Algebraic and homotopical nilpotency

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joint work with Dominique Bourn

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Definition

A group G is *n-nilpotent* if commutators of length $n+1$ vanish:

$$[x_1,[x_2,[x_3,\ldots,[x_n,x_{n+1}]\cdots]]] = e_G \quad \forall x_1,\ldots,x_{n+1} \in G.$$

Definition

A *central group extension* is a surjective group homomorphism $f : G \twoheadrightarrow H$ with kernel $K[f]$ contained in the center of G.

Lemma

A group G is n-nilpotent iff it is an *n-fold central extension* of the trivial group, i.e. $G \overset{f_n}{\twoheadrightarrow} G_{n-1} \overset{f_{n-1}}{\twoheadrightarrow} \cdots G_2 \overset{f_2}{\twoheadrightarrow} G_1 \overset{f_1}{\twoheadrightarrow} \star$ with f_i central.

Problem

In which categories is there a notion of central extension ?

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In which categories is there a notion of central extension ?

Definition (Janelidze-Màrki-Tholen '01)

A category is called *semi-abelian* if it is

- σ-pointed (i.e. with null-object and binary sums);
- exact (Barr '71);
- protomodular (Bourn '91).

Proposition (Bourn '96)

- A pointed category with pullbacks is *protomodular* iff section and kernel of every split epi $f : X \twoheadrightarrow Y$ strongly generate X;
- A σ-pointed category with pullbacks is *protomodular* iff for every split epi $f : X \twoheadrightarrow Y$ with section $s_f : Y \rightarrowtail X$ the morphism $< s_f, i_f >: Y + K[f] \twoheadrightarrow X$ is a strong epimorphism;
- Any protomodular category is a *Mal'cev category* [CKP '93], i.e. reflexive relations are equivalence relations.

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Definition

A *variety* V_T is the cat. of algebras for a *finitary* monad T on *sets*.

Theorem (Lawvere '63)

Each variety V_T is determined by an *algebraic theory* Θ_T where

$$\begin{cases} \mathrm{Ob}\,\Theta_T = \mathbb{N}; \\ \Theta_T(m,n) = \mathrm{Alg}_T(F_T(\{1,\dots,m\}), F_T(\{1,\dots,n\})). \end{cases}$$

Theorem (Mal'cev)

The variety V_T is a Mal'cev category iff $\exists p \in \Theta_T(1,3)$ such that

$$p^*(x,y,y) = x \text{ and } p^*(x,x,y) = y \text{ for any } T\text{-algebra.}$$

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Basic hypothesis (on the ambient category $\mathbb{E}$)

$(\mathbb{E}, \star_{\mathbb{E}})$ is a σ-pointed exact Mal'cev category

e.g. any semi-abelian category or any pointed Mal'cev variety.

Lemma

Exact Mal'cev categories have reflexive coequalizers;
σ-pointed exact Mal'cev categories have all finite colimits.

Examples (of semi-abelian categories)

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A subobject N of X is *central* iff the inclusion of N into X commutes with the identity of X (in the sense of Huq).
An equivalence relation R on X is *central* iff R commutes with the indiscrete equivalence relation on X (in the sense of Smith).

Proposition (Gran-Van der Linden '08)

In a pointed protomodular category with pullbacks, a regular epimorphism $f : X \twoheadrightarrow Y$ has a central kernel $K[f]$ if and only if the kernel pair $R[f] \rightrightarrows X$ is a central equivalence relation.

