Regularity results for the solutions of a non-local model of traffic flow

Florent Berthelin[*] Paola Goatin[†]

June 25, 2018

Abstract

We consider a non-local traffic model involving a convolution product. Unlike other studies, the considered kernel is discontinuous on $\mathbb{R}$. We prove Sobolev estimates and prove the convergence of approximate solutions solving a viscous and regularized non-local equation. It leads to weak, $\mathbf{C}([0,T],\mathbf{L^2}(\mathbb{R}))$, and smooth, $\mathbf{W}^{2,2N}([0,T]\times\mathbb{R})$, solutions for the non-local traffic model.

Key words: Scalar conservation laws; Anisotropic non-local flux; Traffic flow models; Viscous approximation; Sobolev estimates.

1 Introduction

We consider the non-local traffic model introduced in [4, 8] to account for the reaction of drivers to downstream traffic conditions. It consists in the following scalar conservation law, where the traffic velocity depends on a weighted mean of the density:

$$\partial_t\rho + \partial_x(\rho v(\rho * \omega)) = 0, \tag{1.1}$$

where

$$(\rho * \omega)(t,x) = \int_0^{\eta} \rho(t,x+y)\omega(y)\,dy = \int_x^{x+\eta} \rho(t,y)\omega(y-x)\,dy. \tag{1.2}$$

We make the following assumptions for $k = 1,2,3$:

$(\mathbf{A^k_\omega})$ $\omega \in \mathbf{C^k}([0,\eta])$ is non-negative with support in $[0,\eta]$ and is non-increasing on $[0,\eta]$.

$(\mathbf{A^k_v})$ $v \in \mathbf{C^k}(\mathbb{R}^+)$ with v', …, $v^{(k)}$ bounded.

For traffic flow applications, it is reasonable to assume that v is non-increasing, even if monotonicity is not required in this paper. We also recall that a similar model, considering a weighted mean of downstream speeds, has been recently introduced in [7]. More generally, model (1.1) belongs to the class of conservation laws with non-local flux functions, which appear in several applications, see for example [3, 6, 9, 10, 15]. We remark that most of the available well-posedness results concern equations involving smooth convolution kernels [1, 2], and are based on the construction of finite-volume approximations and the use of Kružkov's doubling of variable technique [12]. In particular, these results rely on the concept of entropy solutions. Only recently, alternative proofs based on fixed point theorems have been proposed for specific cases [11, 14], allowing to get rid of the entropy requirement.

In general, solutions to non-local equations may be discontinuous [13], despite the expected regularizing effect of the convolution product. Therefore, given any initial datum $\rho^0 \in \mathbf{L}^\infty(\mathbb{R}) \cap \mathbf{L^1}(\mathbb{R})$, the solutions to the Cauchy problem for (1.1) are usually intended in the following weak form

[*]Laboratoire J. A. Dieudonné, UMR 7351 CNRS, Université Côte d'Azur, LJAD, CNRS, Inria, Université de Nice Sophia-Antipolis, Parc Valrose, 06108 Nice cedex 2, France. E-mail: Florent.Berthelin@unice.fr

[†]Inria Sophia Antipolis - Méditerranée, Université Côte d'Azur, Inria, CNRS, LJAD, 2004 route des Lucioles - BP 93, 06902 Sophia Antipolis Cedex, France. E-mail: paola.goatin@inria.fr

Definition 1. *A function* $\rho \in \left(\mathbf{L}^\infty \cap \mathbf{L}^1\right)(\mathbb{R}^+ \times \mathbb{R})$ *is a solution of* (1.1) *with initial datum* ρ^0 *if*

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} \left(\rho \partial_t \varphi + \rho v(\rho * \omega) \partial_x \varphi\right)(t,x)\, dx dt + \int_{-\infty}^{+\infty} \rho^0(x) \varphi(0,x) dx = 0, \tag{1.3}$$

for all $\varphi \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R}^2)$.

In this paper, we are interested in deriving regularity properties of solutions to (1.1). To this end, we will consider approximate solutions satisfying the viscous and regularized non-local equation

$$\partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon v(\rho_\varepsilon * \omega_\varepsilon)) = \varepsilon \partial^2_{xx} \rho_\varepsilon, \tag{1.4}$$

where, for any $\varepsilon \in]0,1]$, the smooth function ω_ε is an extension of ω with the following regularities:

($\mathbf{A}^\mathbf{k}_{\omega_\varepsilon}$) $\omega_\varepsilon \in \mathbf{C}^\mathbf{k}(\mathbb{R})$ is non-negative with a support in $[-\varepsilon, \eta+\varepsilon]$, is non-decreasing on $[-\varepsilon, x_\varepsilon]$, for some $x_\varepsilon \in]-\varepsilon, 0]$, is non-increasing on $[x_\varepsilon, \eta+\varepsilon]$ and $\omega_\varepsilon = \omega$ on $[0,\eta]$.
We set $W_\varepsilon := \omega_\varepsilon(x_\varepsilon)$ and we assume that $\lim_{\varepsilon\to 0} W_\varepsilon = \omega(0)$. Without loss of generality we can assume

$$W_\varepsilon \leq 2\omega(0). \tag{1.5}$$

($\mathbf{B}^\mathbf{k}_{\omega_\varepsilon}$) $\omega_\varepsilon^{(j)}(-\varepsilon) = \omega_\varepsilon^{(j)}(\eta+\varepsilon) = 0$ for $j = 1, \dots, k$ and $|\omega'_\varepsilon(u)| \leq 2W_\varepsilon/\varepsilon$ on $[-\varepsilon, 0]$ and $|\omega'_\varepsilon(u)| \leq 2\omega(\eta)/\varepsilon$ on $[\eta, \eta+\varepsilon]$.

Remark 1. *Given* ω *satisfying* ($\mathbf{A}^\mathbf{k}_\omega$), *we can construct a function* ω_ε *satisfying* ($\mathbf{A}^\mathbf{k}_{\omega_\varepsilon}$) *and* ($\mathbf{B}^\mathbf{k}_{\omega_\varepsilon}$). *To construct such extensions, for example in the simplest case where the derivatives of* ω *vanish at* 0 *and* η, *we use the function* φ *which is zero for* $x \leq -1$, 1 *for* $x \geq 0$, *non-decreasing and of class* C^∞ *and we define* $\omega_\varepsilon(x) = \omega(0)\varphi(x/\varepsilon)$ *for* $x < 0$ *and* $\omega(0)\varphi((\eta - x)/\varepsilon)$ *for* $x > \eta$.

Notice that a similar approximation was used in [5] to establish a convergence property for the singular limit where the (smooth) convolution kernel is replaced by a Dirac delta, in the viscous case. Here, we will study the properties of smooth solutions ρ_ε of this equation corresponding to a fixed initial datum ρ^0, and then we will recover properties for ρ passing to the limit as $\varepsilon \to 0$.

We have the following result.

Theorem 1. *We assume* ($\mathbf{A}^\mathbf{2}_\omega$)-($\mathbf{A}^\mathbf{3}_\mathbf{v}$). *Let* ρ_ε *be a solution of* (1.4) *with initial datum* ρ^0. *We assume* $\rho^0 \in \mathbf{W}^{1,4}(\mathbb{R}) \cap \mathbf{H}^2(\mathbb{R})$. *Then, for* $T > 0$ *sufficiently small,* ρ_ε *converges in* $\mathbf{L}^\mathbf{2}_\mathbf{loc}([0,T] \times \mathbb{R})$ *to a solution* $\rho \in \mathbf{C}([0,T], \mathbf{L}^\mathbf{2}(\mathbb{R}))$ *to equation* (1.1) *with initial datum* ρ^0. *Furthermore, if* $\rho^0 \in \mathbf{W}^{1,2N}(\mathbb{R})$, $N \in \mathbb{N}^*$, *then* $\rho \in \mathbf{W}^{1,2N}([0,T] \times \mathbb{R})$, *and if* $\rho^0 \in \mathbf{W}^{1,4N}(\mathbb{R}) \cap \mathbf{H}^1(\mathbb{R}) \cap \mathbf{W}^{2,2N}(\mathbb{R})$, *then* $\rho \in \mathbf{W}^{2,2N}([0,T] \times \mathbb{R})$.

In particular, this provides an alternative proof of existence of weak solutions, locally in time. To prove this result, in Section 2 we first establish estimates on the non-local term and we derive $\mathbf{L}^\mathbf{p}(\mathbb{R})$, $p > 1$, estimates for ρ_ε, then we get estimates in $\mathbf{W}^{1,2N}(\mathbb{R})$ for ρ_ε with respect to x. This allows to prove that there exists $T > 0$ such that the sequence ρ_ε is uniformly bounded with respect to ε in $\mathbf{L}^\infty(\mathbb{R})$ on $[0,T]$. Then we prove uniform space estimates in $\mathbf{W}^{2,2N}(\mathbb{R})$ for ρ_ε, which allows to derive estimates on $\partial_t \rho_\varepsilon$. The proof of Theorem 1 is deferred to Section 3.

2 Estimates

Here and in the following sections, we will denote by

$$\|\rho\|_\infty := \|\rho\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R})}$$

and by

$$\big\|\rho(t,\cdot)\big\|_\infty := \big\|\rho(t,\cdot)\big\|_{\mathbf{L}^\infty(\mathbb{R})}.$$

Moreover, notice that we have

$$\begin{aligned}(\rho * \omega_\varepsilon)(t,x) &= \int_{\mathbb{R}} \rho(t,x+y)\omega_\varepsilon(y)\,dy = \int_{\mathbb{R}} \rho(t,y)\omega_\varepsilon(y-x)\,dy \\ &= \int_{-\varepsilon}^{\eta+\varepsilon} \rho(t,x+y)\omega_\varepsilon(y)\,dy = \int_{x-\varepsilon}^{x+\eta+\varepsilon} \rho(t,y)\omega_\varepsilon(y-x)\,dy.\end{aligned}$$

2.1 Estimates of the non-local term

We start by proving the following estimates on the non-local term.

