The cyclic Deligne conjecture for spaces, chain complexes and Hopf algebras\textsuperscript{1}

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\textsuperscript{1}joint work with Michael Batanin (Sydney)
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Hochschild cochains

Definition
For a (unital associative) $K$-algebra $A$ and $A$-bimodule $M$, the Hochschild cochain complex of $A$ with coefficients in $M$ is given by

$$C^n(A; M) = \text{Hom}_K(A^\otimes n, M), \quad n \geq 0,$$

where for $f \in C^n(A; M)$,

$$(\partial_i f)(a_1, \ldots, a_{n+1}) = \begin{cases} a_1 f(a_2, \ldots, a_n) & i = 0; \\ f(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) & i = 1, \ldots, n; \\ f(a_1, \ldots, a_n) a_{n+1} & i = n + 1. \end{cases}$$

$$(s_i f)(a_1, \ldots, a_{n-1}) = f(a_1, \ldots, a_i, 1_A, a_{i+1}, \ldots, a_{n-1}).$$

The Hochschild cohomology $HH^\bullet(A; M)$ is the cohomology of the cochain complex of the cosimplicial $K$-module $C^\bullet(A; M)$. 
Cup and brace operations on $C^\bullet(A; A)$

There is a *cup product*

$$- \cup - : C^m(A; A) \otimes_K C^n(A; A) \to C^{m+n}(A; A)$$

$$(f \cup g)(a_1, \ldots, a_{m+n}) = f(a_1, \ldots, a_m)g(a_{m+1}, \ldots, a_{m+n})$$

and a *brace operation*

$$-\{\cdot\} : C^m(A; A) \otimes_K C^n(A; A) \to C^{m+n-1}(A; A)$$

where $f\{g\}(a_1, \ldots, a_{m+n-1})$ is defined by

$$\sum_{1 \leq i \leq m} (-1)^{(i-1)(n-1)} f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+n-1}), a_{i+n}, \ldots, a_{m+n-1}).$$

The bracket $\{f, g\} = f\{g\} - (-1)^{|f|−1(|g|−1)} g\{f\}$ induces a Lie bracket of degree $-1$ on $HH^\bullet(A; A)$. 
Gerstenhaber structure

**Definition**

A **Gerstenhaber $K$-algebra** $(H, \cup, \{-,-\})$ is a graded-commutative $K$-algebra with Lie bracket of degree $-1$ such that

$$\{f, g \cup h\} = \{f, g\} \cup h + (-1)^{|f|(|g|-1)}g \cup \{f, h\}.$$  

**Proposition (Gerstenhaber '63)**

For any algebra $A$, the Hochschild cohomology $HH^\bullet(A; A)$ is a Gerstenhaber algebra.

**Theorem (F. Cohen '72)**

For any field $K$, the homology $H_\bullet(D_2; K)$ of the little disks operad is the operad for Gerstenhaber $K$-algebras.

**Corollary**

For any based space $(X, \ast)$, the homology $H_\bullet(\Omega^2 X; K)$ is a Gerstenhaber $K$-algebra.
Connes’ coboundary on $C^\bullet(A; A^\ast)$

For $A^\ast = \text{Hom}_K(A, K)$, the adjunction

$$\text{Hom}_K(A^\otimes n, A^\ast) \cong \text{Hom}_K(A^\otimes n+1, K)$$

induces a cyclic operator $\tau_n$ on $C^n(A; A^\ast)$ of order $n + 1$. These cyclic operators are compatible with the simplicial operators:

$$\tau_{n+1} \partial_i = \partial_{i-1} \tau_n \quad i > 0, \quad \tau_{n-1} s_i = s_{i-1} \tau_n \quad i > 0.$$

