right\|_\infty + \left\|\partial^2_{yy}\varphi\right\|_\infty)\sqrt{m_{ijk}} + 3(\left\|\partial_x\varphi\right\|_\infty + \left\|\partial_y\varphi\right\|_\infty) \right) \\
& \qquad\qquad \times \left(\int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} \int_{\frac{j}{k}}^{\frac{j+1}{k}-\frac{1}{k^2}} dx\,dy \right)^{3/2} \\
\leq & \sum_{i,j=-kk_0}^{kk_0-1} \int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} \int_{\frac{j}{k}}^{\frac{j+1}{k}-\frac{1}{k^2}} dx\,dy \left(\left\|\partial_x\varphi\right\|_\infty + \left\|\partial_x\varphi\right\|_\infty \right) \frac{1}{k}
\end{aligned}
$$

$$
\begin{aligned}
&+\|\varphi\|_\infty \sum_{i,j=-kk_0}^{kk_0-1}\left(\frac{1}{k^3}+\frac{1}{k^4}\right)\\
&+\frac{1}{6}\left(\left\|\partial^2_{xx}\varphi\right\|_\infty+\left\|\partial^2_{yy}\varphi\right\|_\infty+3\left\|\partial_x\varphi\right\|_\infty+3\left\|\partial_y\varphi\right\|_\infty\right)\sum_{i,j=-kk_0}^{kk_0-1}\left(\frac{1}{k^2}\right)^{3/2}\\
\leq\ & 4k^2k_0^2\frac{1}{k^2}\left(\|\partial_x\varphi\|_\infty+\|\partial_x\varphi\|_\infty\right)\frac{1}{k}+\|\varphi\|_\infty 4k^2k_0^2\left(\frac{1}{k^3}+\frac{1}{k^4}\right)\\
&+\frac{1}{6}\left(\left\|\partial^2_{xx}\varphi\right\|_\infty+\left\|\partial^2_{yy}\varphi\right\|_\infty+3(\|\partial_x\varphi\|_\infty+\|\partial_y\varphi\|_\infty\right)4k^2k_0^2\frac{1}{k^3}\\
\leq\ & C_\varphi\frac{1}{k}
\end{aligned}
$$

and then $\rho_k^0 \rightharpoonup \rho^0$ holds in the distributional sense. Let us focus now on the convergence of $\rho_k^0 u_k^0$. For $k > k_0$, we have

$$
\begin{aligned}
&\left|\iint\limits_{\mathbb{R}^2}\rho_k^0(x,y)u_k^0(x,y)\varphi(x,y)\,dx\,dy-\iint\limits_{\mathbb{R}^2}\rho^0(x,y)u(x,y)\varphi(x,y)\,dx\,dy\right|\\
\leq\ &\sum_{i,j=-kk_0}^{kk_0-1}\int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}}\int_{\frac{j}{k}}^{\frac{j+1}{k}-\frac{1}{k^2}}\left\|\rho^0\right\|_\infty\left|u^0_{ijk}\,\varphi\left(\frac{i}{k},\frac{j}{k}\right)-u^0(x,y)\varphi(x,y)\right|\,dx\,dy\\
&+\sum_{i,j=-kk_0}^{kk_0-1}\left(\int_{\frac{i+1}{k}-\frac{1}{k^2}}^{\frac{i+1}{k}}\int_{\frac{j}{k}}^{\frac{j+1}{k}}\left\|\rho^0\right\|_\infty\left\|u^0\right\|_\infty|\varphi(x,y)|\,dx\,dy\right.\\
&\qquad\left.+\int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}}\int_{\frac{j+1}{k}-\frac{1}{k^2}}^{\frac{j+1}{k}}\left\|\rho^0\right\|_\infty\left\|u^0\right\|_\infty|\varphi(x,y)|\,dx\,dy\right)\\
&+\frac{1}{6}u^0_{ijk}\sum_{i,j=-kk_0}^{kk_0-1}\left(\left\|\partial^2_{xx}\varphi\right\|_\infty(y_{ijk}-\frac{j}{k})+3\left\|\partial_x\varphi\right\|_\infty\right.\\
&\qquad\left.+3\left\|\partial_y\varphi\right\|_\infty+\left\|\partial^2_{yy}\varphi\right\|_\infty(x_{ijk}-\frac{i}{k})\right)\ m_{ijk}^{3/2}
\end{aligned}
$$

