The nerve theorem and Grothendieck's hypothesis on homotopy types

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A *monadic square* is a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{E}_1' & \xrightarrow{G'} & \mathcal{E}_2' \\ {\scriptstyle U_1}\downarrow & = & \downarrow{\scriptstyle U_2} \\ \mathcal{E}_1 & \xrightarrow[G]{} & \mathcal{E}_2 \end{array}$$

such that

(i) U_1, U_2 are monadic functors with left adjoints F_1, F_2;

(ii) the induced 2-cell $\phi = \epsilon_2 G' F_1 \circ F_2 G \eta_1$ (the "mate")

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Let $(T_1, \mu_1, \eta_1), (T_2, \mu_2, \eta_2)$ be monads on $\mathcal{E}_1, \mathcal{E}_2$ respectively.

A *(strong) monad morphism* $(G, \psi) : (\mathcal{E}_1, T_1) \to (\mathcal{E}_2, T_2)$ is a functor $G : \mathcal{E}_1 \to \mathcal{E}_2$ together with an (invertible) 2-cell $\psi : T_2 G \Rightarrow G T_1$ such that $G\eta_1 = \psi \circ \eta_2 G$ and $\psi \circ \mu_2 G = G\mu_1 \circ G\psi T_1 \circ T_2 \psi G$.

A strong monad morphism (G, ψ) induces a monadic square

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with $G'(X, \xi : T_1 X \to X) = (GX, G\xi \circ \psi : T_2 GX \to GX)$

Conversely, a monadic square induces a strong monad morphism from which it derives up to canonical equivalence.

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Proposition

In any monadic square like above, if G is faithful (resp. fully faithful resp. an equivalence) then so is G'.

Proposition

For a fully faithful functor G, the essential image factorisation of G decomposes the monadic square into two monadic squares

$$\begin{array}{ccccc}
\mathcal{E}_1' & \xrightarrow{\sim} & Im(G) \times_{\mathcal{E}_2} \mathcal{E}_2' & \overset{ff}{\hookrightarrow} & \mathcal{E}_2' \\
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A *category with arities* $(\mathcal{E}, \Theta_0)$ is a category $\mathcal{E}$ equipped with a small dense subcategory $i_0 : \Theta_0 \hookrightarrow \mathcal{E}$, i.e. the induced *nerve functor* $\nu_0 : \mathcal{E} \to \widehat{\Theta}_0 : X \mapsto \mathcal{E}(i_0(-), X)$ is fully faithful.

For each object X of $\mathcal{E}$ the functor $\xi_X : i_0/X \to \Theta_0 \hookrightarrow \mathcal{E}$ induces a colimit cocone $\mathrm{colim}_{i_0/X}\xi_X \xrightarrow{\cong} X$.

A *monad with arities* on $(\mathcal{E}, \Theta_0)$ is a monad T such that the composite functor $\nu_0 \circ T$ preserves the Θ_0-colimit cones.

The *theory* Θ_T induced by a monad with arities T is obtained by factoring $\Theta_0 \xrightarrow{i_0} \mathcal{E} \xrightarrow{F_T} \mathrm{Alg}_T$ into a bijective-on-objects functor $j : \Theta_0 \to \Theta_T$ followed by a fully faithful functor $\Theta_T \to \mathrm{Alg}_T$.

Θ_T is called *homogeneous* if Θ_T admits a *generic/free* factorisation system $\Theta_T = (\Theta_{T,gen}, \Theta_0)$.

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Example (algebraic theories and symmetric operads)

Consider sets with arities $\mathcal{T}_0$ the subcategory of finite sets.

- A monad T has arities $\mathcal{T}_0$ iff T preserves filtered colimits;
- Θ_T is (the dual of) Lawvere's algebraic theory for T-algebras;
- Θ_T is homogeneous iff T is induced by a symmetric operad.

Theorem (B. '02, Leinster '04, Weber '07, Mellies '10)

For a monad with arities T on $(\mathcal{E}, \Theta_0)$, the theory Θ_T is dense in Alg_T. The essential image of $\nu_T : \mathrm{Alg}_T \to \widehat{\Theta}_T$ is spanned by those $X : \Theta_T^{\mathrm{op}} \to \mathrm{Sets}$ whose restriction j^*X belongs to $Im(\nu_0)$.

Remark

If $\mathcal{E} = \widehat{\mathbb{C}}$ and Θ_0 contains the representables, the essential image of $\nu_0 : \widehat{\mathbb{C}} \to \widehat{\Theta}_0$ is spanned by sheaves on Θ_0. The essential image of $\nu_T : \mathrm{Alg}_T \to \widehat{\Theta}_T$ is then given by a *restricted* sheaf condition.

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Proof of the nerve theorem.

Since T is a monad with arities on $(\mathcal{E}, \Theta_0)$ the square

$$\begin{array}{ccc} \mathrm{Alg}_T & \xrightarrow{\nu_T} & \widehat{\Theta}_T \\ {\scriptstyle U_T}\downarrow & & \downarrow{\scriptstyle j^*} \\ \mathcal{E} & \xrightarrow{\nu_0} & \widehat{\Theta}_0 \end{array}$$

is *pseudo*monadic and ν_0 is fully faithful. □

A *theory* on $(\mathcal{E}, \Theta_0)$ is a bijective-on-objects faithful functor $j : \Theta_0 \to \Theta_T$ such that $j^* j_!$ preserves the essential image of ν_0.

