Dold-Kan categories & Catalan monoids

Clemens Berger[1]

Toulouse, 20-24 Juin, 2022

CATS60 – celebrating Carlos Simpson's 60th birthday

[1] joint with Christophe Cazanave and Ingo Waschkies

Theorem (Dold 1958, Kan 1958)

$$M : \underline{\mathrm{Ab}}^{\Delta^{\mathrm{op}}} \simeq \mathrm{Ch}(\mathbb{Z}) : K$$

Remark

The functor K takes homology to homotopy. The K-image of the chain complex $(A, n) = (0 \leftarrow \cdots \leftarrow 0 \leftarrow \underset{n}{A} \leftarrow 0 \leftarrow \cdots)$ is a *simplicial* model for an Eilenberg-MacLane space of type $K(A, n)$.

Purpose of the talk

- categorical explanation for Dold-Kan correspondence
- chain models for $K(A, n)$'s via Joyal's cell categories Θ_n
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Definition (simplex category Δ)

$\mathrm{Ob}\Delta = \{[n] = \{0, 1 \ldots, n\}, n \geq 0\}$, $\mathrm{Mor}\Delta = \{\text{monotone maps}\}$

Remark ($\mathcal{E}$-$\mathcal{M}$ factorisation system)

The category Δ is generated by elementary

- *face operators* $\epsilon_i^n : [n-1] \to [n]$, $0 \leq i \leq n$, and
- *degeneracy operators* $\eta_i^n : [n+1] \to [n]$, $0 \leq i \leq n$.

Every simplicial operator $\phi : [m] \to [n]$ factors as

$$\begin{tikzcd}
{[m]} \arrow[rr, "\phi"] \arrow[dr, "epi"'] & & {[n]} \\
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Definition (Milnor 1957 – geometric realisation)

The functor $\Delta \to \mathrm{Top} : [n] \mapsto \Delta_n$ yields by left Kan extension geometric realisation $|-|_\Delta : \mathrm{Sets}^{\Delta^{\mathrm{op}}} \to \mathrm{Top}$. Each $|X|$ is a *CW-complex* with one cell per *non-degenerate* simplex of X.

Definition (Eilenberg 1944 – simplicial homology)

$$\begin{array}{ccccccc} \mathrm{Sets}^{\Delta^{\mathrm{op}}} & \longrightarrow & \underline{\mathrm{Ab}}^{\Delta^{\mathrm{op}}} & \xrightarrow{C} & \mathrm{Ch}(\mathbb{Z}) & \longrightarrow & \underline{\mathrm{Ab}}^{\mathbb{N}} \\ X_\bullet & \longmapsto & \mathbb{Z}[X_\bullet] & \longmapsto & (C_\bullet(X), d_\bullet) & \longmapsto & H_\bullet(X) \end{array}$$

There are canonical isomorphisms

$$C_n^{cell}(|X|) \cong C_n(X) = \mathbb{Z}[X_n]/\mathbb{Z}[D_n(X)] \cong \bigcap_{0\leq k<n} \ker(\epsilon_k^n) = M_n(X)$$

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$\mathcal{C} = (\mathcal{E}, \mathcal{M}, (-)^*)$ is a DK-category whenever $(-)^* : \mathcal{E}^{\mathrm{op}} \to \mathcal{M}$ is a faithful identity-on-objects functor sth.

(1) $ee^* = 1$ (the idempotent e^*e is called an *$\mathcal{E}$-projector*);
(2) the morphisms f^*e (for $e, f \in \mathcal{E}$) form a subcategory of $\mathcal{C}$;
(3) Inessential $\mathcal{M}$-maps form an ideal in $\mathcal{M}$;
(4) $\mathrm{Proj}_{\mathcal{E}}(A)$ is finite. *Primitive* $\mathcal{E}$-projectors can be enumerated in such a way that $\phi_j\phi_i$ is an $\mathcal{E}$-projector for $i < j$.

Definition (primitive $\mathcal{E}$-projectors e^*e)

Whenever $e = e_2e_1$ then either e_1 or e_2 is invertible.

Definition (essential and inessential $\mathcal{M}$-maps)

An $\mathcal{M}$-map $m : A \to B$ is called *essential* if 1_B is the only $\mathcal{E}$-projector of B fixing m. Otherwise m is called *inessential*.

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Each epi $e : [m] \twoheadrightarrow [n]$ has a *maximal* section $e^* : [n] \to [m]$.
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are precisely the "last" face operators $\epsilon_n^n : [n-1] \rightarrowtail [n]$.

Lemma (quotienting out inessential $\mathcal{M}$-maps)

By axiom (3), there is a *locally pointed* category $\Xi_{\mathcal{C}} = \mathcal{M}/\mathcal{M}_{iness}$.

Remark (description of $\Xi_\Delta = \mathcal{M}/\mathcal{M}_{iness}$)

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For any small category $\mathcal{A}$ the category $\Delta \wr \mathcal{A}$ is defined by

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Put $\Theta_1 = \Delta$ and $\Theta_n = \Delta \wr \Theta_{n-1}$ for $n > 1$.