Proposition 1. *1. We assume* $(\mathbf{A}^{\mathbf{1}}_{\omega_\varepsilon})$ *and that* ρ *is a continuous function. For any* $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$*, we have*

$$|\partial_x(\rho * \omega_\varepsilon)(t,x)| \le 2\big\|\rho(t,\cdot)\big\|_\infty W_\varepsilon. \tag{2.1}$$

2. We assume $(\mathbf{A}^{\mathbf{2}}_{\omega_\varepsilon}) - (\mathbf{B}^{\mathbf{1}}_{\omega_\varepsilon})$ *and that* ρ *is a* C^1 *function. For any* $t \ge 0$*,* $p > 1$*, we have*

$$\begin{aligned}\left(\int_{\mathbb{R}} |\partial^2_{xx}(\rho * \omega_\varepsilon)|^p(t,x)\,dx\right)^{1/p} \le\ & \eta^{1/p}\left(\int_0^\eta |\omega''(u)|^{p/(p-1)}\,du\right)^{1-1/p}\left(\int_{\mathbb{R}} \rho^p(t,y)\,dy\right)^{1/p} \\ & + \big(|\omega'(\eta-)| + |\omega'(0+)|\big)\left(\int_{\mathbb{R}} \rho^p(t,x)\,dx\right)^{1/p} \qquad (2.2) \\ & +2\left(\omega(\eta) + W_\varepsilon\right)\left(\int_{\mathbb{R}} |\partial_x\rho(t,x)|^p\,dx\right)^{1/p}.\end{aligned}$$

3. We assume $(\mathbf{A}^{\mathbf{2}}_{\omega}) - (\mathbf{B}^{\mathbf{2}}_{\omega_\varepsilon})$ *that* ρ *is a* C^2 *function. For any* $t \ge 0$*,* $p > 1$*, we have*

$$\begin{aligned}\left(\int_{\mathbb{R}} |\partial^3_{xxx}(\rho * \omega_\varepsilon)|^p(t,x)\,dx\right)^{1/p} \le\ & \eta^{1/p}\left(\int_0^\eta |\omega''(u)|^{p/(p-1)}\,du\right)^{1-1/p}\left(\int_{\mathbb{R}} |\partial_x\rho(t,y)|^p\,dy\right)^{1/p} \\ & + \big(|\omega'(\eta-)| + |\omega'(0+)|\big)\left(\int_{\mathbb{R}} |\partial_x\rho(t,x)|^p\,dx\right)^{1/p} \qquad (2.3) \\ & +2\left(\omega(\eta) + W_\varepsilon\right)\left(\int_{\mathbb{R}} |\partial^2_{xx}\rho(t,x)|^p\,dx\right)^{1/p}.\end{aligned}$$

Proof. 1. From

$$\begin{aligned}\partial_x(\rho * \omega_\varepsilon)(t,x) &= -\int_{x-\varepsilon}^{x+\eta+\varepsilon} \rho(t,y)\omega'_\varepsilon(y-x)\,dy + \rho(t,x+\eta+\varepsilon)\omega_\varepsilon(\eta+\varepsilon) - \rho(t,x-\varepsilon)\omega_\varepsilon(-\varepsilon) \\ &= -\int_{x-\varepsilon}^{x+\eta+\varepsilon} \rho(t,y)\omega'_\varepsilon(y-x)\,dy = -\int_{-\varepsilon}^{\eta+\varepsilon} \rho(t,u+x)\omega'_\varepsilon(u)\,du,\end{aligned}$$

we obtain

$$\begin{aligned}|\partial_x(\rho*\omega_\varepsilon)(t,x)| &\le \|\rho(t,\cdot)\|_\infty \int_{-\varepsilon}^{\eta+\varepsilon} |\omega_\varepsilon'(u)|\,du \\ &\le \|\rho(t,\cdot)\|_\infty \left(\int_{-\varepsilon}^{x_\varepsilon} \omega_\varepsilon'(u)\,du - \int_{x_\varepsilon}^{\eta+\varepsilon} \omega_\varepsilon'(u)\,du\right) \\ &\le 2\|\rho(t,\cdot)\|_\infty W_\varepsilon.\end{aligned}$$

2. From

$$\begin{aligned}\partial_{xx}^2(\rho*\omega_\varepsilon)(t,x) &= \int_{x-\varepsilon}^{x+\eta+\varepsilon} \rho(t,y)\omega_\varepsilon''(y-x)\,dy - \rho(t,x+\eta+\varepsilon)\omega_\varepsilon'(\eta+\varepsilon) + \rho(t,x-\varepsilon)\omega_\varepsilon'(-\varepsilon) \\ &= \int_{x-\varepsilon}^{x+\eta+\varepsilon} \rho(t,y)\omega_\varepsilon''(y-x)\,dy = \int_{-\varepsilon}^{\eta+\varepsilon} \rho(t,x+u)\omega_\varepsilon''(u)\,du \\ &= \int_{-\varepsilon}^{0} \rho(t,x+u)\omega_\varepsilon''(u)\,du + \int_0^\eta \rho(t,x+u)\omega_\varepsilon''(u)\,du + \int_\eta^{\eta+\varepsilon} \rho(t,x+u)\omega_\varepsilon''(u)\,du \\ &= \rho(t,x)\omega_\varepsilon'(0) - \rho(t,x-\varepsilon)\omega_\varepsilon'(-\varepsilon) - \int_{-\varepsilon}^0 \partial_x\rho(t,x+u)\omega_\varepsilon'(u)\,du \\ &\quad + \int_0^\eta \rho(t,x+u)\omega_\varepsilon''(u)\,du \\ &\quad + \rho(t,x+\eta+\varepsilon)\omega_\varepsilon'(\eta+\varepsilon) - \rho(t,x+\eta)\omega_\varepsilon'(\eta) - \int_\eta^{\eta+\varepsilon} \partial_x\rho(t,x+u)\omega_\varepsilon'(u)\,du \\ &= \rho(t,x)\omega'(0+) - \rho(t,x+\eta)\omega'(\eta-) + \int_0^\eta \rho(t,x+u)\omega_\varepsilon''(u)\,du \\ &\quad - \int_{-\varepsilon}^0 \partial_x\rho(t,x+u)\omega_\varepsilon'(u)\,du - \int_\eta^{\eta+\varepsilon} \partial_x\rho(t,x+u)\omega_\varepsilon'(u)\,du,\end{aligned}$$

we have

$$\begin{aligned}&\left(\int_{\mathbb{R}} |\partial_{xx}^2(\rho*\omega_\varepsilon)|^p(t,x)\,dx\right)^{1/p} \\ \le& \left(\int_{\mathbb{R}} \left|\int_{-\varepsilon}^0 \partial_x\rho(t,x+u)\omega_\varepsilon'(u)\,du\right|^p dx\right)^{1/p} \\ &+ \left(\int_{\mathbb{R}} \left|\int_\eta^{\eta+\varepsilon} \partial_x\rho(t,x+u)\omega_\varepsilon'(u)\,du\right|^p dx\right)^{1/p} \\ &+ \left(\int_{\mathbb{R}} \left|\int_0^\eta \rho(t,x+u)\omega_\varepsilon''(u)\,du\right|^p dx\right)^{1/p} \\ &+ \left(\int_{\mathbb{R}} \rho(t,x)^p|\omega'(0+)|^p\,dx\right)^{1/p} + \left(\int_{\mathbb{R}} \rho(t,x+\eta)^p|\omega'(\eta-)|^p\,dx\right)^{1/p}.\end{aligned}$$

Notice that

$$\left(\int_{\mathbb{R}} \left|\int_{-\varepsilon}^0 \partial_x\rho(t,x+u)\omega_\varepsilon'(u)\,du\right|^p dx\right)^{1/p}$$

$$
\begin{aligned}
&\leq \left(\int_{\mathbb{R}} \left(\int_{-\varepsilon}^{0} |\partial_x \rho(t,x+u)|^p \, du\right) \left(\int_{-\varepsilon}^{0} |\omega_\varepsilon'(u)|^q \, dy\right)^{p/q} dx\right)^{1/p} \\
&\leq \left(\int_{-\varepsilon}^{0} |\omega_\varepsilon'(u)|^{p/(p-1)} \, du\right)^{1-1/p} \left(\int_{\mathbb{R}} \left(\int_{-\varepsilon}^{\eta+\varepsilon} |\partial_x \rho(t,x+u)|^p \, du\right) dx\right)^{1/p} \\
&\leq \left(\int_{-\varepsilon}^{0} \left(2\frac{W_\varepsilon}{\varepsilon}\right)^{p/(p-1)} du\right)^{1-1/p} \left(\int_{\mathbb{R}} \int_{x-\varepsilon}^{x} |\partial_x \rho(t,y)|^p \, dy \, dx\right)^{1/p} \\
&\leq 2\frac{W_\varepsilon}{\varepsilon} \varepsilon^{1-1/p} \left(\int_{\mathbb{R}} \int_{y}^{y+\varepsilon} dx \, |\partial_x \rho(t,y)|^p \, dy\right)^{1/p} \\
&\leq 2\frac{W_\varepsilon}{\varepsilon} \varepsilon^{1-1/p} \varepsilon^{1/p} \left(\int_{\mathbb{R}} |\partial_x \rho(t,y)|^p \, dy\right)^{1/p} = 2W_\varepsilon \left(\int_{\mathbb{R}} |\partial_x \rho(t,y)|^p \, dy\right)^{1/p}
\end{aligned}
$$

using Hölder's inequality with $q = p/(p-1)$ the conjugated exponent of p. Similarly

$$
\left(\int_{\mathbb{R}} \left|\int_{\eta}^{\eta+\varepsilon} \partial_x \rho(t,x+u) \omega_\varepsilon'(u) \, du\right|^p dx\right)^{1/p} \leq 2\omega(\eta) \left(\int_{\mathbb{R}} |\partial_x \rho(t,y)|^p \, dy\right)^{1/p}.
$$