and the main difference with to regard to the first convergence is the first term. We write

$$
\begin{aligned}
&\sum_{i,j=-kk_0}^{kk_0-1}\int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}}\int_{\frac{j}{k}}^{\frac{j+1}{k}-\frac{1}{k^2}}\left\|\rho^0\right\|_\infty\left|u^0_{ijk}\,\varphi\left(\frac{i}{k},\frac{j}{k}\right)-u^0(x,y)\varphi(x,y)\right|\,dx\,dy\\
\leq\ &\sum_{i,j=-kk_0}^{kk_0-1}\int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}}\int_{\frac{j}{k}}^{\frac{j+1}{k}-\frac{1}{k^2}}\left\|\rho^0\right\|_\infty\left|u^0(x,y)\right|\left|\varphi\left(\frac{i}{k},\frac{j}{k}\right)-\varphi(x,y)\right|\,dx\,dy\\
&+\sum_{i,j=-kk_0}^{kk_0-1}\int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}}\int_{\frac{j}{k}}^{\frac{j+1}{k}-\frac{1}{k^2}}\left\|\rho^0\right\|_\infty\left|u^0_{ijk}-u^0(x,y)\right|\,\varphi\left(\frac{i}{k},\frac{j}{k}\right)\,dx\,dy
\end{aligned}
$$

and the main new term is in fact the last one. We control it the following way:

$$
\sum_{i,j=-kk_0}^{kk_0-1}\int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}}\int_{\frac{j}{k}}^{\frac{j+1}{k}-\frac{1}{k^2}}\left\|\rho^0\right\|_\infty\left|u^0_{ijk}-u^0(x,y)\right|\,\varphi\left(\frac{i}{k},\frac{j}{k}\right)\,dx\,dy
$$

$$\leq \sum_{i,j=-kk_0}^{kk_0-1} \int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} \int_{\frac{j}{k}}^{\frac{j+1}{k}-\frac{1}{k^2}} \left\|\rho^0\right\|_\infty \left|\Delta_{ijk}^{u^0}(x,y)\right| \|\varphi\|_\infty \, dx\, dy,$$

where

$$\Delta_{ijk}^{u^0}(x,y) = k^2 \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} (u^0(\tilde{x},\tilde{y}) - u^0(x,y))\, d\tilde{x}\, d\tilde{y}.$$

Now $u^0 \in BV(\mathbb{R}^2)$, then u^0 is continuous and then uniformly continuous on compacts. Let $\varepsilon > 0$, there exists $\eta > 0$ such that for any $(x,y),(\tilde{x},\tilde{y}) \in [-k_0,k_0]^2$, if $|x-\tilde{x}| \leq \eta$, $|y-\tilde{y}| \leq \eta$, then $|u^0(\tilde{x},\tilde{y}) - u^0(x,y)| \leq \varepsilon$. Now for $i,j \in \mathbb{Z} \cap [-kk_0, kk_0-1]$ and $x,\tilde{x} \in [\frac{i}{k}, \frac{i+1}{k}]$, $y,\tilde{y} \in [\frac{j}{k}, \frac{j+1}{k}]$, then $(x,y),(\tilde{x},\tilde{y}) \in [-k_0,k_0]^2$. Thus for $\frac{1}{k} < \eta$, we have

$$\sum_{i,j=-kk_0}^{kk_0-1} \int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} \int_{\frac{j}{k}}^{\frac{j+1}{k}-\frac{1}{k^2}} \left\|\rho^0\right\|_\infty \left|u_{ijk}^0 - u^0(x,y)\right| \varphi\left(\frac{i}{k},\frac{j}{k}\right) dx\, dy$$

$$\leq \sum_{i,j=-kk_0}^{kk_0-1} \int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} \int_{\frac{j}{k}}^{\frac{j+1}{k}-\frac{1}{k^2}} \left\|\rho^0\right\|_\infty \left|k^2 \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} \varepsilon\, d\tilde{x}\, d\tilde{y}\right| \|\varphi\|_\infty \, dx\, dy$$

$$\leq \varepsilon \sum_{i,j=-kk_0}^{kk_0-1} \frac{1}{k^2} = 4k_0^2\varepsilon.$$