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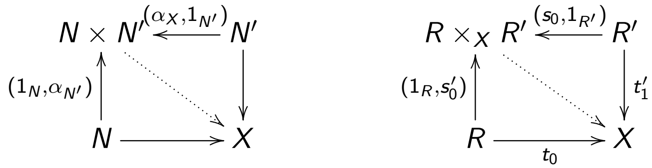

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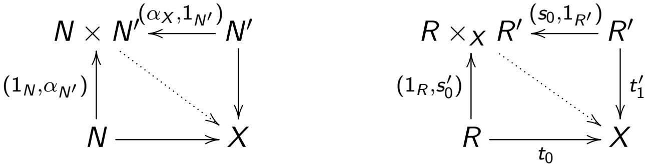

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Definition (Nilpotency)

- Central extension = regular epi with central kernel pair;
- An *n-nilpotent object* is an n-fold central extension of $\star_{\mathbb{E}}$;
- $\mathrm{Nil}^n(\mathbb{E})$ is the subcategory spanned by the n-nilpotent objects;
- A category is *n-nilpotent* iff all its objects are n-nilpotent.

Remark

The abstract notion of n-nilpotent object yields for groups (Lie algebras) the classical notion of n-nilpotent group (Lie algebra).

Proposition

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A reflective subcategory $\mathbb{D}$ of $\mathbb{E}$ is a *Birkhoff subcategory* iff $\mathbb{D}$ is closed under taking subobjects and quotients in $\mathbb{E}$.

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This is the case iff for each regular epi $f : X \twoheadrightarrow Y$ the reflection $I : \mathbb{E} \to \mathbb{D}$ induces a *cocartesian* naturality square of reg. epi's

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The unit of a central reflection is pointwise an *affine extension*. Any morphism inverted by a central reflection is *affine*.

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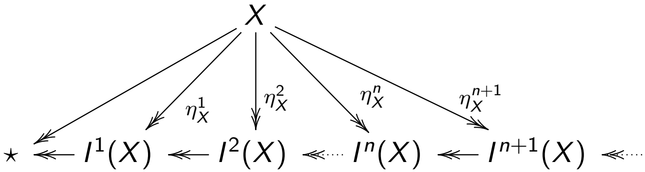

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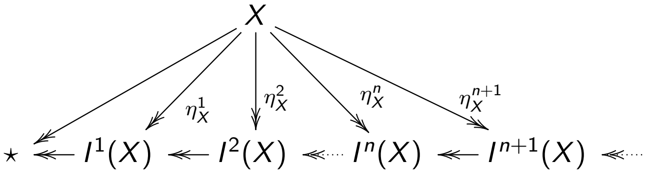

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Definition (Fibration of "points", Bourn '96)

Let $\mathrm{Pt}_X(\mathbb{E})$ be the category of split epis of $\mathbb{E}$ with codomain X. Each $f : X \to Y$ induces an adjunction $f_! : \mathrm{Pt}_X(\mathbb{E}) \leftrightarrows \mathrm{Pt}_Y(\mathbb{E}) : f^*$. A morphism f is *affine* iff $(f_!, f^*)$ is an adjoint equivalence.

Example (split epimorphisms in groups)

A split epimorphism $f : G \twoheadrightarrow X$ in groups

- exhibits G as a *semi-direct product* $X \ltimes_\phi K[f]$;
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For $\mathbb{E}$ =(groups) or $\mathbb{E}$ =(Lie algebras), pointed base-change $f^* : \mathrm{Pt}_{\mathbb{E}}(Y) \to \mathrm{Pt}_{\mathbb{E}}(X)$ has a left adjoint $f_!$ *and* a right adjoint f_*.

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Proposition

Pointed base-change along regular epi's is fully faithful.

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A regular epi $f : X \twoheadrightarrow Y$ is an *affine extension* iff for all objects Z either of the following two squares is cartesian

$$\begin{array}{ccc} X+Z & \xrightarrow{f+Z} & Y+Z \\ \downarrow{\scriptstyle \pi^Z_X} & & \downarrow{\scriptstyle \pi^Z_Y} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad \begin{array}{ccc} X+Z & \xrightarrow{f+Z} & Y+Z \\ \downarrow{\scriptstyle \theta_{X,Z}} & & \downarrow{\scriptstyle \theta_{Y,Z}} \\ X\times Z & \xrightarrow[f\times Z]{} & Y\times Z \end{array}$$

Corollary (for semi-abelian categories)

$f : X \twoheadrightarrow Y$ affine $\iff$ $f \diamond Z : X \diamond Z \cong Y \diamond Z$ invertible $\forall Z$
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$\mathbb{E}$ is n-nilpotent iff for all X, Y the map $\theta_{X,Y} : X + Y \to X \times Y$ exhibits $X + Y$ as an $(n-1)$-fold central extension of $X \times Y$.