Then we get

$$
\begin{aligned}
&\left(\int_{\mathbb{R}} |\partial_{xx}^2 (\rho * \omega_\varepsilon)|^p (t,x) \, dx\right)^{1/p} \\
\leq &2\left(\omega(\eta) + W_\varepsilon\right) \left(\int_{\mathbb{R}} |\partial_x \rho(t,y)|^p \, dy\right)^{1/p} \\
&+ \left(\int_{\mathbb{R}} \left(\int_{x}^{x+\eta} \rho^p(t,y) \, dy\right) \left(\int_{0}^{\eta} |\omega_\varepsilon''(u)|^q \, du\right)^{p/q} dx\right)^{1/p} \\
&+ \left(|\omega'(\eta+)| + |\omega'(0-)|\right) \left(\int_{\mathbb{R}} \rho^p(t,x) \, dx\right)^{1/p}.
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
&\int_{\mathbb{R}} \left(\int_{x}^{x+\eta} \rho^p(t,y) \, dy\right) \left(\int_{0}^{\eta} |\omega''(u)|^q \, dy\right)^{p-1} dx \\
&\leq \left(\int_{0}^{\eta} |\omega''(u)|^{p/(p-1)} \, du\right)^{p-1} \int_{\mathbb{R}} \int_{y-\eta}^{y} \rho^p(t,y) \, dx \, dy \\
&\leq \eta \left(\int_{0}^{\varepsilon} |\omega''(u)|^{p/(p-1)} \, du\right)^{p-1} \int_{\mathbb{R}} \rho^p(t,y) \, dy,
\end{aligned}
$$

then we get the announced formula.

3. Remark that, since $\omega_\varepsilon''(-\varepsilon) = \omega_\varepsilon''(\eta+\varepsilon) = 0$, we have

$$
\partial_{xxx}^3 (\rho * \omega_\varepsilon)(t,x) = -\int_{-\varepsilon}^{\eta+\varepsilon} \rho(t,x+u) \omega_\varepsilon^{(3)}(u) \, du = \int_{-\varepsilon}^{\eta+\varepsilon} \partial_x \rho(t,x+u) \omega_\varepsilon''(u) \, du
$$

$$= \partial^2_{xx}(\partial_x \rho * \omega_\varepsilon)(t,x),$$

then applying 2., we get

$$\begin{aligned}\left(\int_{\mathbb{R}} |\partial^3_{xxx}(\rho*\omega)|^p(t,x)\,dx\right)^{1/p} &\le \eta^{1/p}\left(\int_0^\eta |\omega''(u)|^{p/(p-1)}\,du\right)^{1-1/p}\left(\int_{\mathbb{R}} |\partial_x\rho(t,y)|^p\,dy\right)^{1/p}\\ &\quad+\left(|\omega'(\eta-)|+|\omega'(0+)|\right)\left(\int_{\mathbb{R}}|\partial_x\rho(t,x)|^p\,dx\right)^{1/p}\\ &\quad+2\left(\omega(\eta)+W_\varepsilon\right)\left(\int_{\mathbb{R}}|\partial^2_{xx}\rho(t,x)|^p\,dx\right)^{1/p}.\end{aligned}$$

□

2.2 $\mathbf{L^p}$ estimates for the viscous case

We turn now to estimates on solutions solving the viscous and regularized non-local equation. First, we deal with $\mathbf{L^p}$ estimates.

Proposition 2. *We assume* $(\mathbf{A^1_\omega})$-$(\mathbf{A^1_v})$. *Let* ρ_ε *be the solution of (1.4) with initial datum* $\rho^0 \in \mathbf{L^p}(\mathbb{R})$. *If* $\rho_\varepsilon \in \mathbf{L}^\infty([0,T]\times\mathbb{R})$ *for some* $T>0$, *then*

$$\rho_\varepsilon \in \mathbf{L}^\infty([0,T],\mathbf{L^p}(\mathbb{R})) \cap \mathbf{L^p}([0,T]\times\mathbb{R}).$$

Proof. The equation (1.4) can be rewritten as

$$\partial_t\rho_\varepsilon + v(\rho_\varepsilon*\omega_\varepsilon)\partial_x\rho_\varepsilon + \rho_\varepsilon v'(\rho_\varepsilon*\omega_\varepsilon)\partial_x(\rho_\varepsilon*\omega_\varepsilon) = \varepsilon\partial^2_{xx}\rho_\varepsilon. \tag{2.4}$$

Multiplying (2.4) by ρ_ε^{p-1}, then integrating with respect to x, we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}}\rho_\varepsilon^p(t,x)\,dx &= -\int_{\mathbb{R}}\rho_\varepsilon^{p-1}(t,x)v((\rho_\varepsilon*\omega_\varepsilon)(t,x))\partial_x\rho_\varepsilon(t,x)\,dx\\ &\quad-\int_{\mathbb{R}}\rho_\varepsilon^p(t,x)v'((\rho_\varepsilon*\omega_\varepsilon)(t,x))\partial_x(\rho_\varepsilon*\omega_\varepsilon)(t,x)\,dx\\ &\quad+\varepsilon\int_{\mathbb{R}}\rho_\varepsilon^{p-1}(t,x)\partial^2_{xx}\rho_\varepsilon(t,x)\,dx.\end{aligned}$$

We observe that

$$\begin{aligned}\int_{\mathbb{R}}\rho_\varepsilon^{p-1}\,v(\rho_\varepsilon*\omega_\varepsilon)\,\partial_x\rho_\varepsilon\,dx &= \int_{\mathbb{R}}\partial_x\left(\frac{\rho_\varepsilon^p}{p}\right)v(\rho_\varepsilon*\omega_\varepsilon)\,dx\\ &= -\int_{\mathbb{R}}\frac{\rho_\varepsilon^p}{p}\,\partial_x(v(\rho_\varepsilon*\omega_\varepsilon))\,dx\\ &= -\int_{\mathbb{R}}\frac{\rho_\varepsilon^p}{p}\,v'(\rho_\varepsilon*\omega_\varepsilon)\partial_x(\rho_\varepsilon*\omega_\varepsilon)\,dx\end{aligned}$$

and

$$\int_{\mathbb{R}}\rho_\varepsilon^{p-1}\partial^2_{xx}\rho_\varepsilon\,dx = -(p-1)\int_{\mathbb{R}}\rho_\varepsilon^{p-2}\left(\partial_x\rho_\varepsilon\right)^2\,dx \le 0,$$

therefore

$$\frac{d}{dt}\int_{\mathbb{R}} \rho_\varepsilon^p(t,x)\,dx \le (1-p)\int_{\mathbb{R}} \rho_\varepsilon^p(t,x) v'((\rho_\varepsilon * \omega_\varepsilon)(t,x))\partial_x(\rho_\varepsilon * \omega_\varepsilon)(t,x)\,dx. \tag{2.5}$$

We use (2.1) to control the right hand side of (2.5) and we get

$$\frac{d}{dt}\int_{\mathbb{R}} \rho_\varepsilon^p(t,x)\,dx \le C_1^{\varepsilon,p}\int_{\mathbb{R}} \rho_\varepsilon^p(t,x)\,dx, \tag{2.6}$$

which implies

$$\int_{\mathbb{R}} \rho_\varepsilon^p(t,x)\,dx \le e^{C_1^{\varepsilon,p}t}\int_{\mathbb{R}} \rho_\varepsilon^p(0,x)\,dx, \tag{2.7}$$

with

$$C_1^{\varepsilon,p} = 2(p-1)\|\rho_\varepsilon\|_\infty \, W_\varepsilon \,\|v'\|_\infty.$$

It gives

$$\sup_{t\in[0,T]}\int_{\mathbb{R}} \rho_\varepsilon^p(t,x)\,dx \le e^{C_1^{\varepsilon,p}T}\int_{\mathbb{R}} \rho_\varepsilon^p(0,x)\,dx. \tag{2.8}$$

By integration of (2.7) with respect to $t\in[0,T]$, we get

$$\int_0^T\int_{\mathbb{R}} \rho_\varepsilon^p(t,x)\,dx\,dt \le \frac{1}{C_1^{\varepsilon,p}}\left(e^{C_1^{\varepsilon,p}T}-1\right)\int_{\mathbb{R}} \rho_\varepsilon^p(0,x)\,dx. \tag{2.9}$$

□

2.3 $\mathbf{W}^{1,p}$ estimates for $p = 2N$ in the viscous case

We turn now to Sobolev estimates. Let $N\in\mathbb{N}^*$ and set $p=2N$.