It gives the limit of the new term and we get that $\rho_k^0 u_k^0 \rightharpoonup \rho^0 u^0$ holds in the distributional sense. Similarly we obtain the convergence of $\rho_k^0 v_k^0$.

7 Appendix B: technical results

The first result is to help us passing to the limit in the products. It is an extension in dimension two of a similar lemma in dimension one proved in [2]. Notice that we also have to consider locally BV bounds.

Lemma 7.1 *Consider for any $k \in \mathbb{N}$, some functions $\gamma_k \in L^\infty(]0,T[\times\mathbb{R}^2)$, $\omega_k \in L^\infty(]0,T[, BV_{loc}(\mathbb{R}^2))$ and $\gamma \in L^\infty(]0,T[\times\mathbb{R}^2)$, $\omega \in L^\infty(]0,T[, BV_{loc}(\mathbb{R}^2))$. Let us assume that $(\gamma_k)_{k\in\mathbb{N}}$ is a bounded sequence in $L^\infty(]0,T[\times\mathbb{R}^2)$ that tends to γ in $L^\infty_{w*}(]0,T[\times\mathbb{R}^2)$, and satisfies, for any $\Gamma \in C_c^\infty(\mathbb{R}^2)$,*

$$\iint_{\mathbb{R}^2} (\gamma_k - \gamma)(t,x,y)\Gamma(x,y)\, dx\, dy \underset{k\to+\infty}{\to} 0, \tag{7.1}$$

either i) a.e. $t \in]0,T[$ or ii) in $L^1_t(]0,T[)$. Let us also assume that $(\omega_k)_{k\in\mathbb{N}}$ is a bounded sequence in $L^\infty(]0,T[\times\mathbb{R}^2)$ that tends to ω in $L^\infty_{w*}(]0,T[\times\mathbb{R}^2)$, and assume that, for any $a > 0$, there exists $C_a > 0$ such that, for any $t \in [0,T]$,

$$\iint_{[-a,a]^2} |\partial_x \omega_k(t,x,y)| \leq C_a, \qquad \iint_{[-a,a]^2} |\partial_y \omega_k(t,x,y)| \leq C_a, \qquad \text{for any } k. \tag{7.2}$$

Then $\gamma_k\omega_k \rightharpoonup \gamma\omega$ in $L^\infty_{w*}(]0,T[\times\mathbb{R}^2)$, as $k \to +\infty$.

Proof. We detail case $i)$, the proof being very similar for case $ii)$. First, we notice that we also have

$$\iint\limits_{[-a,a]^2} |\partial_x \omega(t,x,y)| \leq C_a, \qquad \iint\limits_{[-a,a]^2} |\partial_y \omega(t,x,y)| \leq C_a. \tag{7.3}$$

Let ζ_ε be a sequence of mollifiers in $\mathbb{R}^2$. We shall use the notation $z^\varepsilon = z \underset{xy}{*} \zeta_\varepsilon$. Let us write the decomposition

$$\gamma_k\omega_k - \gamma\omega = \gamma_k(\omega_k - \omega_k^\varepsilon) + (\gamma_k - \gamma)\omega_k^\varepsilon + \gamma(\omega_k^\varepsilon - \omega^\varepsilon) + \gamma(\omega^\varepsilon - \omega). \tag{7.4}$$

We are first going to control the first and fourth terms of this decomposition for ε small enough and uniformly in k. Then, fixing ε, we shall pass to the limit, when k tends towards infinity, in the second and third terms. Let $\varphi \in C_c^\infty(]0,T[\times\mathbb{R}^2)$ and let $a > 0$ be such that the support of φ is a subset of $]0,T[\times[-a,a]^2$.

