Convection, optimal transport, Zeldovich systems and magnetic relaxation

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Outline

1. From convection theory to optimal transportation, through the Navier-Sokes Boussinesq equations

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4. A fully nonlinear version of Zeldovich gravitational model
5. A stringy generalization of the DB model leading to a magnetic relaxation model à la Arnold-Moffatt

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6. The cross-Burgers equation: a model of magnetic reversal

The NS-Boussinesq model

Let D be a smooth bounded domain $\mathrm{D} \subset \mathrm{R}^{\mathbf{3}}$ in which moves an incompressible fluid of velocity $\mathrm{v}(\mathrm{t}, \mathrm{x})$ at $\mathrm{x} \in \mathrm{D}, \quad \mathrm{t} \geq 0$, subject to the Navier-Stokes equations

$$\mathrm{NSB} \quad \epsilon(\partial_{\mathbf{t}}\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}) + \mathbf{K}\mathbf{v} + \nabla \mathbf{p} = \mathbf{y} \qquad \nabla \cdot \mathbf{v} = \mathbf{0}$$

where $\boxed{\mathbf{K}\mathbf{v} = \alpha\mathbf{v} - \nu\boldsymbol{\Delta}\mathbf{v}}$ with $\alpha, \epsilon, \nu > 0$ and $\mathbf{v} = 0$ along $\partial\mathrm{D}$.

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The force field $\mathbf{y}$ is subject to the advection equation

$$\partial_{\mathbf{t}}\mathbf{y} + (\mathbf{v}\cdot\nabla)\mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{y})$$

where G is a given smooth function with bounded derivatives.

Classical Convection Theory

Classical Convection Theory corresponds to the special case:

$$\boxed{\mathbf{K} = -\boldsymbol{\Delta}, \quad \mathbf{G} = \mathbf{0}, \quad \mathbf{y}//\mathbf{e_3}, \quad \mathbf{y} = \eta\mathbf{e_3}, \quad \eta = \eta(\mathbf{t}, \mathbf{x}) \in \mathbf{R}}$$

namely:

$$\boxed{\epsilon(\partial_{\mathbf{t}}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) - \boldsymbol{\Delta}\mathbf{v} + \nabla\mathbf{p} = \eta\mathbf{e_3}, \quad \nabla\cdot\mathbf{v} = \mathbf{0}}$$

$$\boxed{\partial_{\mathbf{t}}\eta + (\mathbf{v}\cdot\nabla)\eta = \mu\boldsymbol{\Delta}\eta}$$

with $\mu \geq \mathbf{0}$**.**

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Three limits of the NSB model

While keeping unchanged

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and dropping inertia terms, we consider three possible limits:

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$$\boxed{\textbf{HYDROSTATIC HB}: \quad \epsilon = \nu = \alpha = \mathbf{0} \quad \Rightarrow \quad \nabla\mathbf{p} = \mathbf{y}}$$

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Each of these limits, as $\epsilon \to 0$, can be rigorously justified as long as the limit equation admits smooth solutions, YB 2007.
The last case is more subtle and requires an additional CONVEXITY condition that will be discussed below, YB 2008.

The AHT model

In the case $G = 0$, the Darcy-Boussinesq model

$$\text{DB} \quad \partial_t \mathbf{y} + (\mathbf{v} \cdot \nabla)\mathbf{y} = \mathbf{0} \qquad \nabla \cdot \mathbf{v} = \mathbf{0}, \quad \mathbf{v} + \nabla \mathbf{p} = \mathbf{y}$$

coincides with the Angenent/Haker/Tannenbaum model which was introduced for the numerical resolution of the polar factorization problem arising in optimal transport theory (cf. Y.B. CPAM 1991).

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$$\frac{\mathbf{d}}{\mathbf{dt}} \int_{\mathbf{D}} |\mathbf{y}(\mathbf{t}, \mathbf{x}) - \mathbf{x}|^2 \mathbf{dx} = -2 \int_{\mathbf{D}} |\mathbf{v}(\mathbf{t}, \mathbf{x})|^2 \, \mathbf{dx}$$

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So, we expect, as $\mathrm{t} \to +\infty$, $\mathbf{v} \to \mathbf{0}$, $(\mathbf{y}, \mathbf{p}) \to (\mathbf{y}^\infty, \mathbf{p}^\infty)$, so that

$$\mathbf{y}^\infty = \nabla \mathbf{p}^\infty$$

is a curl-free 'rearrangement' of the given initial vector field $\mathbf{y}_0$.