Proof.

$$\begin{array}{ccccc} X \diamond X & \rightarrowtail & X + X & \xrightarrow{\theta_{X,X}} & X \times X \\ \downarrow & & \downarrow{\scriptstyle \delta^2_X} & & \downarrow \\ [X,X] & \rightarrowtail & X & \xrightarrow{\eta^1_X} & I^1(X) \end{array}$$

$\theta_{X,X}$ $(n-1)$-fold central ext. $\implies$ η^1_X $(n-1)$-fold central ext. □

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$\mathbb{E}$ 2-nilpotent iff $\theta_{X,Y}$ central extension iff $\theta_{X,Y}$ affine extension.

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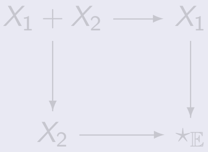

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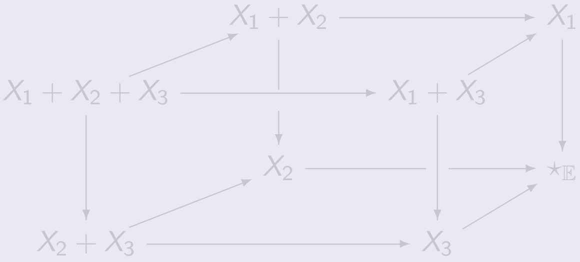

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$$\Xi_{X_1,X_2} \qquad \begin{array}{ccc} X_1 + X_2 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & \star_{\mathbb{E}} \end{array}$$

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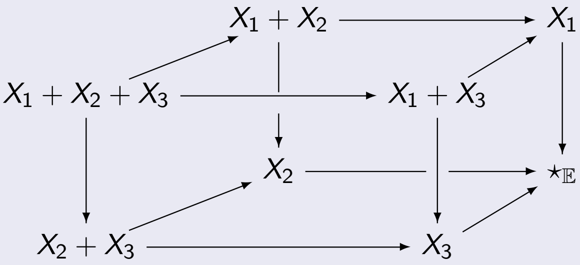

Definition (cubical cross-effects)

- $P_{X_1,\dots,X_n} = \varprojlim_{[0,1]^n \setminus \{(0,\dots,0)\}} \check{\Xi}$ (limit of the punctured cube);
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If $\mathbb{E}$ has an identity functor of degree $\leq n$ then $\mathbb{E}$ is n-nilpotent.

Theorem

$\mathbb{E}$ has a quadratic identity functor iff $\mathbb{E}$ is 2-nilpotent and moreover one of the following two conditions is satisfied for all X, Y, Z:

- $(X \times Y) + Z \cong (X + Z) \times_Z (Y + Z)$ (alg. codistributivity)
- $(X \times Z) +_Z (Y \times Z) \cong (X + Y) \times Z$ (alg. distributivity)

Corollary

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Proposition

If $\mathbb{E}$ has an identity functor of degree $\leq n$ then $\mathbb{E}$ is n-nilpotent.

Theorem

$\mathbb{E}$ has a quadratic identity functor iff $\mathbb{E}$ is 2-nilpotent and moreover one of the following two conditions is satisfied for all X, Y, Z:

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Definition (n-additivity)

$$\begin{array}{ccccc}
cr_{n+1}(X,\ldots,X) & \rightarrowtail & X+\cdots+X & \overset{\theta_{X,\ldots,X}}{\twoheadrightarrow} & P_{X,\ldots,X} \\
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X is *n-additive* iff δ_X^{n+1} factors through $\theta_{X,\ldots,X}$. For semi-abelian $\mathbb{E}$, this amounts to vanishing Higgins commutator of length $n+1$.