– Let $\eta > 0$. The term $\displaystyle\int_0^T \iint\limits_{\mathbb{R}^2} \gamma(\omega^\varepsilon - \omega)\varphi\, dx\, dy\, dt$ is controlled in the following way. For $|\tilde{x}|, |\tilde{y}| \leq 1$, we have

$$\begin{aligned}
&\iint\limits_{[-a,a]^2} |\omega(t, x-\tilde{x}, y-\tilde{y}) - \omega(t,x,y)|\, dx\, dy \\
\leq\ & |\tilde{x}| \iint\limits_{[-a-1,a+1]^2} |\partial_x\omega(t,.,.)| + |\tilde{y}| \iint\limits_{[-a-1,a+1]^2} |\partial_y\omega(t,.,.)| \\
\leq\ & C_a(|\tilde{x}| + |\tilde{y}|),
\end{aligned}$$

hence for $\varepsilon < 1$,

$$\begin{aligned}
&\|\omega^\varepsilon(t,.,.) - \omega(t,.,.)\|_{L^1([-a,a]^2)} \\
=\ & \iint\limits_{[-a,a]^2} \left| \iint\limits_{\mathbb{R}^2} (\omega(t, x-\tilde{x}, y-\tilde{y}) - \omega(t,x,y))\zeta_\varepsilon(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y} \right| dx\, dy \\
=\ & \iint\limits_{\mathbb{R}^2} \iint\limits_{[-a,a]^2} |\omega(t, x-\tilde{x}, y-\tilde{y}) - \omega(t,x,y)|\, dx\, dy\, \zeta_\varepsilon(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y} \\
\leq\ & C_a \int_{B(0,\varepsilon)} (|\tilde{x}| + |\tilde{y}|)\zeta_\varepsilon(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y} \leq 2\varepsilon C_a.
\end{aligned}$$

Thus we get

$$\left| \int_0^T \iint\limits_{\mathbb{R}^2} \gamma(\omega^\varepsilon - \omega)\varphi\, dx\, dy\, dt \right| \leq 2\varepsilon C_a T \|\varphi\|_\infty \|\gamma\|_\infty.$$

This is less than η if ε is small enough. We have the same bound uniformly in k for $(\gamma_k)_{k\geq 0}$ and $(\omega_k)_{k\geq 0}$, thus for such ε,

$$\left| \int_0^T \iint\limits_{\mathbb{R}^2} \gamma(\omega^\varepsilon - \omega)\varphi\, dx\, dy\, dt \right| \leq \eta \ \text{ and } \ \left| \int_0^T \iint\limits_{\mathbb{R}^2} \gamma_k(\omega_k^\varepsilon - \omega_k)\varphi\, dx\, dy\, dt \right| \leq \eta, \ \ \forall k \in \mathbb{N}.$$

– Let now ε be fixed as above. For the third term of the decomposition (7.4), obviously $\omega_k^\varepsilon - \omega^\varepsilon \rightharpoonup 0$ in $L^\infty_{w*}(]0,T[\times\mathbb{R}^2)$, thus $\gamma(\omega_k^\varepsilon - \omega^\varepsilon) \rightharpoonup 0$.
It remains to establish the convergence $(\gamma_k - \gamma)\omega_k^\varepsilon \rightharpoonup 0$ in $L^\infty_{w*}(]0,T[\times\mathbb{R}^2)$. In order to do this, we only need to consider a test function $\varphi \in C_c^\infty(]0,T[\times\mathbb{R}^2)$, $\varphi(t,x,y) = \varphi_1(t)\varphi_2(x,y)$, $\varphi_1 \in C_c^\infty(]0,T[)$, $\varphi_2 \in C_c^\infty(\mathbb{R}^2)$. In order to prove that

$$\int_0^T \iint_{\mathbb{R}^2} (\gamma_k - \gamma)(t,x,y)\omega_k^\varepsilon(t,x,y)\varphi(t,x,y)\,dx\,dy\,dt \to 0, \quad k \to \infty,$$

we write this integral as $\displaystyle\int_0^T \iint_{\mathbb{R}^2} I_k(t,\tilde{x},\tilde{y})\,d\tilde{x}\,d\tilde{y}\,dt$ where

$$I_k(t,\tilde{x},\tilde{y}) = \omega_k(t,\tilde{x},\tilde{y}) \left(\iint_{\mathbb{R}^2} (\gamma_k - \gamma)(t,x,y)\zeta_\varepsilon(x-\tilde{x}, y-\tilde{y})\varphi(t,x,y)\,dx\,dy \right).$$