It results a covariant functor on Connes’ cyclic category

$$\Delta C \to \text{Mod}_K : [n] \mapsto C^n(A; A^\ast).$$

In particular, $C^\bullet(A; A^\ast)$ is a mixed complex

$$C^0(A; A^\ast) \leftrightarrow C^1(A, A^\ast) \leftrightarrow C^2(A; A^\ast) \leftrightarrow \cdots$$

and $HH^\bullet(A; A^\ast)$ has a differential $\Delta$ of degree $-1$:

$$\Delta^n : HH^n(A, A^\ast) \to HH^{n-1}(A; A^\ast).$$
Batalin-Vilkovisky structure

Definition
A *Batalin-Vilkovisky algebra* is a Gerstenhaber algebra 
\((H, \cup, \{-, -\})\) with a differential \(\Delta\) of degree \(-1\) such that

\[
(-1)^{|f|}\{f, g\} = \Delta(f \cup g) - (\Delta f \cup g) - (-1)^{|f|}(f \cup \Delta g).
\]

A *symmetric* \(K\)-algebra \(A\) is a \(K\)-algebra equipped with an isomorphism of \(A\)-bimodules \(A \cong A^*\), i.e. a symmetric exact pairing \(\langle -, -\rangle: A \otimes_K A \to K\) such that \(\langle ab, c\rangle = \langle a, bc\rangle\).

Proposition (Menichi '04)
For any symmetric algebra \(A\), the Hochschild cohomology \(HH^\bullet(A, A)\) is a Batalin-Vilkovisky algebra.

Theorem (Getzler '94)
For any field \(K\), the homology \(H_\bullet(fD_2, K)\) of the framed little disks operad is the operad for Batalin-Vilkovisky \(K\)-algebras.
The dg- Deligne conjecture

**Theorem (MS ’02, KS ’02, Vo ’02, Ta ’04, BF ’04)**
The Hochschild cochain complex of an algebra $A$ admits a $C_\bullet(D_2)$-action inducing the Gerstenhaber structure on $HH^\bullet(A; A)$.

**Theorem (KS ’06, TZ ’06, Ka ’07, BB ’09)**
The Hochschild cochain complex of a symmetric algebra $A$ admits a $C_\bullet(fD_2)$-action inducing the $BV$-structure on $HH^\bullet(A; A)$.

**Proposition (Gerstenhaber-Voronov ’95, Menichi ’04)**
The Hochschild cochain complex of $A$ is isomorphic to the deformation complex of the endomorphism operad $\text{End}_A$ of $A$. If $A$ is symmetric, then $\text{End}_A$ is multiplicative cyclic.

**Proof.**
$C^n(A; A) = \text{Hom}(A^\otimes n, A) = \text{End}_A(n) \ni \mu_n$. For $f \in \text{End}_A(n)$, $\partial_0 f = \mu_2 \circ_1 f$, $\partial_n f = \mu_2 \circ_0 f$, $\partial_i f = f \circ_i \mu_2$ if $0 < i < n$. If $A$ is symmetric then $\text{End}_A$ is cyclic and $\tau_n(\mu_n) = \mu_n$. \qed
Multiplicative operads

Definition
A multiplicative (cyclic) operad is a non-symmetric (cyclic) operad $\mathcal{O}$ equipped with a map of (cyclic) operads $\text{Ass} \to \mathcal{O}$.
A multiplicative (cyclic) operad $\mathcal{O}$ has an underlying cosimplicial (cocyclic) object $\mathcal{O}^\bullet$. In a closed monoidal category $\mathcal{E}$ equipped with $\delta : \Delta \to \mathcal{E}$, the deformation complex of $\mathcal{O}$ is $\text{Hom}_\Delta(\delta^\bullet, \mathcal{O}^\bullet)$.