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Lemma

$$(c) \implies (b) \implies (a)$$

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$F : \mathbb{E} \to \mathbb{E}'$ is θ-linear iff $F(\theta_{X,Y})$ invertible $\forall X, Y$.

Example

Abelianization $\mathbb{E} \to \mathrm{Nil}^1(\mathbb{E})$ is θ-linear.

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If $\mathbb{E}$ is alg extensive and has multi-θ-linear n-th cross-effect then $\mathbb{E}$ has an identity functor of degree $\leq n$.

Proposition

Any alg extensive n-nilpotent $\mathbb{E}$ has multi-θ-linear n-th cross-effect.

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If $\mathbb{E}$ is alg extensive then $\mathrm{Nil}^n(\mathbb{E})$ has an identity of degree $\leq n$. Each n-nilpotent object is n-additive (iterated Huq=Higgins).

Examples

This is the case for the category of groups, resp. Lie algebras.

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Definition (Quillen model category)

A Quillen model structure on a bicomplete $\mathbb{E}$ consists of three composable classes of morphisms $\mathrm{cof}_{\mathbb{E}}, \mathrm{we}_{\mathbb{E}}, \mathrm{fib}_{\mathbb{E}}$ such that

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Which constructions of $\mathbb{E}$ carry over to $\mathrm{Ho}(\mathbb{E})$?

Example (pullback vs homotopy pullback)

Even if α, β, γ are we's, δ is NOT a we in general. Yet, if moreover f, g, f', g' are fibrations between fibrant objects, then δ is a we !

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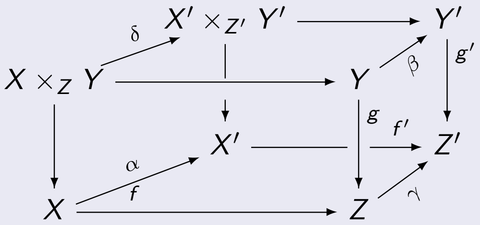

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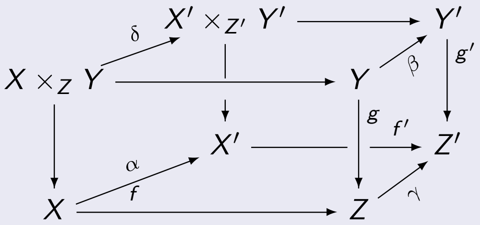

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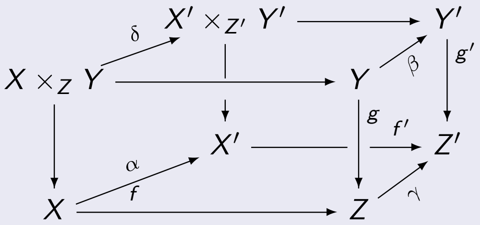

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For each pointed Mal'cev variety V_T $\exists$ *model structure* on sV_T sth

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For each pointed Mal'cev variety V_T $\exists$ *model structure* on sV_T sth

- we's are the maps inducing a quasi-iso on *Moore complexes*;
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For cofibrant objects $X_1, \ldots, X_n$ in sV_T the n-th "algebraic" cross-effect $cr_n(X_1, \ldots, X_n)$ is *homotopy-invariant*.

Definition (Homotopical nilpotency degrees)

Let X be a cofibrant object in sV_T.

- $\mathrm{nil}_1^T(X) = n$ iff n is the least integer for which $\eta_X^n : X \to I^n(X)$ is a trivial fibration;
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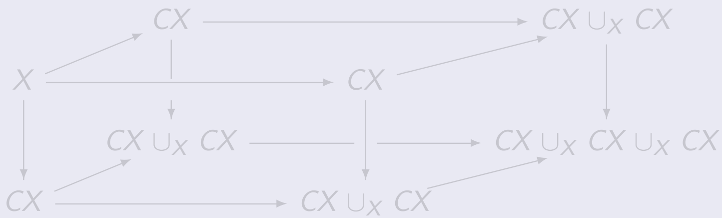
A homotopy functor is n-excisive if it takes cofibration $(n+1)$-cubes to h-cartesian $(n+1)$-cubes.