A convexity condition for the HB model

The Hydrostatic Boussinesq HB system

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Let us consider, for simplicity, the case of 2 space variables $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2})$ and write $\mathrm{v} = (-\partial_1\theta, \partial_2\theta)$, where $\theta(\mathrm{t}, \mathbf{x_1}, \mathbf{x_2}) \in \mathrm{R}$.

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Take the 2D curl of the evolution equation in $\mathbf{y} = (\partial_1\mathbf{p}, \partial_2\mathbf{p})$:

$$\partial_{11}\mathbf{p}\,\partial_{22}\theta + \partial_{22}\mathbf{p}\,\partial_{11}\theta - \mathbf{2}\partial_{12}\mathbf{p}\,\partial_{12}\theta = \partial_1(\mathbf{G_2}) - \partial_2(\mathbf{G_1})$$

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$$\partial_{11}\mathbf{p}\,\partial_{22}\theta + \partial_{22}\mathbf{p}\,\partial_{11}\theta - 2\partial_{12}\mathbf{p}\,\partial_{12}\theta = \partial_1(\mathbf{G}_2) - \partial_2(\mathbf{G}_1)$$

a well posed linear elliptic equation in θ whenever $\mathbf{D}^2_{\mathbf{x}}\mathbf{p}(\mathbf{t},\mathbf{x}) > \mathbf{0}$

Observables in Boussinesq systems

For each suitable test function f, consider the 'observable'

$$\mathbf{t} \rightarrow \rho_{\mathbf{f}}(\mathbf{t}) = \int_{\mathbf{D}} \mathbf{f}(\mathbf{y}(\mathbf{t},\mathbf{x}))\mathbf{dx}$$

where y is solution to one of the Boussinesq systems NSB,SB,DB,HB.

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Then, we get

$$\frac{\mathbf{d}}{\mathbf{dt}} \int_{\mathbf{D}} \mathbf{f}(\mathbf{y}(\mathbf{t},\mathbf{x}))\mathbf{dx} = \int_{\mathbf{D}} (\nabla \mathbf{f})(\mathbf{y}(\mathbf{t},\mathbf{x})) \cdot \mathbf{G}(\mathbf{x},\mathbf{y}(\mathbf{t},\mathbf{x}))\mathbf{dx}$$

since $\boxed{\partial_{\mathbf{t}}\mathbf{y} + (\mathbf{v} \cdot \nabla)\mathbf{y} = \mathbf{G}(\mathbf{x},\mathbf{y})}$ where $\boxed{\nabla \cdot \mathbf{v} = \mathbf{0}, \quad \mathbf{v} // \partial \mathbf{D}.}$

Recovery from Observables

In the **HB** model, the field $\mathbf{y}$ is required to be a gradient $\boxed{\mathbf{y} = \nabla \mathbf{p}}$

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If we a priori assume

$$\mathbf{p}(\mathbf{t}, \mathbf{x}) \text{ is a CONVEX function of } \mathbf{x} \in \mathbf{D},$$

then the field $\mathbf{y}$ is completely determined by the knowledge of all observables

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then the field y is completely determined by the knowledge of all observables

$$\boxed{t \to \rho_f(t) = \int_D f(y(t, x))dx}$$

NB: This is a typical result of OPTIMAL TRANSPORT THEORY

YB, CRAS Paris 1987 and CPAM 1991, Smith and Knott, JOTA 1987, cf. Villani, Topics in optimal transportation, AMS, 2003, see also papers, lecture notes and books by Rachev-Rüschendorf, Evans, Caffarelli, Urbas, Gangbo-McCann, Otto, Ambrosio-Savaré, Villani, Trudinger-Wang and many others contributions...

Global solutions to the HB system

THEOREM (YB 2007, following YB 2002 (unpublished))

Assume G to be a smooth function with bounded first derivatives. Let C be the convex cone of all maps $y \in L^2(D, R^3)$ such that $y(x) = \nabla p(x)$ a.e. in D for some CONVEX function p. We say that $(t \to y(t,\cdot)) \in C^0(R_+, L^2(D, R^3))$ valued in the cone C is a solution to the HB system if

$$\frac{d}{dt}\int_D f(y(t,x))dx = \int_D (\nabla f)(y(t,x)) \cdot G(x, y(t,x))dx, \quad \forall f$$

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Then, for each $y_0 \in C$, there is always a GLOBAL solution such that $y(t=0,\cdot) = y_0$