Proposition 3. *We assume* $(\mathbf{A_\omega^2})$-$(\mathbf{A_v^2})$. *Let ρ_ε be the solution of (1.4) with initial datum $\rho^0 \in \mathbf{W}^{1,2N}(\mathbb{R})$. If $\rho_\varepsilon\in\mathbf{L}^\infty([0,T]\times\mathbb{R})$ for some $T>0$, then*

$$\rho_\varepsilon \in \mathbf{L}^\infty\left([0,T],\mathbf{W}^{1,2N}(\mathbb{R})\right) \quad \text{and} \quad \rho_\varepsilon, \partial_x\rho_\varepsilon \in \mathbf{L}^\infty\left([0,T],\mathbf{L^{2N}}(\mathbb{R})\right)\cap \mathbf{L^{2N}}([0,T]\times\mathbb{R}).$$

Proof. We differentiate (2.4) with respect to x, it gives

$$\begin{aligned}
&\partial_t\partial_x\rho_\varepsilon + 2v'(\rho_\varepsilon*\omega_\varepsilon)\,\partial_x(\rho_\varepsilon*\omega_\varepsilon)\,\partial_x\rho_\varepsilon + v(\rho_\varepsilon*\omega_\varepsilon)\,\partial^2_{xx}\rho_\varepsilon\\
&+\rho_\varepsilon v''(\rho_\varepsilon*\omega_\varepsilon)\,(\partial_x(\rho_\varepsilon*\omega_\varepsilon))^2 + \rho_\varepsilon v'(\rho_\varepsilon*\omega_\varepsilon)\partial^2_{xx}(\rho_\varepsilon*\omega_\varepsilon) \quad = \quad \varepsilon\partial^3_{xxx}\rho_\varepsilon.
\end{aligned} \tag{2.10}$$

Multiplying this relation by $(\partial_x\rho_\varepsilon)^{p-1}$, then integrating with respect to x, we have

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}}(\partial_x\rho_\varepsilon)^p\,dx + 2\int_{\mathbb{R}} v'(\rho_\varepsilon*\omega_\varepsilon)\,\partial_x(\rho_\varepsilon*\omega_\varepsilon)\,(\partial_x\rho_\varepsilon)^p\,dx + \int_{\mathbb{R}} v(\rho_\varepsilon*\omega_\varepsilon)\,\partial^2_{xx}\rho_\varepsilon\,(\partial_x\rho_\varepsilon)^{p-1}\,dx\\
&+\int_{\mathbb{R}}\rho_\varepsilon\, v''(\rho_\varepsilon*\omega_\varepsilon)\,(\partial_x(\rho_\varepsilon*\omega_\varepsilon))^2\,(\partial_x\rho_\varepsilon)^{p-1}\,dx + \int_{\mathbb{R}}\rho_\varepsilon\, v'(\rho_\varepsilon*\omega_\varepsilon)\,\partial^2_{xx}(\rho_\varepsilon*\omega_\varepsilon)\,(\partial_x\rho_\varepsilon)^{p-1}\,dx\\
= \quad &\varepsilon\int_{\mathbb{R}}(\partial_x\rho_\varepsilon)^{p-1}\partial^3_{xxx}\rho_\varepsilon\,dx.
\end{aligned}$$

Notice that

$$\int_{\mathbb{R}} v(\rho_\varepsilon*\omega_\varepsilon)\,\partial^2_{xx}\rho_\varepsilon\,(\partial_x\rho_\varepsilon)^{p-1}\,dx \quad = \quad \frac{1}{p}\int_{\mathbb{R}} v(\rho_\varepsilon*\omega_\varepsilon)\,\partial_x\left((\partial_x\rho_\varepsilon)^p\right)\,dx$$

$$= \; -\frac{1}{p}\int_{\mathbb{R}} v'(\rho_\varepsilon * \omega_\varepsilon)\,\partial_x(\rho_\varepsilon * \omega_\varepsilon)\,(\partial_x\rho_\varepsilon)^p\,dx$$

and, since p is even,

$$\int_{\mathbb{R}} (\partial_x\rho_\varepsilon)^{p-1}\partial^3_{xxx}\rho_\varepsilon\,dx \;=\; -(p-1)\int_{\mathbb{R}} (\partial^2_{xx}\rho_\varepsilon)^2(\partial_x\rho_\varepsilon)^{p-2}\,dx \le 0,$$

thus

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}(\partial_x\rho_\varepsilon)^p\,dx \;&\le\; (1-2p)\int_{\mathbb{R}} v'(\rho_\varepsilon * \omega_\varepsilon)\,\partial_x(\rho_\varepsilon * \omega_\varepsilon)\,(\partial_x\rho_\varepsilon)^p\,dx\\ &\quad -p\int_{\mathbb{R}}\rho_\varepsilon\, v''(\rho_\varepsilon * \omega_\varepsilon)\,(\partial_x(\rho_\varepsilon * \omega_\varepsilon))^2\,(\partial_x\rho_\varepsilon)^{p-1}\,dx\\ &\quad -p\int_{\mathbb{R}}\rho_\varepsilon\, v'(\rho_\varepsilon * \omega_\varepsilon)\,\partial^2_{xx}(\rho_\varepsilon * \omega_\varepsilon)\,(\partial_x\rho_\varepsilon)^{p-1}\,dx\\ &=: I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon.\end{aligned}$$

We estimate now each of these terms.

- By (2.1) we get

$$\left|I_1^\varepsilon\right| = \left|(1-2p)\int_{\mathbb{R}} v'(\rho_\varepsilon * \omega_\varepsilon)\,\partial_x(\rho_\varepsilon * \omega_\varepsilon)\,(\partial_x\rho_\varepsilon)^p\,dx\right| \le 2(2p-1)\|v'\|_\infty\,\|\rho_\varepsilon\|_\infty\,W_\varepsilon\int_{\mathbb{R}}|\partial_x\rho_\varepsilon|^p\,dx.$$

- Again by (2.1) we get

$$\begin{aligned}\left|I_2^\varepsilon\right| &= \left|p\int_{\mathbb{R}}\rho_\varepsilon\, v''(\rho_\varepsilon * \omega_\varepsilon)\,(\partial_x(\rho_\varepsilon * \omega_\varepsilon))^2\,(\partial_x\rho_\varepsilon)^{p-1}\,dx\right|\\ &\le p\|v''\|_\infty\,(2\|\rho_\varepsilon\|_\infty\,W_\varepsilon)^2\int_{\mathbb{R}}\rho_\varepsilon\,|\partial_x\rho_\varepsilon|^{p-1}\,dx\\ &\le 4\|v''\|_\infty\,\|\rho_\varepsilon\|^2_\infty\,W_\varepsilon^2\left(\int_{\mathbb{R}}\rho_\varepsilon^p\,dx + (p-1)\int_{\mathbb{R}}|\partial_x\rho_\varepsilon|^p\,dx\right),\end{aligned}$$

  where we have used Young's inequality $uv \le \frac{1}{p}u^p + \frac{1}{q}v^q$ with $q = p/(p-1)$.

- Similarly,

$$\begin{aligned}\left|I_3^\varepsilon\right| &= \left|p\int_{\mathbb{R}}\rho_\varepsilon\, v'(\rho_\varepsilon * \omega_\varepsilon)\,\partial^2_{xx}(\rho_\varepsilon * \omega_\varepsilon)\,(\partial_x\rho_\varepsilon)^{p-1}\,dx\right|\\ &\le p\|v'\|_\infty\,\|\rho_\varepsilon\|_\infty\int_{\mathbb{R}}|\partial^2_{xx}(\rho_\varepsilon * \omega_\varepsilon)\,|\partial_x\rho_\varepsilon|^{p-1}|\,dx\\ &\le \|v'\|_\infty\,\|\rho_\varepsilon\|_\infty\left(\int_{\mathbb{R}}|\partial^2_{xx}(\rho_\varepsilon * \omega_\varepsilon)|^p\,dx + (p-1)\int_{\mathbb{R}}|\partial_x\rho_\varepsilon|^p\,dx\right).\end{aligned}$$

  We now observe that

$$(u+v+w)^p \le 3^p(u^p+v^p+w^p) \qquad \text{for any } u,v,w>0 \text{ and } p>0. \tag{2.11}$$

  Indeed, from the binomial expansion we get

$$(u+v+w)^p \;=\; (u+(v+w))^p$$

$$= \sum_{k=0}^{p} \binom{p}{k} u^k (v+w)^{p-k}$$
$$= \sum_{k=0}^{p} \binom{p}{k} u^k \sum_{l=0}^{p-k} \binom{p-k}{l} v^l w^{p-k-l}.$$

Observing that $u^k v^l w^{p-k-l} \leq u^p + v^p + w^p$ and that $\sum_{l=0}^{p-k} \binom{p-k}{l} = (1+1)^{p-k}$ and $\sum_{k=0}^{p} \binom{p}{k} 1^k 2^{p-k} = 3^p$, we get the result.