Enough to check the following cofibration cubes ($n = 3$):



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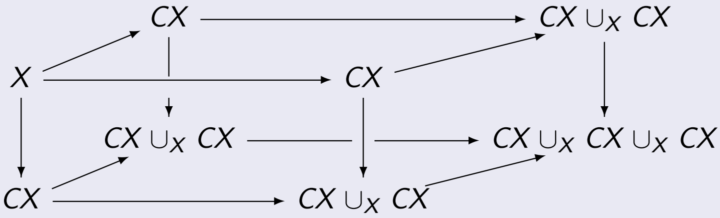

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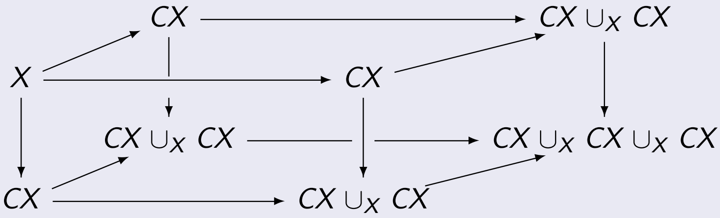

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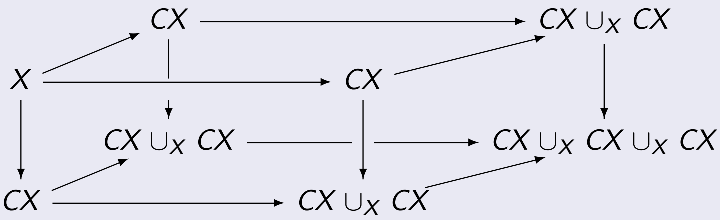

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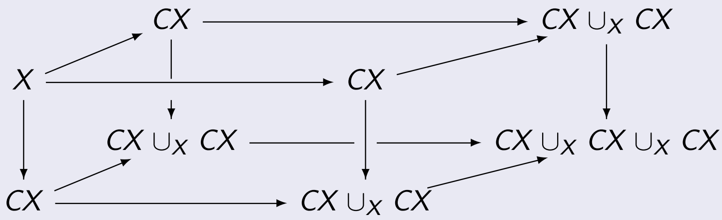

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Theorem (special case $V_T = \text{(groups)}$, Kan '56, Quillen '66)

The free-forgetful adjunction $U : sGr \leftrightarrows s\mathrm{Sets}_* : F$ breaks into two adjunctions

$$sGr \underset{G}{\overset{\overline{W}}{\rightleftarrows}} s\mathrm{Sets}_{red} \underset{\Sigma}{\overset{\Omega}{\rightleftarrows}} s\mathrm{Sets}_*$$

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Simplicial groups model loop spaces/loop maps, resp. based *connected* spaces and based maps. The free simpl. group gen. by a pointed simpl. set X is a model of $\Omega\Sigma|X|$, cf. James.

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For a reduced simplical set X one has

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For any based connected space X one has

$$\mathrm{nil}_{BG}(\Omega X) \leq \mathrm{cocat}_{Hov}(X) \leq \mathrm{nil}_{BD}(\Omega X)$$

Remark (Lusternik-Schnirelmann '34, Whitehead '56)

$\mathrm{cat}_{LS}(X) \leq n \iff \exists$ open cover $X \subset U_1 \cup \cdots \cup U_{n+1}$ sth. $U_i \hookrightarrow X$ null-homotopic $\forall\, i \iff$ the diagonal $X \to X^{n+1}$ can be deformed into the fat wedge $Q_{X,\ldots,X} = \{x \in X^{n+1} \,|\, \exists\, i : x_i = \star\}$.