Example
For $\mathcal{E} = \text{Ch}(\mathbb{Z})$ and $\delta_\mathbb{Z} : \Delta \to \text{Ch}(\mathbb{Z}) : [n] \mapsto N_\bullet(\Delta[n]; \mathbb{Z})$ we get

$$C^\bullet(A; A) = \text{Hom}_\Delta(\delta^\bullet_\mathbb{Z}, \text{End}_A^\bullet).$$

Theorem (Kaufmann ’07, BB ’09)
For any multiplicative chain operad $\mathcal{O}$, the deformation complex of $\mathcal{O}$ admits a $C_\bullet(D_2)$-action. If $\mathcal{O}$ is multiplicative cyclic, this action extends to a $C_\bullet(fD_2)$-action.
The coloured operad for multiplicative operads

Let $L_2(n_1, \ldots, n_k; n)$ be the set of iso-classes of planar rooted trees with $n$ leaves and a bipartite vertex-set such that:

1. one part of the vertex-set is in bijection with $\{1, \ldots, k\}$;
2. the vertex with label $i$ has arity $n_i$;
3. each edge has at least one labelled extremity;
4. unlabelled vertices have arity $\neq 1$.

Let $C[n] = \mathbb{Z}/(n + 1)\mathbb{Z}$ and put

$$L_2^{cyc}(n_1, \ldots, n_k; n) = L_2(n_1, \ldots, n_k; n) \times C[n_1] \times \cdots \times C[n_k].$$

$L_2$ and $L_2^{cyc}$ are $\mathbb{N}$-coloured operads for an evident substitution of trees into trees; in $L_2^{cyc}$, the cyclic permutations distinguish for each labelled vertex one of its incident edges, the neutral element stands for the edge closest to the root of the tree.

**Lemma**

$L_2$-algebras are multiplicative operads; $L_2^{cyc}$-algebras are multiplicative cyclic operads. The category of unary operations of $L_2$ (resp. $L_2^{cyc}$) is $\Delta$ (resp. $\Delta C$).
Condensation of coloured operads

Unary operations of a coloured operad act covariantly on inputs and contravariantly on the output; therefore:

\[ L_2(-, \cdots, -; -) : \Delta^{\text{op}} \times \cdots \times \Delta^{\text{op}} \times \Delta \to \text{Sets}. \]

Given \( \delta_Z : \Delta \to \text{Ch}(\mathbb{Z}) \) we can realize multisimplicially, and totalize the resulting cosimplicial chain complex. This yields

\[
\xi(L_2, \delta_Z)(k) := \text{Hom}_{\Delta}(\delta^\bullet, |L_2(-, \cdots, -; \bullet)|_{\delta^\otimes k}), \quad k \geq 0.
\]

**Proposition (Day-Street '03, McClure-Smith '04, BB '09)**

\( \xi(L_2, \delta_Z) \) (resp. \( \xi(L_2^{\text{cyc}}, \delta_Z^{\text{cyc}}) \)) is a chain operad acting on the deformation complex of any multiplicative (cyclic) operad.

**Theorem (BB '09)**

As chain operads we have \( \xi(L_2, \delta_Z) \sim C_\bullet(D_2) \) and \( \xi(L_2^{\text{cyc}}, \delta_Z^{\text{cyc}}) \sim C_\bullet(fD_2) \).
The cobar complex of a bialgebra

**Theorem (cf. Gerstenhaber-Schack '92, Menichi '04)**

The cobar complex $\Omega A$ of a bialgebra (resp. involutive Hopf algebra) $A$ has an action by $\xi(L_2, \delta_Z)$ (resp. $\xi(L_2^{cyc}, \delta_Z^{cyc})$). Its homology $H_*(\Omega A; \mathbb{Z})$ is a Gerstenhaber (resp. BV-) algebra.

**Proof.**

The bialgebra $A$ is a comonoid in the monoidal category of $A$-modules. Therefore: $(\Omega A)_n = A^{\otimes n} \cong \text{Hom}_A(A, A^{\otimes n})$. This $\mathbb{Z}$-linear operad is multiplicative via the diagonal of $A$. If $A$ has an involutive antipode then the operad is multiplicative cyclic.

**Remark**

(a) $\Omega C_*(\Omega X; \mathbb{Z}) \sim C_*(\Omega^2 X; \mathbb{Z})$ (Adams). The $\xi(L_2, \delta_Z)$-action on $\Omega C_*(\Omega X; \mathbb{Z})$ corresponds to the $C_*(D_2)$-action on $C_*(\Omega^2 X; \mathbb{Z})$.