Estimate (2.2) of Proposition 1 and inequality (2.11) give

$$\begin{aligned}\int_{\mathbb{R}} |\partial^2_{xx}(\rho_\varepsilon * \omega)|^p \, dx \quad &\leq \quad 3^p \eta \left(\int_0^\eta |\omega''(u)|^{p/(p-1)} \, du \right)^{p-1} \int_{\mathbb{R}} \rho_\varepsilon^p(t,x) \, dx \\ &\quad + 3^p \left(|\omega'(\eta-)| + |\omega'(0+)| \right)^p \int_{\mathbb{R}} \rho_\varepsilon^p(t,x) \, dx \\ &\quad + 6^p \left(\omega(\eta) + W_\varepsilon \right)^p \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^p \, dx, \end{aligned} \tag{2.12}$$

thus

$$\left| p \int_{\mathbb{R}} \rho_\varepsilon \, v'(\rho * \omega_\varepsilon) \, \partial^2_{xx}(\rho_\varepsilon * \omega_\varepsilon) \, (\partial_x \rho_\varepsilon)^{p-1} \, dx \right| \leq C_2^{\varepsilon,p} \int_{\mathbb{R}} \rho_\varepsilon^p \, dx + C_3^{\varepsilon,p} \int_{\mathbb{R}} |\partial_x \rho_\varepsilon|^p \, dx,$$

with

$$C_2^{\varepsilon,p} = \|v'\|_\infty \, \|\rho_\varepsilon\|_\infty 3^p \left[\eta \left(\int_0^\eta |\omega''(u)|^{p/(p-1)} \, du \right)^{p-1} + \left(|\omega'(\eta-)| + |\omega'(0+)| \right)^p \right]$$

and

$$C_3^{\varepsilon,p} = \|v'\|_\infty \, \|\rho_\varepsilon\|_\infty \left[p - 1 + 6^p \left(\omega(\eta) + W_\varepsilon \right)^p \right].$$

These bounds give finally the estimate

$$\frac{d}{dt} \int_{\mathbb{R}} (\partial_x \rho_\varepsilon(t,x))^p \, dx \quad \leq \quad C_4^{\varepsilon,p} \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^p \, dx + C_5^{\varepsilon,p} \int_{\mathbb{R}} \rho_\varepsilon^p(t,x) \, dx.$$

with

$$C_4^{\varepsilon,p} = 2(2p-1) \|v'\|_\infty \, \|\rho_\varepsilon\|_\infty \, W_\varepsilon + 4(p-1) \|v''\|_\infty \, \|\rho_\varepsilon\|_\infty^2 \, W_\varepsilon^2 + C_3^\varepsilon$$

and

$$C_5^{\varepsilon,p} = 4 \|v''\|_\infty \, \|\rho_\varepsilon\|_\infty^2 \, W_\varepsilon^2 + C_2^\varepsilon.$$

With (2.6), we get

$$\begin{aligned} &\frac{d}{dt} \left(\int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^{2N} \, dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t,x) \, dx \right) \\ &\leq C_6^{\varepsilon,p} \left(\int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^{2N} \, dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t,x) \, dx \right), \end{aligned} \tag{2.13}$$

with $C_6^{\varepsilon,p} = \max(C_4^{\varepsilon,p}, C_5^{\varepsilon,p} + C_1^{\varepsilon,p})$, which implies

$$\int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^{2N} \, dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t,x) \, dx$$

$$\leq \quad e^{C_6^{\varepsilon,p}t}\left(\int_{\mathbb{R}} |\partial_x \rho_\varepsilon(0,x)|^{2N}\,dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(0,x)\,dx\right). \tag{2.14}$$

Then

$$\begin{aligned}
&\sup_{t\in[0,T[} \left(\int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^{2N}\,dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t,x)\,dx\right) \\
\leq \quad & e^{C_6^{\varepsilon,p}T}\left(\int_{\mathbb{R}} |\partial_x \rho_\varepsilon(0,x)|^{2N}\,dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(0,x)\,dx\right).
\end{aligned} \tag{2.15}$$

Integrating (2.14) with respect to t on $[0,T]$, we get

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^{2N}\,dx\,dt + \int_0^T \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t,x)\,dx\,dt \\
\leq \quad & \frac{1}{C_6^{\varepsilon,p}}\left(e^{C_6^{\varepsilon,p}T} - 1\right)\left(\int_{\mathbb{R}} |\partial_x \rho_\varepsilon(0,x)|^{2N}\,dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(0,x)\,dx\right).
\end{aligned} \tag{2.16}$$

□

2.4 $\mathbf{L}^\infty$ bound on an interval $[0,T]$

With the previous estimates, we are now able to prove an $\mathbf{L}^\infty$ bounds for the sequence $\{\rho_\varepsilon\}_\varepsilon$ on an interval $[0,T]$.

Proposition 4. *We assume* $(\mathbf{A^2_\omega})$-$(\mathbf{A^2_v})$. *Let* ρ_ε *be the solution of (1.4) with initial datum* $\rho^0 \in \mathbf{H}^1$. *Then there exists a constant* $\bar{T} > 0$ *such that* $\rho_\varepsilon \in \mathbf{L}^\infty\left([0,T]\times\mathbb{R}\right)$ *for any* $\varepsilon > 0$, $T < \bar{T}$. *Furthermore*

$$\rho_\varepsilon \in \mathbf{L}^\infty\left([0,T], \mathbf{W}^{1,2N}(\mathbb{R})\right) \qquad \text{and} \qquad \rho_\varepsilon, \partial_x\rho_\varepsilon \in \mathbf{L}^\infty\left([0,T], \mathbf{L^{2N}}(\mathbb{R})\right) \cap \mathbf{L^{2N}}([0,T]\times\mathbb{R}) \tag{2.17}$$

and this sequence is uniformly bounded in these spaces with respect to ε.

Proof. Let ρ_ε be a smooth solution of (1.4) with the same initial datum $\rho^0 \in \mathbf{H}^1$. The relation (2.13) for $N = 1$ gives

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^2\,dx + \int_{\mathbb{R}} \rho_\varepsilon^2(t,x)\,dx\right) \\
&\leq C \max\left\{1, \|\rho_\varepsilon(t,\cdot)\|_\infty^2\right\}\left(\int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^2\,dx + \int_{\mathbb{R}} \rho_\varepsilon^2(t,x)\,dx\right),
\end{aligned}$$

for some constant C that does not depend on ε (since W_ε is uniformly bounded). If no uniform $\mathbf{L}^\infty$-bound on ρ_ε is available, we can use the Sobolev injection of $\mathbf{H}^1(\mathbb{R})$ in $\mathbf{L}^\infty(\mathbb{R})$ and get

$$\frac{d}{dt}\|\rho_\varepsilon(t,\cdot)\|_{\mathbf{H}^1}^2 \leq C\|\rho_\varepsilon(t,\cdot)\|_{\mathbf{H}^1}^2 + C\|\rho_\varepsilon(t,\cdot)\|_{\mathbf{H}^1}^4,$$

eventually updating the constant C. We set $u_\varepsilon(t) = \|\rho_\varepsilon(t,\cdot)\|_{\mathbf{H}^1}^2$, then $u_\varepsilon' \leq C(u_\varepsilon + u_\varepsilon^2)$, which leads to

$$\frac{u_\varepsilon'}{u_\varepsilon} - \frac{u_\varepsilon'}{1+u_\varepsilon} \leq C.$$

We obtain

$$u_\varepsilon(t) \leq \frac{C_0 e^{Ct}}{1 - C_0 e^{Ct}}, \qquad \text{for any} \qquad 0 \leq t < -\frac{\ln C_0}{C},$$

with $C_0 = \dfrac{u_0}{1+u_0} < 1$, $u_0 = \|\rho^0\|^2_{\mathbf{H}^1}$. Notice that the initial datum is the same for all the sequence and then u_0 and C_0 do not depend on ε. Setting $T < \bar{T} := -\dfrac{\ln C_0}{C}$, we have

$$\|\rho_\varepsilon(t,\cdot)\|^2_{\mathbf{H}^1} \le C\|\rho^0\|^2_{\mathbf{H}^1}, \qquad \text{for any} \quad 0 \le t \le T, \quad \varepsilon > 0.$$

Therefore, by Sobolev injection, $\rho_\varepsilon \in \mathbf{L}^\infty([0,T]\times\mathbb{R})$. Using the estimates of Propositions 2 and 3, we get (2.17) with bounds independents of ε. □

2.5 $\mathbf{W}^{2,p}$ estimate for $p = 2N$

To pass to the limit, we need also estimates in $\mathbf{W}^{2,p}$, which will provide, in the next section, with the help of the equation, the necessary regularity in time. As in Section 2.3, let $N \in \mathbb{N}^*$ and set $p = 2N$.

Proposition 5. *We assume* $(\mathbf{A^2_\omega})$-$(\mathbf{A^3_v})$. *Let* ρ_ε *be the solution of (1.4) with initial datum* $\rho^0 \in \mathbf{W}^{1,4N}(\mathbb{R}) \cap \mathbf{H}^1(\mathbb{R}) \cap \mathbf{W}^{2,2N}(\mathbb{R})$. *Let* $T > 0$ *as in Proposition 4. Then*

$$\rho_\varepsilon \in \mathbf{L}^\infty\Big([0,T], \mathbf{W}^{2,2N}(\mathbb{R})\Big) \qquad \text{and} \qquad \rho_\varepsilon, \partial_x\rho_\varepsilon, \partial^2_{xx}\rho_\varepsilon \in \mathbf{L}^\infty\Big([0,T], \mathbf{L^{2N}}(\mathbb{R})\Big) \cap \mathbf{L^{2N}}([0,T]\times\mathbb{R})$$

and this sequence is bounded in these spaces with respect to ε.