A Monge-Ampère formulation for HB

Under the CONVEXITY assumption, the HB system

$$\mathrm{HB}: \ \partial_t \mathbf{y} + (\mathbf{v} \cdot \nabla)\mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{y}), \quad \mathbf{y} = \nabla \mathbf{p}, \quad \nabla \cdot \mathbf{v} = \mathbf{0}$$

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is (formally) equivalent to

The coupled MONGE-AMPERE system

$$\partial_t \rho + \nabla\cdot(\rho\mathbf{w}) = \mathbf{0}, \quad \mathbf{w} = \mathbf{G}(\nabla \mathbf{p}^*(\mathbf{t},\mathbf{x}),\mathbf{x}), \quad \rho = \det(\mathbf{D}^2\mathbf{p}^*(\mathbf{t},\mathbf{x}))$$

where p^* is the LEGENDRE transform

$$\mathbf{p}^*(\mathbf{t},\mathbf{x}) = \sup_{\tilde{\mathbf{x}}\in\mathbf{D}} \ \mathbf{x}\cdot\tilde{\mathbf{x}} - \mathbf{p}(\mathbf{t},\tilde{\mathbf{x}})$$

Justification

Use the change of variable

$$\tilde{\mathbf{x}} = \nabla \mathbf{p}^*(\mathbf{t}, \mathbf{x}), \quad \mathbf{d}\tilde{\mathbf{x}} = \mathbf{det}(\mathbf{D}^2 \mathbf{p}^*(\mathbf{t}, \mathbf{x}))\mathbf{dx}$$

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Get

$$\int_{\mathbf{D}} \mathbf{f}(\mathbf{y}(\mathbf{t}, \tilde{\mathbf{x}}))\mathbf{d}\tilde{\mathbf{x}} = \int \mathbf{f}(\mathbf{x})\mathbf{det}(\mathbf{D}^2\mathbf{p}^*(\mathbf{t}, \mathbf{x}))\mathbf{dx} = \int \mathbf{f}(\mathbf{x})\rho(\mathbf{t}, \mathbf{x})\mathbf{dx}$$

$$\int_{\mathbf{D}} (\nabla \mathbf{f})(\mathbf{y}(\mathbf{t}, \tilde{\mathbf{x}})) \cdot \mathbf{G}(\tilde{\mathbf{x}}, \mathbf{y}(\mathbf{t}, \tilde{\mathbf{x}}))\mathbf{d}\tilde{\mathbf{x}} = \int (\rho \mathbf{v})(\mathbf{t}, \mathbf{x}) \cdot \nabla \mathbf{f}(\mathbf{x})\mathbf{dx}$$

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Thus

$$\frac{\mathbf{d}}{\mathbf{dt}} \int \mathbf{f}(\mathbf{x})\rho(\mathbf{t}, \mathbf{x})\mathbf{dx} - \int (\rho \mathbf{v})(\mathbf{t}, \mathbf{x}) \cdot \nabla \mathbf{f}(\mathbf{x})\mathbf{dx} =$$

$$= \frac{\mathbf{d}}{\mathbf{dt}} \int_{\mathbf{D}} \mathbf{f}(\mathbf{y}(\mathbf{t}, \tilde{\mathbf{x}}))\mathbf{d}\tilde{\mathbf{x}} - \int_{\mathbf{D}} (\nabla \mathbf{f})(\mathbf{y}(\mathbf{t}, \tilde{\mathbf{x}})) \cdot \mathbf{G}(\tilde{\mathbf{x}}, \mathbf{y}(\mathbf{t}, \tilde{\mathbf{x}}))\mathbf{d}\tilde{\mathbf{x}} = \mathbf{0}$$

A gravitational model à la Zeldovich

In the special case: $\mathbf{G(x,y) = (y - x)/\beta}$, **where** $\beta > \mathbf{0}$ **is a constant, the HB system, written in coupled Monge-Ampère formulation, reads:**

$$\partial_{\mathbf{t}}\rho + \nabla \cdot (\rho \mathbf{w}) = \mathbf{0}, \quad \mathbf{w} = \nabla\psi(\mathbf{t},\mathbf{x}), \quad \rho = \mathbf{det}(\mathbf{I} - \beta \mathbf{D^2}\psi(\mathbf{t},\mathbf{x})),$$

where $\nabla\psi(\mathbf{t},\mathbf{x}) = (\nabla \mathbf{p}^*(\mathbf{t},\mathbf{x}) - \mathbf{x})/\beta$

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where $\nabla\psi(\mathbf{t},\mathbf{x}) = (\nabla\mathbf{p}^*(\mathbf{t},\mathbf{x}) - \mathbf{x})/\beta$

This is a fully nonlinear correction (as $\beta << 1$**) of**

$$\partial_{\mathbf{t}}\rho + \nabla\cdot(\rho\mathbf{w}) = \mathbf{0}, \quad \mathbf{w} = \nabla\psi(\mathbf{t},\mathbf{x}), \quad \rho = \mathbf{1} - \beta\boldsymbol{\Delta}\psi(\mathbf{t},\mathbf{x}))$$

which is a gravitational system à la Zeldovich.