Theorem (B. '02, Mellies '10)

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Each *finite level tree* S defines a globular set S_* with $\mathrm{ht}(S) = \dim(S_*)$ (Batanin's star-construction '98).

The category of arities Θ_0 is the full subcategory of $\widehat{\mathbb{G}}$ spanned by the S_* where S runs through the set of finite level trees.

The Grothendieck topology on Θ_0 induced by the nerve $\nu_0 : \widehat{\mathbb{G}} \to \widehat{\Theta}_0$ has the caracteristic property that a presheaf X on Θ_0 is a sheaf if and only if X transforms the canonical colimit cones

$$\mathrm{colim}_{\sigma \in el(S_*)} \sigma \xrightarrow{\cong} S_*$$

into limit cones.

A theory Θ_A on $(\mathbb{G}, \Theta_0)$ is called *globular*. The presheaves X such that j^*X is a sheaf are called Θ_A-*models*. According to the nerve theorem they correspond to $\underline{A}$-algebras for a monad $\underline{A}$ on $\widehat{\mathbb{G}}$.

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$$\begin{array}{ccccc} \mathrm{Alg}_{\underline{A}} & \xrightarrow{\sim} & \mathrm{Mod}_{\Theta_A} & \hookrightarrow & \widehat{\Theta}_A \\ {\scriptstyle U_{\underline{A}}}\downarrow & & \downarrow & & \downarrow{\scriptstyle j^*} \\ \widehat{\mathbb{G}} & \xrightarrow{\sim} & \mathrm{Sh}(\Theta_0) & \hookrightarrow & \widehat{\Theta}_0 \end{array}$$

A globular theory Θ_A is called *homogeneous* if there is a factorisation system $\Theta_A = (\Theta_{A,gen}, \Theta_0)$ such that each *generic* operator $\phi : S \to T$ satisfies $\mathrm{ht}(S) \geq \mathrm{ht}(T)$.

Theorem (Makkai-Zawadowski '01, B. '02)

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Corollary

- There is a canonical one-to-one correspondence between homogeneous n-globular theories and globular n-operads;
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Example (n=1, Segal condition)

The terminal graphical theory is the simplex category Δ.

$$\begin{array}{ccccc} \mathrm{Cat} & \xrightarrow{\sim} & \mathrm{Mod}_\Delta & \hookrightarrow & \widehat{\Delta} \\ {\scriptstyle U}\downarrow & & \downarrow & & \downarrow{\scriptstyle j^*} \\ \widehat{\mathbb{G}_1} & \xrightarrow{\sim} & \mathrm{Sh}(\Delta_0) & \hookrightarrow & \widehat{\Delta_0} \end{array}$$

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The *wreath product* $\Delta \wr \mathcal{A}$ is the category

- with objects $([m], a_1, \ldots, a_m) \in \Delta \times \mathcal{A}^m$, $m \geq 0$;
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Proposition (B. '07, Steiner '07, Oury '10)

$$\Theta_{n+1} = \Delta \wr \Theta_n \quad (n \geq 1)$$

This identification is compatible with the theory structures.

Remark (Batanin's category of quasi-bijections '10)

- If $\mathcal{A}$ is augmented over Segal's category Γ then so is $\Delta \wr \mathcal{A}$.
- There is thus a canonical functor $\gamma_n : \Theta_n \to \Gamma$ for each $n \geq 1$.
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Each topological space X is (weakly) homotopy equivalent to the inverse limit of its Postnikov tower

$$\cdots \longrightarrow X_{\leq n+1} \longrightarrow X_{\leq n} \longrightarrow \cdots$$

In principle this allows to reconstruct the homotopy type of X through cohomological invariants, called *Postnikov invariants* of X.

The fundamental groupoid $\Pi_1(X)$ captures the homotopy type of the Postnikov section $X_{\leq 1}$, but it is known that for $n \geq 3$ there cannot exist a strict fundamental n-groupoid capturing the homotopy type of $X_{\leq n}$ for all X.

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Strict n-categories are Θ_n-sets fulfilling a restricted sheaf condition.

"Weak" n-categories are Θ_n-spaces which are *fibrant* for a Quillen model structure on Θ_n-spaces, introduced by Rezk '10.

These fibrant Θ_n-spaces (the Rezk n-categories) are essentially those Θ_n-spaces X for which j^*X is a *homotopy sheaf* on $\Theta_{n,0}$.

Rezk proves Grothendieck's hypothesis for his n-groupoids.

There are discrete versions Rezk's n-categories:

- Segal n-categories, i.e. fibrant objects for a suitable model structure on Θ_n-spaces which are discrete on Θ_{n-1};
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These discretesized model structures have been shown to exist only for $n = 1$, cf. Joyal-Tierney '07 !!

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