Proof. We differentiate (2.10) with respect to x, which gives

$$\begin{aligned} \partial_t\partial^2_{xx}\rho_\varepsilon + 3v''(\rho_\varepsilon * \omega_\varepsilon)\,(\partial_x(\rho_\varepsilon * \omega_\varepsilon))^2\,\partial_x\rho_\varepsilon + 3v'(\rho_\varepsilon * \omega_\varepsilon)\,\partial^2_{xx}(\rho_\varepsilon * \omega_\varepsilon)\,\partial_x\rho_\varepsilon \\ +3v'(\rho_\varepsilon * \omega_\varepsilon)\,\partial_x(\rho_\varepsilon * \omega_\varepsilon)\,\partial^2_{xx}\rho_\varepsilon + v(\rho_\varepsilon * \omega_\varepsilon)\,\partial^3_{xxx}\rho_\varepsilon + \rho_\varepsilon v^{(3)}(\rho_\varepsilon * \omega_\varepsilon)\,(\partial_x(\rho_\varepsilon * \omega_\varepsilon))^3 \\ +3\rho_\varepsilon v''(\rho_\varepsilon * \omega_\varepsilon)\,\partial_x(\rho_\varepsilon * \omega_\varepsilon)\,\partial^2_{xx}(\rho_\varepsilon * \omega_\varepsilon) + \rho_\varepsilon v'(\rho_\varepsilon * \omega_\varepsilon)\partial^3_{xxx}(\rho_\varepsilon * \omega_\varepsilon) \;=\; \varepsilon\partial^4_{xxxx}\rho_\varepsilon. \end{aligned} \tag{2.18}$$

Multiplying this relation by $(\partial^2_{xx}\rho_\varepsilon)^{p-1}$, then integrating with respect to x, we obtain

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}}(\partial^2_{xx}\rho_\varepsilon)^p\,dx + 3\int_{\mathbb{R}} v''(\rho_\varepsilon * \omega_\varepsilon)\,(\partial_x(\rho_\varepsilon * \omega_\varepsilon))^2\,\partial_x\rho_\varepsilon\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx \\ &+3\int_{\mathbb{R}} v'(\rho_\varepsilon * \omega_\varepsilon)\,\partial^2_{xx}(\rho_\varepsilon * \omega_\varepsilon)\,\partial_x\rho_\varepsilon\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx + 3\int_{\mathbb{R}} v'(\rho_\varepsilon * \omega_\varepsilon)\,\partial_x(\rho_\varepsilon * \omega_\varepsilon)\,(\partial^2_{xx}\rho_\varepsilon)^p\,dx \\ &+\int_{\mathbb{R}} v(\rho_\varepsilon * \omega_\varepsilon)\,\partial^3_{xxx}\rho_\varepsilon\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx + \int_{\mathbb{R}} \rho_\varepsilon v^{(3)}(\rho_\varepsilon * \omega_\varepsilon)\,(\partial_x(\rho_\varepsilon * \omega_\varepsilon))^3\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx \\ &+3\int_{\mathbb{R}} \rho_\varepsilon v''(\rho_\varepsilon * \omega_\varepsilon)\,\partial_x(\rho_\varepsilon * \omega_\varepsilon)\,\partial^2_{xx}(\rho_\varepsilon * \omega_\varepsilon)\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx \\ &+\int_{\mathbb{R}} \rho_\varepsilon v'(\rho_\varepsilon * \omega_\varepsilon)\partial^3_{xxx}(\rho_\varepsilon * \omega_\varepsilon)\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx \\ =\;& \varepsilon\int_{\mathbb{R}} \partial^4_{xxxx}\rho_\varepsilon\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx. \end{aligned}$$

Now

$$\begin{aligned} \int_{\mathbb{R}} v(\rho_\varepsilon * \omega_\varepsilon)\,\partial^3_{xxx}\rho_\varepsilon\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx &= \frac{1}{p}\int_{\mathbb{R}} v(\rho_\varepsilon * \omega_\varepsilon)\,\partial_x\Big((\partial^2_{xx}\rho_\varepsilon)^p\Big)\,dx \\ &= -\frac{1}{p}\int_{\mathbb{R}} v'(\rho_\varepsilon * \omega_\varepsilon)\,\partial_x(\rho_\varepsilon * \omega_\varepsilon)\,(\partial^2_{xx}\rho_\varepsilon)^p\,dx, \end{aligned}$$

therefore

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}}(\partial^2_{xx}\rho_\varepsilon)^p\,dx &= -3p\int_{\mathbb{R}} v''(\rho_\varepsilon*\omega_\varepsilon)\,(\partial_x(\rho_\varepsilon*\omega_\varepsilon))^2\,\partial_x\rho_\varepsilon\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx\\
&\quad -3p\int_{\mathbb{R}} v'(\rho_\varepsilon*\omega_\varepsilon)\,\partial^2_{xx}(\rho_\varepsilon*\omega_\varepsilon)\,\partial_x\rho_\varepsilon\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx\\
&\quad +(1-3p)\int_{\mathbb{R}} v'(\rho_\varepsilon*\omega_\varepsilon)\,\partial_x(\rho_\varepsilon*\omega_\varepsilon)\,(\partial^2_{xx}\rho_\varepsilon)^{p}\,dx\\
&\quad -p\int_{\mathbb{R}} \rho_\varepsilon v^{(3)}(\rho_\varepsilon*\omega_\varepsilon)\,(\partial_x(\rho_\varepsilon*\omega_\varepsilon))^3\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx\\
&\quad -3p\int_{\mathbb{R}} \rho_\varepsilon v''(\rho_\varepsilon*\omega_\varepsilon)\,\partial_x(\rho_\varepsilon*\omega_\varepsilon)\,\partial^2_{xx}(\rho_\varepsilon*\omega_\varepsilon)\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx\\
&\quad -p\int_{\mathbb{R}} \rho_\varepsilon v'(\rho_\varepsilon*\omega_\varepsilon)\partial^3_{xxx}(\rho_\varepsilon*\omega_\varepsilon)\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx\\
&\quad +\varepsilon p\int_{\mathbb{R}} \partial^4_{xxxx}\rho_\varepsilon\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx\\
&=: J_1+J_2+J_3+J_4+J_5+J_6+J_7.
\end{aligned}
$$

We estimate now each of these terms.

- Using (2.1) and Young's inequality, we get

$$
\begin{aligned}
|J_1| &= 3p\left|\int_{\mathbb{R}} v''(\rho_\varepsilon*\omega_\varepsilon)\,(\partial_x(\rho_\varepsilon*\omega_\varepsilon))^2\,\partial_x\rho_\varepsilon\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx\right|\\
&\le 3p\,\|v''\|_\infty\,\big(2\|\rho_\varepsilon\|_\infty\,W_\varepsilon\big)^2\int_{\mathbb{R}}|\partial_x\rho_\varepsilon|\,|\partial^2_{xx}\rho_\varepsilon|^{p-1}\,dx\\
&\le 12\,\|v''\|_\infty\,\|\rho_\varepsilon\|^2_\infty\,W^2_\varepsilon\left(\int_{\mathbb{R}}|\partial_x\rho_\varepsilon|^p dx+(p-1)\int_{\mathbb{R}}|\partial^2_{xx}\rho_\varepsilon|^p dx\right)
\end{aligned}
$$

-

$$
\begin{aligned}
|J_2| &= 3p\left|\int_{\mathbb{R}} v'(\rho_\varepsilon*\omega_\varepsilon)\,\partial^2_{xx}(\rho_\varepsilon*\omega_\varepsilon)\,\partial_x\rho_\varepsilon\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx\right|\\
&\le 3p\,\|v'\|_\infty\int_{\mathbb{R}}|\partial^2_{xx}(\rho_\varepsilon*\omega_\varepsilon)|\,|\partial_x\rho_\varepsilon|\,|\partial^2_{xx}\rho_\varepsilon|^{p-1}\,dx\\
&\le 3\,\|v'\|_\infty\left(\frac{1}{2}\int_{\mathbb{R}}|\partial^2_{xx}(\rho_\varepsilon*\omega_\varepsilon)|^{2p}\,dx+\frac{1}{2}\int_{\mathbb{R}}|\partial_x\rho_\varepsilon|^{2p}\,dx+(p-1)\int_{\mathbb{R}}|\partial^2_{xx}\rho_\varepsilon|^p\,dx\right)
\end{aligned}
$$

  using the inequality

$$
uvw\le\frac{1}{p_1}u^{p_1}+\frac{1}{p_2}v^{p_2}+\frac{1}{p_3}w^{p_3},\qquad\text{with}\quad\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1, \tag{2.19}
$$

  setting $p_1=2p=p_2$ and $p_3=p/(p-1)$. Estimate (2.19) can be derived applying twice the classical Young's inequality to $uvw=u(vw)$. Using now the relation (2.12) with $2p$ at the place of p, we get

$$
|J_2| \le \frac{3^{2p+1}}{2}\,\|v'\|_\infty\,\eta\left(\int_0^\eta|\omega''(u)|^{2p/(2p-1)}\,du\right)^{2p-1}\int_{\mathbb{R}}\rho_\varepsilon^{2p}\,dx
$$

$$
\begin{aligned}
&+\frac{3^{2p+1}}{2}\,\|v'\|_\infty \left(|\omega'(\eta-)|+|\omega'(0+)|\right)^{2p} \int_{\mathbb{R}} \rho_\varepsilon^{2p}\,dx \\
&+3^{2p+1}2^{2p-1}\,\|v'\|_\infty \left(\omega(\eta)+2\omega(0)\right)^{2p} \int_{\mathbb{R}} |\partial_x\rho_\varepsilon|^{2p}\,dx \\
&+\frac{3}{2}\,\|v'\|_\infty \int_{\mathbb{R}} |\partial_x\rho_\varepsilon|^{2p}\,dx + 3(p-1)\,\|v'\|_\infty \int_{\mathbb{R}} |\partial^2_{xx}\rho_\varepsilon|^p\,dx.
\end{aligned}
$$

-

$$
\begin{aligned}
|J_3| &= (3p-1)\left|\int_{\mathbb{R}} v'(\rho_\varepsilon*\omega_\varepsilon)\,\partial_x(\rho_\varepsilon*\omega_\varepsilon)\,(\partial^2_{xx}\rho_\varepsilon)^p\,dx\right| \\
&\le 2(3p-1)\,\|v'\|_\infty\,\|\rho_\varepsilon\|_\infty\,W_\varepsilon \int_{\mathbb{R}} |\partial^2_{xx}\rho_\varepsilon|^p\,dx.
\end{aligned}
$$

-

$$
\begin{aligned}
|J_4| &= p\left|\int_{\mathbb{R}} \rho_\varepsilon v^{(3)}(\rho_\varepsilon*\omega_\varepsilon)\,(\partial_x(\rho_\varepsilon*\omega_\varepsilon))^3\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx\right| \\
&\le p\,\|v^{(3)}\|_\infty \left(2\|\rho_\varepsilon\|_\infty\,W_\varepsilon\right)^3 \int_{\mathbb{R}} \rho_\varepsilon|\partial^2_{xx}\rho_\varepsilon|^{p-1}\,dx \\
&\le 8\,\|v^{(3)}\|_\infty\,\|\rho_\varepsilon\|^3_\infty\,W_\varepsilon^3 \left(\int_{\mathbb{R}} \rho_\varepsilon^p\,dx + (p-1)\int_{\mathbb{R}} |\partial^2_{xx}\rho_\varepsilon|^p\,dx\right)
\end{aligned}
$$

using Young's inequality.