The fully nonlinear version will be called FLNZ SYSTEM

Integrability of the FNLZ model

Written in HB formulation, the FNL Zeldovich system reads

$$\partial_{\mathbf{t}}\mathbf{y} + (\mathbf{v}\cdot\nabla)\mathbf{y} = \frac{\mathbf{y}-\mathbf{x}}{\beta}, \quad \mathbf{y} = \nabla\mathbf{p}, \quad \nabla\cdot\mathbf{v} = \mathbf{0}$$

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and has trivial solutions

$$\mathbf{y}(\mathbf{t},\mathbf{x}) = \mathbf{x} + (\mathbf{y_0}(\mathbf{x}) - \mathbf{x})\exp(\frac{\mathbf{t}}{\beta}), \qquad \mathbf{v} = \mathbf{0}$$

corresponding to motion along straight lines.
Notice that y has a CONVEX potential only for SHORT times.
Loss of convexity corresponds to collisions and mass concentration.

Global solutions to the FNLZ system

THEOREM

For each initial condition $\mathbf{y_0 = \nabla p_0 \in L^2(D, R^3)}$ with convex potential, the FNLZ system admits a GLOBAL solution with convex potential $\mathbf{(t \to y(t,\cdot) = (\nabla p)(t,\cdot)) \in C^0(R_+, L^2(D, R^3))}$, in the sense:

$$\mathbf{\frac{d}{dt}\int_D f(y(t,x))dx = \beta^{-1}\int_D (\nabla f)(y(t,x)) \cdot (y(t,x) - x)dx, \quad \forall f}$$

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Notice that this solution is not known to be unique. The globality of the solution means that collision and mass concentration are taken into account.

A Stringy Boussinesq model

A natural generalization of the DB model amounts to consider the set of curves ('strings') valued in $\mathrm{VPM}(\mathrm{D})$ (the set of all volume preserving maps of D)

$$\mathbf{X}: \ \mathbf{s} \in [\mathbf{0}, \mathbf{1}] \rightarrow \mathbf{X}(\mathbf{s}, \cdot) \in \mathbf{VPM}(\mathbf{D})$$

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and the corresponding gradient flow for the functional

$$\boxed{\mathbf{J}[\mathbf{X}] = \frac{\mathbf{1}}{\mathbf{2}} \int_{\mathbf{0}}^{\mathbf{1}} \int_{\mathbf{D}} |\partial_{\mathbf{s}} \mathbf{X}(\mathbf{s}, \mathbf{a})|^{\mathbf{2}} \mathbf{d}\mathbf{a}\mathbf{d}\mathbf{s}}$$

A Stringy Boussinesq model

A natural generalization of the DB model amounts to consider the set of curves ('strings') valued in $\mathrm{VPM}(\mathrm{D})$ (the set of all volume preserving maps of D)

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The resulting equation reads

$$\partial_{\mathbf{t}} \mathbf{X}(\mathbf{t}, \mathbf{s}, \mathbf{a}) + (\nabla \mathbf{p})(\mathbf{t}, \mathbf{s}, \mathbf{X}(\mathbf{t}, \mathbf{s}, \mathbf{a})) = \partial_{\mathbf{ss}} \mathbf{X}(\mathbf{t}, \mathbf{s}, \mathbf{a})$$

where p is a Lagrange multiplier for the incompressibility constraint.