(b) If $A$ is involutive, the $\xi(L_2^{cyc}, \delta_Z^{cyc})$-action induces a cocyclic structure on $\Omega A$ yielding $HC^*(A)$ of Connes-Moscovici '99.

(c) $\xi(L_2, \delta_Z)$ contains the second filtration stage of the surjection operad of MS '03, BF '04 as a suboperad. Cyclic extension?
The topological Deligne conjecture

There is a cosimplicial resp. cocommutative space

\[ \delta_{\text{top}} : \Delta \to \text{Top} : [n] \mapsto \Delta^n \text{ resp. } \delta_{\text{cyc}}^{\text{top}} : \Delta \text{C} \to \text{Top} : [n] \mapsto \Delta^n \times S^1. \]

**Theorem (McClure-Smith '04, Salvatore '09, BB '09)**

The operad \( \xi(\mathcal{L}_2, \delta_{\text{top}}) \) is weakly equivalent to \( D_2 \) and acts on the deformation complex of multiplicative operads in spaces.

The operad \( \xi(\mathcal{L}_2^{\text{cyc}}, \delta_{\text{cyc}}^{\text{top}}) \) is weakly equivalent to \( fD_2 \) and acts on the deformation complex of multiplicative cyclic operads in spaces.

**Remark (cf. Markl '99, Salvatore-Wahl '03, Salvatore '09)**

\[ fD_2(k) \cong D_2(k) \times (S^1)^k, \xi(\mathcal{L}_2^{\text{cyc}}, \delta_{\text{cyc}}^{\text{top}})(k) \cong \xi(\mathcal{L}_2, \delta_{\text{top}})(k) \times (S^1)^k. \]

For \( n = 1 \):

\[ fD(1) \cong D(1) \times S^1, \text{Hom}_{\Delta \text{C}}(\delta_{\text{top}}^{\text{cyc}}, \delta_{\text{top}}^{\text{cyc}}) \cong \text{Hom}_{\Delta}(\delta_{\text{top}}, \delta_{\text{top}}) \boxtimes S^1. \]

**Proposition (Sinha '06)**

The simplicial 2-sphere \( S^2 = \Delta[2]/\partial \Delta[2] \) is an \( \mathcal{L}_2 \)-coalgebra in finite pointed sets. For a based space \((X, \ast)\), \( \Omega^2 X \) is the deformation complex of the multiplicative operad \((X, \ast)(S^2, \ast)\).
Braid and ribbon-braid groups

$\mathfrak{S}_k$ denotes the *permutation group* on $k$ letters. $\mathfrak{S}_k^\pm$ denotes the *signed permutation group* on $k$ letters.

$\mathfrak{S}_k^\pm = \mathfrak{S}_k \wr \mathfrak{S}_2 = \mathfrak{S}_k \ltimes (\mathfrak{S}_2)^k$ acts on $fD_2(k) = D_2(k) \times (S^1)^k$.

**Definition (Braid and ribbon-braid groups on $k$ strands)**

$B_k = \pi_1(D_2(k)/\mathfrak{S}_k)$  
$RB_k = \pi_1(fD_2(k)/\mathfrak{S}_k^\pm)$  
$PB_k = \pi_1(D_2(k))$  
$PRB_k = \pi_1(fD_2(k))$

**Proposition (Asphericity of $D_2(k)$ and $fD_2(k)$)**

$D_2(k)/\mathfrak{S}_k = K(B_k, 1)$  
$fD_2(k)/\mathfrak{S}_k^\pm = K(RB_k, 1)$  
$D_2(k) = K(PB_k, 1)$  
$fD_2(k) = K(PRB_k, 1)$

**Corollary**

The coverings $D_2(k) \to D_2(k)/\mathfrak{S}_k$ and $fD_2(k) \to fD_2(k)/\mathfrak{S}_k^\pm$ are classified by the short exact sequences $1 \to PB_k \to B_k \to \mathfrak{S}_k \to 1$ and $1 \to PRB_k \to RB_k \to \mathfrak{S}_k^\pm \to 1$.