-

$$
\begin{aligned}
|J_5| &= 3p\left|\int_{\mathbb{R}} \rho_\varepsilon v''(\rho_\varepsilon*\omega_\varepsilon)\,\partial_x(\rho_\varepsilon*\omega_\varepsilon)\,\partial^2_{xx}(\rho_\varepsilon*\omega_\varepsilon)\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx\right| \\
&\le 6p\,\|v''\|_\infty\,\|\rho_\varepsilon\|^2_\infty\,W_\varepsilon \int_{\mathbb{R}} |\partial^2_{xx}(\rho_\varepsilon*\omega_\varepsilon)|\,|\partial^2_{xx}\rho_\varepsilon|^{p-1}\,dx \\
&\le 6\,\|v''\|_\infty\,\|\rho_\varepsilon\|^2_\infty\,W_\varepsilon \left(\int_{\mathbb{R}} |\partial^2_{xx}(\rho_\varepsilon*\omega_\varepsilon)|^p\,dx + (p-1)\int_{\mathbb{R}} |\partial^2_{xx}\rho_\varepsilon|^p\,dx\right) \\
&\le 6\,\|v''\|_\infty\,\|\rho_\varepsilon\|^2_\infty\,W_\varepsilon \Bigg(3^p\eta\left(\int_0^\eta |\omega''(u)|^{p/(p-1)}\,du\right)^{p-1} \int_{\mathbb{R}} \rho_\varepsilon^p\,dx \\
&\qquad + 3^p\left(|\omega'(\eta-)|+|\omega'(0+)|\right)^p \int_{\mathbb{R}} \rho_\varepsilon^p\,dx \\
&\qquad + 6^p\left(\omega(\eta)+2\omega(0)\right)^p \int_{\mathbb{R}} |\partial_x\rho_\varepsilon|^p\,dx + (p-1)\int_{\mathbb{R}} |\partial^2_{xx}\rho_\varepsilon|^p\,dx\Bigg)
\end{aligned}
$$

using relation (2.12).

-

$$
\begin{aligned}
|J_6| &= p\left|\int_{\mathbb{R}} \rho_\varepsilon v'(\rho_\varepsilon*\omega_\varepsilon)\partial^3_{xxx}(\rho_\varepsilon*\omega_\varepsilon)\,(\partial^2_{xx}\rho_\varepsilon)^{p-1}\,dx\right| \\
&\le p\,\|\rho_\varepsilon\|_\infty\|v'\|_\infty \int_{\mathbb{R}} |\partial^3_{xxx}(\rho_\varepsilon*\omega_\varepsilon)|\,|\partial^2_{xx}\rho_\varepsilon|^{p-1}\,dx
\end{aligned}
$$

$$\leq \|\rho_\varepsilon\|_\infty \|v'\|_\infty \left(\int_{\mathbb{R}} |\partial^3_{xxx}(\rho_\varepsilon * \omega_\varepsilon)|^p \, dx + (p-1) \int_{\mathbb{R}} |\partial^2_{xx}\rho_\varepsilon|^p \, dx \right).$$

Estimate (2.3) of Proposition 1 and the inequality (2.11) give

$$\begin{aligned} \int_{\mathbb{R}} |\partial^3_{xxx}(\rho_\varepsilon * \omega_\varepsilon)|^p(t,x) \, dx \leq\ & 3^p \eta \left(\int_0^\eta |\omega''(u)|^{p/(p-1)} \, du \right)^{p-1} \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^p \, dx \\ & + 3^p \left(|\omega'(\eta-)| + |\omega'(0+)| \right)^p \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^p \, dx \\ & + 6^p \left(\omega(\eta) + 2\omega(0) \right)^p \int_{\mathbb{R}} |\partial^2_{xx}\rho_\varepsilon(t,x)|^p \, dx. \end{aligned}$$

Then

$$\begin{aligned} |J_6| \leq\ & \|\rho_\varepsilon\|_\infty \|v'\|_\infty \Bigg(3^p \eta \left(\int_0^\eta |\omega''(u)|^{p/(p-1)} \, du \right)^{p-1} \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^p \, dx \\ & + 3^p \left(|\omega'(\eta-)| + |\omega'(0+)| \right)^p \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^p \, dx \\ & + 6^p \left(\omega(\eta) + 2\omega(0) \right)^p \int_{\mathbb{R}} |\partial^2_{xx}\rho_\varepsilon(t,x)|^p \, dx \\ & + (p-1) \int_{\mathbb{R}} |\partial^2_{xx}\rho_\varepsilon(t,x)|^p \, dx \Bigg). \end{aligned}$$

- $$J_7 = \varepsilon p \int_{\mathbb{R}} \partial^4_{xxxx}\rho_\varepsilon \, (\partial^2_{xx}\rho_\varepsilon)^{p-1} \, dx = -\varepsilon p(p-1) \int_{\mathbb{R}} \left(\partial^3_{xxx}\rho_\varepsilon \right)^2 (\partial^2_{xx}\rho_\varepsilon)^{2(N-1)} \, dx \leq 0.$$

The above estimates give an estimate of the form

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} (\partial^2_{xx}\rho_\varepsilon(t,x))^p \, dx \leq\ & C_7^{\varepsilon,p} \Bigg(\int_{\mathbb{R}} |\partial^2_{xx}\rho_\varepsilon(t,x)|^p \, dx + \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^p \, dx + \int_{\mathbb{R}} \rho_\varepsilon^p(t,x) \, dx \\ & \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^{2p} \, dx + \int_{\mathbb{R}} \rho_\varepsilon^{2p}(t,x) \, dx \Bigg), \end{aligned}$$

where $C_7^p = C_7^p \left(p, \|v'\|_\infty, \|v''\|_\infty, \|v^{(3)}\|_\infty, \sup_\varepsilon \left\{ \|\rho_\varepsilon\|_\infty W_\varepsilon \right\}, C_\omega^p, C_\omega^{2p} \right)$ and

$$C_\omega^p = \max \left\{ 3^p \eta \left(\int_0^\eta |\omega''(u)|^{p/(p-1)} \, du \right)^{p-1}, 3^p \left(|\omega'(\eta-)| + |\omega'(0+)| \right)^p, 6^p \left(\omega(\eta) + 2\omega(0) \right)^p \right\}.$$

Note that C_7^p is a constant since $\|\rho_\varepsilon\|_\infty W_\varepsilon$ is bounded with respect to ε thanks to Proposition 4 and 1.5. This estimate, combined with (2.13) and (2.14), give then

$$\frac{d}{dt} \left(\int_{\mathbb{R}} |\partial^2_{xx}\rho_\varepsilon(t,x)|^{2N} \, dx + \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^{2N} \, dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t,x) \, dx \right)$$

$$
\begin{aligned}
\leq\ & C_8^{2N}\left(\int_{\mathbb{R}}|\partial_{xx}^2\rho_\varepsilon(t,x)|^{2N}\,dx+\int_{\mathbb{R}}|\partial_x\rho_\varepsilon(t,x)|^{2N}\,dx+\int_{\mathbb{R}}\rho_\varepsilon^{2N}(t,x)\,dx\right)\\
&+C_7^{2N}e^{C_9^{2N}t}\left(\int_{\mathbb{R}}|\partial_x\rho(0,x)|^{4N}\,dx+\int_{\mathbb{R}}|\rho(0,x)|^{4N}\,dx\right),
\end{aligned}
$$

where

$$
C_8^{2N}=C_7^{2N}+\sup_\varepsilon C_6^{\varepsilon,2N},\qquad C_9^{2N}=\sup_\varepsilon C_6^{\varepsilon,2N}.
$$