The stringy DB model

Let us move to Eulerian coordinates by setting

$$\partial_t X(t,s,a) = v(t,s,X(t,s,a)), \quad \partial_s X(t,s,a) = b(t,s,X(t,s,a))$$

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First, we get two differential constraints (since $\mathrm{X}(\mathrm{t},\mathrm{s},\cdot)$ is volume preserving) and a compatibility condition (using $\partial_t \partial_s \mathbf{X} = \partial_s \partial_t \mathbf{X}$):

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{b} = \mathbf{0}\ , \quad \partial_t \mathbf{b} + (\mathbf{v} \cdot \nabla)\mathbf{b} = \partial_s \mathbf{v} + (\mathbf{b} \cdot \nabla)\mathbf{v}$$

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Then, the equation

$$\partial_t \mathbf{X}(\mathbf{t},\mathbf{s},\mathbf{a}) + (\nabla \mathbf{p})(\mathbf{t},\mathbf{s},\mathbf{X}(\mathbf{t},\mathbf{s},\mathbf{a})) = \partial_{ss} \mathbf{X}(\mathbf{t},\mathbf{s},\mathbf{a})$$

becomes, in Eulerian coordinates, $\mathbf{v} + \nabla \mathbf{p} = \partial_s \mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{b}$

Magnetic relaxation

We may interpret the stringy DB model (in Eulerian coordinates)

$$\mathbf{v} + \nabla\mathbf{p} = \partial_{\mathbf{s}}\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{b} \ , \quad \partial_{\mathbf{t}}\mathbf{b} + (\mathbf{v}\cdot\nabla)\mathbf{b} = \partial_{\mathbf{s}}\mathbf{v} + (\mathbf{b}\cdot\nabla)\mathbf{v} \ ,$$

as a MAGNETIC RELAXATION MODEL A LA ARNOLD-MOFFATT (cf. Arnold-Khesin, Springer 1998, Moffatt JFM 1985).

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As $\mathrm{t} \longrightarrow +\infty$, we expect 'equilibrium states'
$(\mathrm{B}, \mathrm{P})(\mathrm{s}, \mathrm{x}) = (\mathrm{b}, \mathrm{p})(\mathrm{t} = \infty, \mathrm{s}, \mathrm{x})$ to solve the Euler equation

$$\nabla \mathbf{P} = \partial_{\mathbf{s}} \mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{B}, \quad \nabla \cdot \mathbf{B} = \mathbf{0}$$

The cross-Burgers equation

In the special case when D is the unit ball of $\mathrm{R}^{\mathbf{3}}$, we look for special solutions of the stringy DB model

$$\mathbf{X}(\mathbf{t},\mathbf{s},\mathbf{a}) = \mathbf{U}(\mathbf{t},\mathbf{s})\mathbf{a}, \quad \forall \mathbf{a} \in \mathbf{D}$$

where $\mathrm{U}(\mathrm{t},\mathrm{s})$ is valued in THE ORTHOGONAL GROUP $\mathbf{O}(\mathbf{3})$.

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where $\mathrm{U}(\mathrm{t}, \mathrm{s})$ is valued in THE ORTHOGONAL GROUP $\mathbf{O(3)}$.
We find $\boxed{\partial_{\mathbf{t}}\mathbf{U}(\mathbf{t}, \mathbf{s}) + \mathbf{S}(\mathbf{t}, \mathbf{s})\mathbf{U}(\mathbf{t}, \mathbf{s}) = \partial_{\mathbf{ss}}\mathbf{U}(\mathbf{t}, \mathbf{s})}$ where each $\mathrm{S}(\mathrm{t}, \mathrm{s})$ should be a real symmetric 3×3 matrix.

The cross-Burgers equation

In the special case when D is the unit ball of R^3, we look for special solutions of the stringy DB model

$$\boxed{\mathbf{X}(\mathbf{t},\mathbf{s},\mathbf{a}) = \mathbf{U}(\mathbf{t},\mathbf{s})\mathbf{a}, \quad \forall \mathbf{a} \in \mathbf{D}}$$

where $\mathrm{U}(\mathrm{t},\mathrm{s})$ is valued in THE ORTHOGONAL GROUP $\mathbf{O}(\mathbf{3})$.
We find $\boxed{\partial_\mathbf{t}\mathbf{U}(\mathbf{t},\mathbf{s}) + \mathbf{S}(\mathbf{t},\mathbf{s})\mathbf{U}(\mathbf{t},\mathbf{s}) = \partial_{\mathbf{ss}}\mathbf{U}(\mathbf{t},\mathbf{s})}$ where each $\mathrm{S}(\mathrm{t},\mathrm{s})$ should be a real symmetric 3×3 matrix.
Introducing for each (t,s) the unique vector $\mathbf{B}(\mathbf{t},\mathbf{s}) \in \mathbf{R}^3$ such that