**Problem**

Describe the operad structure of $D_2$ (resp. $fD_2$) in terms of the pure braid (resp. ribbon-braid) groups.
Coxeter geometry of permutation groups

\[ B_k = \langle s_1, \ldots, s_{k-1} \mid (s_i s_j)^2 = 1 \text{ if } |i - j| > 1 \text{ and } (s_i s_{i+1})^3 = 1 \rangle \]

The pure Artin group \( PB_k = \text{Ker}(B_k \to \mathfrak{S}_k) \cong \pi_1(\mathbb{C}^k - A_{\mathfrak{S}_k}) \)
where \( A_{\mathfrak{S}_k} \) is the complexified braid arrangement.

The Salvetti complex \( \text{Sal}_\mathfrak{S}_k \) is a partially ordered set of the same equivariant homotopy type as \( \mathbb{C}^k - A_{\mathfrak{S}_k} \).

\[ \text{Sal} : (\text{Coxeter groups}) \to (\text{posets}) \]

is a functor commuting with finite products. Thus, \((PB_k)_{k \geq 0}\) is a categorical operad. Similarly, \((PRB_k)_{k \geq 0}\) is a categorical operad.

**Proposition**

\( D_2 \sim K(PB, 1) \) and \( fD_2 \sim K(PRB, 1) \) as operads. Moreover, \( PB \)-algebras are braided strict monoidal categories; \( PRB \)-algebras are ribbon-braided (i.e. balanced) strict monoidal categories.

**Corollary (B ’98, Salvatore-Wahl ’03)**

The nerve of a braided (resp. ribbon-braided) strict monoidal category is \( E_2 \) (resp. framed \( E_2 \)).
The categorical Deligne conjecture

Consider the cosimplicial category

$$\delta_{\text{Cat}} : \Delta \to \text{Cat} : [n] \mapsto [n][n]^{-1}$$

Proposition

There are weak equivalences of categorical operads

$$PB \sim \xi(L_2, \delta_{\text{Cat}}) \quad \text{and} \quad PRB \sim \xi(L_2^{\text{cyc}}, \delta_{\text{Cat}}).$$

Definition

A central element of a monoidal category $E$ is a pair $(A, c_A)$ where

$$c_{A,-} : A \otimes - \cong - \otimes A \quad \text{and} \quad c_{A,B \otimes C} = (1_B \otimes c_{A,C}) \circ (c_{A,B} \otimes 1_C).$$

The center $\mathcal{Z}E$ is the category of central elements.

Proposition

For $E = \text{Mod}_H$, $\mathcal{Z}E \simeq \text{Mod}_{DH}$ where $DH$ is the Drinfeld double of the Hopf algebra $H$. 
The Drinfeld double of a Hopf algebra

Proposition (Street ’04)
\[ \mathcal{Z} \mathcal{E} = \text{Hom}_\Delta(\delta_{\text{Cat}}, \text{End}\mathcal{E}) \]

Corollary
The center of a monoidal category is braided monoidal; in particular, the Drinfeld double of a Hopf algebra is “braided”.

Definition
An involutive category is a closed monoidal category \( \mathcal{E} \) such that the duality functor \( (\cdot)^* = \text{Hom}(\cdot, I) \) is self-adjoint. A Hopf algebra \( H \) is called quasi-involutive if \( \text{Mod}_H \) is involutive.

Proposition
The category \( \mathcal{E}_f \) of symmetric duality objects of an involutive category \( \mathcal{E} \) has a multiplicative cyclic endomorphism-operad \( \text{End}\mathcal{E}_f \).

Corollary
The center of \( \mathcal{E}_f \) is ribbon-braided; in particular, the Drinfeld double of a quasi-involutive Hopf algebra is “ribbon-braided”.