Note that $C_6^{\varepsilon,2N}$ is bounded with respect to ε thanks to Proposition 4 and 1.5. Since an inequality of the form

$$
u'(t)\leq K_1u(t)+K_2e^{K_3t}
$$

implies the estimate

$$
u(t)\leq u(0)e^{K_1t}+K_2e^{K_1t}\int_0^t e^{(K_3-K_1)s}\,ds\leq u(0)e^{K_1t}+\frac{K_2}{K_3}\left(e^{(K_1+K_3)t}-e^{K_1t}\right),
$$

we get the estimate

$$
\begin{aligned}
&\left(\int_{\mathbb{R}}|\partial_{xx}^2\rho_\varepsilon(t,x)|^{2N}\,dx+\int_{\mathbb{R}}|\partial_x\rho_\varepsilon(t,x)|^{2N}\,dx+\int_{\mathbb{R}}\rho_\varepsilon^{2N}(t,x)\,dx\right)\\
\leq\ &\left(\int_{\mathbb{R}}|\partial_{xx}^2\rho(0,x)|^{2N}\,dx+\int_{\mathbb{R}}|\partial_x\rho(0,x)|^{2N}\,dx+\int_{\mathbb{R}}\rho^{2N}(0,x)\,dx\right)e^{C_8^{2N}t}\\
&+\frac{C_7^{2N}}{C_9^{2N}}\left(\int_{\mathbb{R}}|\partial_x\rho(0,x)|^{4N}\,dx+\int_{\mathbb{R}}|\rho(0,x)|^{4N}\,dx\right)\left(e^{(C_8^{2N}+C_9^{2N})t}-e^{C_8^{2N}t}\right),
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\sup_{t\in[0,T]}\left(\int_{\mathbb{R}}|\partial_{xx}^2\rho_\varepsilon(t,x)|^{2N}\,dx+\int_{\mathbb{R}}|\partial_x\rho_\varepsilon(t,x)|^{2N}\,dx+\int_{\mathbb{R}}\rho_\varepsilon^{2N}(t,x)\,dx\right)\\
\leq\ &\left(\int_{\mathbb{R}}|\partial_{xx}^2\rho(0,x)|^{2N}\,dx+\int_{\mathbb{R}}|\partial_x\rho(0,x)|^{2N}\,dx+\int_{\mathbb{R}}\rho^{2N}(0,x)\,dx\right)e^{C_8^{2N}T}\\
&+\frac{C_7^{2N}}{C_9^{2N}}\left(\int_{\mathbb{R}}|\partial_x\rho(0,x)|^{4N}\,dx+\int_{\mathbb{R}}|\rho(0,x)|^{4N}\,dx\right)e^{(C_8^{2N}+C_9^{2N})T}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\int_0^T\int_{\mathbb{R}}|\partial_{xx}^2\rho_\varepsilon(t,x)|^{2N}\,dx\,dt+\int_0^T\int_{\mathbb{R}}|\partial_x\rho_\varepsilon(t,x)|^{2N}\,dx\,dt+\int_0^T\int_{\mathbb{R}}\rho_\varepsilon^{2N}(t,x)\,dx\,dt\right)\\
\leq\ &\left(\int_{\mathbb{R}}|\partial_{xx}^2\rho(0,x)|^{2N}\,dx+\int_{\mathbb{R}}|\partial_x\rho(0,x)|^{2N}\,dx+\int_{\mathbb{R}}\rho^{2N}(0,x)\,dx\right)Te^{C_8^{2N}T}\\
&+\frac{C_7^{2N}}{C_9^{2N}}\left(\int_{\mathbb{R}}|\partial_x\rho(0,x)|^{4N}\,dx+\int_{\mathbb{R}}|\rho(0,x)|^{4N}\,dx\right)e^{(C_8^{2N}+C_9^{2N})T}T.
\end{aligned}
$$

□

3 Proof of Theorem 1

In this section, we pass to the limit as $\varepsilon \to 0$ and we show that the limit function ρ satisfies equation (1.1).

Using Proposition 2, the sequence $\{\rho_\varepsilon\}_\varepsilon$ is bounded in $\mathbf{L}^\infty([0,T],\mathbf{L^2}(\mathbb{R}))$. Using Proposition 3, the sequence $\{\partial_x\rho_\varepsilon\}_\varepsilon$ is bounded in $\mathbf{L}^\infty([0,T],\mathbf{L^2}(\mathbb{R}))$. Using Propositions 1 and 4, the sequences $\big\{v(\rho_\varepsilon * \omega_\varepsilon)\big\}_\varepsilon$, $\big\{v'(\rho_\varepsilon * \omega_\varepsilon)\big\}_\varepsilon$ and $\big\{\partial_x(\rho_\varepsilon * \omega_\varepsilon)\big\}_\varepsilon$ are bounded in $\mathbf{L}^\infty([0,T]\times\mathbb{R})$. Then

$$\partial_x(\rho_\varepsilon v(\rho_\varepsilon * \omega_\varepsilon)) = \partial_x\rho_\varepsilon \cdot v(\rho_\varepsilon * \omega_\varepsilon) + \rho_\varepsilon\, v'(\rho_\varepsilon * \omega_\varepsilon)\, \partial_x(\rho_\varepsilon * \omega_\varepsilon)$$

is bounded in $\mathbf{L}^\infty([0,T],\mathbf{L^2}(\mathbb{R}))$. Using Proposition 5, we also have a bound with respect to ε for $\partial^2_{xx}\rho_\varepsilon$ in the space $\mathbf{L}^\infty([0,T],\mathbf{L^2}(\mathbb{R}))$, then

$$\partial_t\rho_\varepsilon = \varepsilon\partial^2_{xx}\rho_\varepsilon - \partial_x(\rho_\varepsilon v(\rho_\varepsilon * \omega_\varepsilon)) \in \mathbf{L}^\infty([0,T],\mathbf{L^2}(\mathbb{R}))$$

uniformly with respect to ε. In particular, $\rho_\varepsilon \in \mathbf{C}([0,T],\mathbf{L^2}(\mathbb{R}))$ and the sequence is bounded in this space. Since $\partial_t\rho_\varepsilon, \partial_x\rho_\varepsilon \in \mathbf{L}^\infty([0,T],\mathbf{L^2}(\mathbb{R}))$ with uniform bounds with respect to ε, then $\{\rho_\varepsilon\}_\varepsilon$ is bounded in $\mathbf{H^1_{loc}}([0,T]\times\mathbb{R})$. Up to the extraction of a subsequence, the sequence $\{\rho_\varepsilon\}_\varepsilon$ converges to some ρ in $\mathbf{L^2_{loc}}([0,T]\times\mathbb{R})$ and a.e. We have now to prove that the limit $\rho \in \mathbf{C}([0,T],\mathbf{L^2}(\mathbb{R}))$ is a solution of (1.1). Since

$$\begin{aligned}(\rho_\varepsilon * \omega_\varepsilon)(t,x) - (\rho * \omega)(t,x) \;&=\; \int_{-\varepsilon}^{0} \rho_\varepsilon(t,x+y)\omega_\varepsilon(y)\,dy + \int_0^\eta (\rho_\varepsilon - \rho)(t,x+y)\omega(y)\,dy \\ &\quad + \int_\eta^{\eta+\varepsilon} \rho_\varepsilon(t,x+y)\omega_\varepsilon(y)\,dy\end{aligned}$$

tends to 0 when ε goes to zero, we have

$$\rho_\varepsilon v(\rho_\varepsilon * \omega_\varepsilon) \to \rho v(\rho * \omega) \;\text{ a.e.}$$

Therefore using dominated convergence Theorem, we get $\rho_\varepsilon v(\rho_\varepsilon * \omega_\varepsilon) \to \rho v(\rho * \omega)$ in $\mathbf{L^1_{loc}}([0,T]\times\mathbb{R})$, implying that ρ is a solution of (1.1).

References

[1] A. Aggarwal, R. M. Colombo, and P. Goatin. Nonlocal systems of conservation laws in several space dimensions. *SIAM J. Numer. Anal.*, 53(2):963–983, 2015.

[2] P. Amorim, R. Colombo, and A. Teixeira. On the numerical integration of scalar nonlocal conservation laws. *ESAIM M2AN*, 49(1):19–37, 2015.

[3] F. Betancourt, R. Bürger, K. H. Karlsen, and E. M. Tory. On nonlocal conservation laws modelling sedimentation. *Nonlinearity*, 24(3):855–885, 2011.

[4] S. Blandin and P. Goatin. Well-posedness of a conservation law with non-local flux arising in traffic flow modeling. *Numer. Math.*, 132(2):217–241, 2016.

[5] M. Colombo, G. Crippa, and L. V. Spinolo. On the singular local limit for conservation laws with nonlocal fluxes. *ArXiv e-prints*, Oct. 2017.

[6] R. M. Colombo, M. Garavello, and M. Lécureux-Mercier. A class of nonlocal models for pedestrian traffic. *Mathematical Models and Methods in Applied Sciences*, 22(04):1150023, 2012.

[7] J. Friedrich, O. Kolb, and S. Göttlich. A Godunov type scheme for a class of LWR traffic flow models with non-local flux. *ArXiv e-prints*, Feb. 2018.

[8] P. Goatin and S. Scialanga. Well-posedness and finite volume approximations of the LWR traffic flow model with non-local velocity. *Netw. Heterog. Media*, 11(1):107–121, 2016.

[9] S. Göttlich, S. Hoher, P. Schindler, V. Schleper, and A. Verl. Modeling, simulation and validation of material flow on conveyor belts. *Applied Mathematical Modelling*, 38(13):3295 – 3313, 2014.

[10] M. Gröschel, A. Keimer, G. Leugering, and Z. Wang. Regularity theory and adjoint-based optimality conditions for a nonlinear transport equation with nonlocal velocity. *SIAM J. Control Optim.*, 52(4):2141–2163, 2014.

[11] A. Keimer and L. Pflug. Existence, uniqueness and regularity results on nonlocal balance laws. *J. Differential Equations*, 263(7):4023–4069, 2017.

[12] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[13] D. Li and T. Li. Shock formation in a traffic flow model with Arrhenius look-ahead dynamics. *Networks and Heterogeneous Media*, 6(4):681–694, 2011.

[14] E. Rossi and R. M. Colombo. Non Local Conservation Laws in Bounded Domains. *ArXiv e-prints*, Nov. 2017.

[15] A. Sopasakis and M. A. Katsoulakis. Stochastic modeling and simulation of traffic flow: asymmetric single exclusion process with Arrhenius look-ahead dynamics. *SIAM J. Appl. Math.*, 66(3):921–944 (electronic), 2006.