$$\boxed{\partial_\mathbf{s}\mathbf{U}(\mathbf{t},\mathbf{s})\mathbf{a} = \mathbf{B}(\mathbf{t},\mathbf{s}) \times (\mathbf{U}(\mathbf{t},\mathbf{s})\mathbf{a}), \quad \forall \mathbf{a} \in \mathbf{D}}$$

we get for $\mathrm{B}(\mathrm{t},\mathrm{s})$ what we call THE CROSS-BURGERS EQUATION

$$\boxed{\partial_\mathbf{t}\mathbf{B}(\mathbf{t},\mathbf{s}) + \mathbf{B}(\mathbf{t},\mathbf{s}) \times \partial_\mathbf{s}\mathbf{B}(\mathbf{t},\mathbf{s}) = \partial_{\mathbf{ss}}\mathbf{B}(\mathbf{t},\mathbf{s})}$$

Magnetic Reversal

An exact solution of the CROSS-BURGERS EQUATION is

$$\mathbf{B}(t, s) = (f(t)\cos(2\pi s), f(t)\sin(2\pi s), g(t))$$

where

$$\frac{df}{dt} = 2\pi(g - 2\pi)f, \quad \frac{dg}{dt} = -2\pi f^2$$

For $g(t=0) > 4\pi, \quad f(t=0) \neq 0$ **, we can check that** $g(t=+\infty) = 4\pi - g(0) < 0, \quad f(t=+\infty) = 0$**, even when** $|f(t=0)| << 1$ **.**

$$\text{This looks like a magnetic reversal}$$

Magnetic Reversal 2

$$\mathbf{g(0) = 36.2830925,\ \ f(0) = 0.075,\ \ g(+\infty) = -23.7167}$$

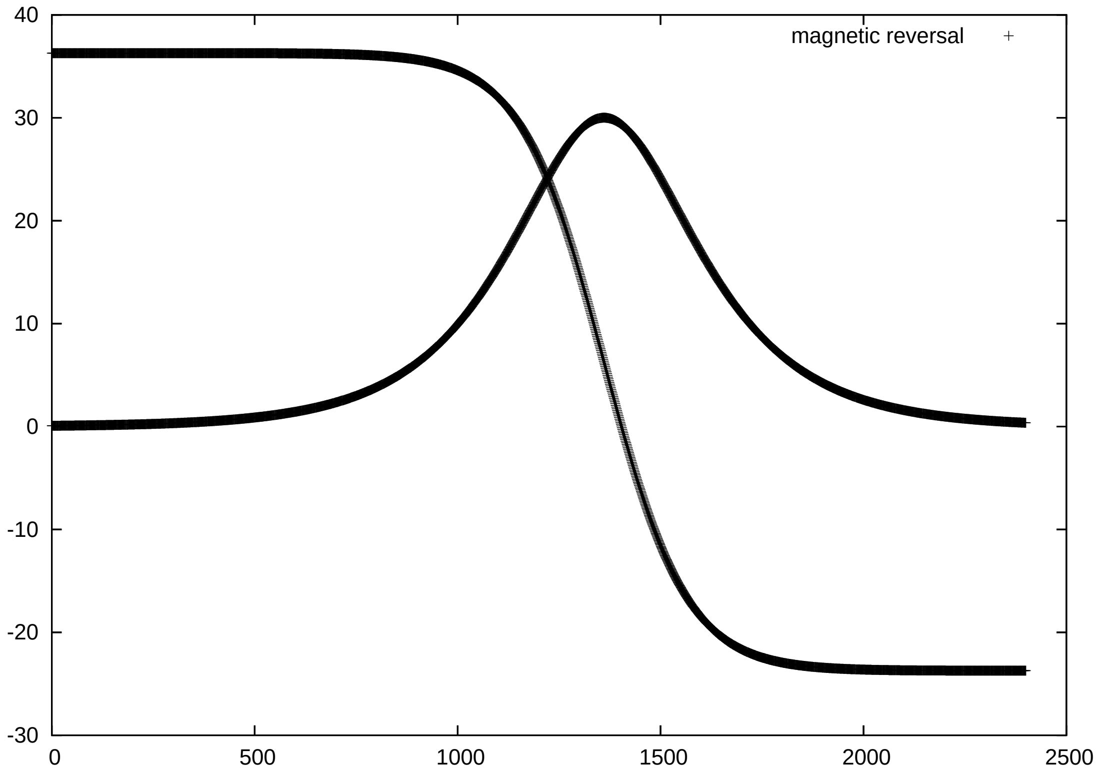

Figure 1: g(t) and f(t) versus t