We are going to prove the convergence of this integral using Lebesgue's theorem. Since, at $(\tilde{x},\tilde{y})$ being fixed, $(x,y) \mapsto \zeta_\varepsilon(x-\tilde{x}, y-\tilde{y})\varphi_2(x,y) \in C_c^\infty(\mathbb{R}^2)$ and together with the fact that $(\omega_k)_{k\geq 0}$ is bounded in $L^\infty(]0,T[\times\mathbb{R}^2)$, we deduce from the property of γ, that for *a.e.* t, x, y, $I_k(t,x,y) \to 0$ as $k \to \infty$. We also have the following estimate,

$$|I_k(t,\tilde{x},\tilde{y})| \leq \sup_k \|\omega_k\|_{L^\infty}(\sup_k \|\gamma_k\|_{L^\infty} + \|\gamma\|_{L^\infty})J(t,\tilde{x},\tilde{y}),$$

where $J : (t,\tilde{x},\tilde{y}) \mapsto \displaystyle\iint_{\mathbb{R}^2} \zeta_\varepsilon(x-\tilde{x}, y-\tilde{y})|\varphi(t,x,y)|\,dx\,dy \in L^1(]0,T[\times\mathbb{R}^2)$. Therefore, by dominated convergence, we have that $I_k(t,x,y) \to 0$ in $L^1(]0,T[\times\mathbb{R}^2)$, which gives the desired convergence.
– Finally, we can conclude that $\gamma_n\omega_n - \gamma\omega \rightharpoonup 0$ in $L^\infty_{w*}(]0,T[\times\mathbb{R})$. $\square$

Remark 7.1 This is a result of compensated compactness, which uses the compactness in (x,y) for $(\omega_k)_k$ given by (7.2) and the weak compactness in t for $(\gamma_k)_k$ given by (7.1) to pass to the weak limit in the product $\gamma_k\omega_k$.

The second result gives some continuity in time. The proof is an easy adaptation in dimension two of lemma 4.4 of [5]. The main idea is to use a countable dense set in $C_c^\infty(\mathbb{R}^2)$ for the L^1-norm and Ascoli's theorem. Since there is no new difficulty, we skip the proof.

Lemma 7.2 *Let $(n_k)_{k\in\mathbb{N}^*}$ be a bounded sequence in $L^\infty(]0,T[\times\mathbb{R}^2)$ which satisfies:*
for all $\varphi \in C_c^\infty(\mathbb{R}^2)$, the sequence $(\int_{\mathbb{R}} n_k(t,x,y)\varphi(x,y)\,dx\,dy)_k$ is uniformly Lipschitz continuous on $[0,T]$, i.e. $\exists C_\varphi > 0$, $\forall k \in \mathbb{N}^*$, $\forall s,t \in [0,T]$,

$$\left| \iint_{\mathbb{R}^2} (n_k(t,x,y) - n_k(s,x,y))\varphi(x,y)\,dx\,dy \right| \leq C_\varphi|t-s|.$$

Then, up to a subsequence, it exists $n \in L^\infty(]0,T[\times\mathbb{R}^2)$ *such that* $n_k \to n$ *in* $C([0,T], L^\infty_{w*}(\mathbb{R}^2))$*, i.e.*

$$\forall \Gamma \in L^1(\mathbb{R}^2), \qquad \sup_{t\in[0,T]} \left| \iint_{\mathbb{R}^2} (n_k(t,x,y) - n(t,x,y))\Gamma(x,y)\,dx\,dy \right| \underset{k\to+\infty}{\to} 0.$$

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