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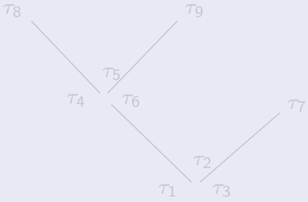

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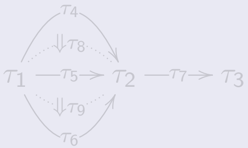

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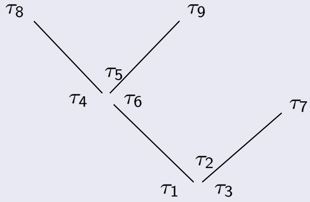

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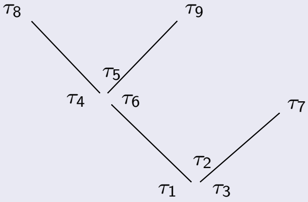

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Any presheaf $X : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ on a geometric DK-category has *CW*-realisation $|X|$ whose chain complex $C_*^{cell}(|X|)$ is isomorphic to the "totalisation" of the Moore normalisation $M_{\mathcal{C}}(\mathbb{Z}[X])$.

Proposition (B 2007, Bergner-Rezk 2017, BCW 2022)

If $\mathcal{A}$ is a geometric DK-category then so is $\Delta \wr \mathcal{A}$.
For instance, Joyal's cell category Θ_n is a geometric DK-category.

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Proposition (CW-realisation)

Any presheaf $X : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ on a geometric DK-category has CW-realisation $|X|$ whose chain complex $C_*^{cell}(|X|)$ is isomorphic to the "totalisation" of the Moore normalisation $M_{\mathcal{C}}(\mathbb{Z}[X])$.

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Example (# cells of $K(\mathbb{Z}/2\mathbb{Z}, n)$ = generalised Fibonacci number)

# cells in dim	0	1	2	3	4	5	6	7	8	9	10
$K(\mathbb{Z}/2\mathbb{Z}, 1)$	1	1	1	1	1	1	1	1	1	1	1
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Example (action of Ξ_{Θ_2} on $C_*^{cell}(K(Z/2,2))$ for $2 \leq * \leq 6$)

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\begin{tikzcd}
 & & & & (1;5) \arrow[dl, "0"'] \\
 & & & (1;4) \arrow[dl, "0"'] & (2;3,1) \arrow[l, squiggly] \\
(1;1) & (1;2) \arrow[l, "0"'] & (1;3) \arrow[l, "0"'] & (2;2,1) \arrow[l, Rightarrow] \arrow[dl] & (2;2,2) \arrow[l, "0"'] \arrow[ul, squiggly] \arrow[dl, "0"'] \\
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Theorem (Serre 1953)

$$H^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[Sq^J(\iota_2),\ J\,\text{admissible},\ e(J) < n]$$

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Proposition

Let $(x_i)_{1\leq i\leq n}$ be a family of projectors of an R-module X such that $x_ix_jx_i = x_ix_j = x_jx_ix_j$ for $i<j$. Then we get a direct sum decomposition $X = N_X \oplus D_X := \bigcap_{1\leq i\leq n}\ker(x_i) \oplus \sum_{1\leq i\leq n}\mathrm{im}(x_i)$.

Corollary

Let $X : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ be a presheaf on a Dold-Kan category $\mathcal{C}$ with $\mathcal{A}$ abelian. Then, for each object A of $\mathcal{C}$, we get

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Definition

Let Γ by a finite quiver with $V(\Gamma) = \{1, \ldots, n\}$ and edge set $E(\Gamma) \subset V(\Gamma) \times V(\Gamma)$ such that if $(i,j) \in E(\Gamma)$ then $i < j$. The *Catalan monoid* C_Γ is generated by $x_i, i \in V(\Gamma)$, with relations:

- $x_i^2 = x_i$ for $i \in V(\Gamma)$;
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Proposition (Kudryatseva-Mazorchuk 2009)

Every Catalan monoid C_Γ is finite and has $2^{\#V(\Gamma)}$ idempotents. The unit of $\mathbb{Q}[C_\Gamma]$ is a sum of $2^{\#V(\Gamma)}$ pairwise orth. idempotents:

$$1 = \sum_{\{i_1,\ldots,i_k\} \sqcup \{j_1,\ldots,j_{n-k}\} = \underline{n}} x_{i_k} \cdots x_{i_2} x_{i_1} (1 - x_{j_1})(1 - x_{j_2}) \cdots (1 - x_{j_{n-k}}).$$

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Remark (Catalan monoid rings are semi-perfect)

The idempotents of C_Γ induce the simple modules while the decomposition of 1 induces the irreducible components of $\mathbb{Q}[C_\Gamma]$.

Example (Catalan monoids inside Δ)

- The submonoid $C_{[n]} \subset \Delta([n],[n])$ generated by the primitive projectors $x_i = \epsilon_i \eta_i$ $(0 \leq i < n)$ is the Catalan monoid C_{L_n} of the *linear quiver* because $x_i x_j = x_j x_i$ if $|i-j| \geq 2$.
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Remark (Kiselman monoids C_{K_n})

The cardinalities of C_{K_n} for the complete quivers K_n are not known.

Remark (Catalan monoid rings are semi-perfect)

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