Derived modular operads and metric ribbon graphs

Clemens Berger[1]

University of Nice

Homotopical Algebra and Higher Structures
Oberwolfach, September 20-25, 2021

[1] joint with Ralph Kaufmann (Purdue University)

Definition (moduli space for oriented surfaces/ribbon graphs)

- $\mathcal{M}_{g,n}$ moduli space of *hyperbolic metrics* on a closed oriented surface of *genus g* with *n punctures*.
- $\mathcal{MRG}_{g,n}$ moduli space of *admissible metrics* on a ribbon graph of type (g, n).

Theorem (Mumford, Harer, Penner, Strebel, Kontsevich)

$\mathcal{M}_{g,n} \simeq \mathcal{MRG}_{g,n}$ if $n > 0$ and $2 - 2g < n$.

Purpose of the talk (cf. Igusa, Costello, Giansiracusa)

Realise $\mathcal{MRG}_{g,n}$ as a *"derived" modular envelope* of the cyclic operad CycAss of planar structures.

Definition (moduli space for oriented surfaces/ribbon graphs)

- $\mathcal{M}_{g,n}$ moduli space of *hyperbolic metrics* on a closed oriented surface of *genus g* with *n punctures*.
- $\mathcal{MRG}_{g,n}$ moduli space of *admissible metrics* on a ribbon graph of type (g, n).

Theorem (Mumford, Harer, Penner, Strebel, Kontsevich)

$\mathcal{M}_{g,n} \simeq \mathcal{MRG}_{g,n}$ if $n > 0$ and $2 - 2g < n$.

Purpose of the talk (cf. Igusa, Costello, Giansiracusa)

Realise $\mathcal{MRG}_{g,n}$ as a *"derived" modular envelope* of the cyclic operad CycAss of planar structures.

Definition (moduli space for oriented surfaces/ribbon graphs)

- $\mathcal{M}_{g,n}$ moduli space of *hyperbolic metrics* on a closed oriented surface of *genus* g with n *punctures*.
- $\mathcal{MRG}_{g,n}$ moduli space of *admissible metrics* on a ribbon graph of type (g, n).

Theorem (Mumford, Harer, Penner, Strebel, Kontsevich)

$\mathcal{M}_{g,n} \simeq \mathcal{MRG}_{g,n}$ if $n > 0$ and $2 - 2g < n$.

Purpose of the talk (cf. Igusa, Costello, Giansiracusa)

Realise $\mathcal{MRG}_{g,n}$ as a *"derived" modular envelope* of the cyclic operad CycAss of planar structures.

Definition (moduli space for oriented surfaces/ribbon graphs)

- $\mathcal{M}_{g,n}$ moduli space of *hyperbolic metrics* on a closed oriented surface of *genus g* with *n punctures*.
- $\mathcal{MRG}_{g,n}$ moduli space of *admissible metrics* on a ribbon graph of type (g, n).

Theorem (Mumford, Harer, Penner, Strebel, Kontsevich)

$\mathcal{M}_{g,n} \simeq \mathcal{MRG}_{g,n}$ if $n > 0$ and $2 - 2g < n$.

Purpose of the talk (cf. Igusa, Costello, Giansiracusa)

Realise $\mathcal{MRG}_{g,n}$ as a *"derived" modular envelope* of the cyclic operad CycAss of planar structures.

Definition (moduli space for oriented surfaces/ribbon graphs)

- $\mathcal{M}_{g,n}$ moduli space of *hyperbolic metrics* on a closed oriented surface of *genus g* with *n punctures*.
- $\mathcal{MRG}_{g,n}$ moduli space of *admissible metrics* on a ribbon graph of type (g, n).

Theorem (Mumford, Harer, Penner, Strebel, Kontsevich)

$\mathcal{M}_{g,n} \simeq \mathcal{MRG}_{g,n}$ if $n > 0$ and $2 - 2g < n$.

Purpose of the talk (cf. Igusa, Costello, Giansiracusa)

Realise $\mathcal{MRG}_{g,n}$ as a *"derived" modular envelope* of the cyclic operad CycAss of planar structures.

Definition (moduli space for oriented surfaces/ribbon graphs)

- $\mathcal{M}_{g,n}$ moduli space of *hyperbolic metrics* on a closed oriented surface of *genus g* with *n punctures*.
- $\mathcal{MRG}_{g,n}$ moduli space of *admissible metrics* on a ribbon graph of type (g, n).

Theorem (Mumford, Harer, Penner, Strebel, Kontsevich)

$\mathcal{M}_{g,n} \simeq \mathcal{MRG}_{g,n}$ if $n > 0$ and $2 - 2g < n$.

Purpose of the talk (cf. Igusa, Costello, Giansiracusa)

Realise $\mathcal{MRG}_{g,n}$ as a *"derived" modular envelope* of the cyclic operad CycAss of planar structures.

Definition (graph)

A *graph* is quadruple (V, F, ∂, ι) consisting of a vertex-set V, a flag-set F, a boundary map $\partial : F \to V$ and an involution $\iota : F \to F$

A two-element orbit of ι is called an *edge*, any fixpoint of ι an *outer flag*. For each vertex $v \in V$ the set $\partial^{-1}(v)$ is the flag-set of the *star* $*_v$ at v. Each graph decomposes into stars.

Definition (ribbon graph)

- A *ribbon graph* is a graph equipped with a cyclic flag-ordering for each vertex-star, i.e. equipped with a permutation $\sigma_0 : F \to F$ whose orbits correspond to the vertex-stars.
- A *boundary cycle* of a ribbon graph is an orbit of $\sigma_\infty = \sigma_0\iota$.
- A ribbon graph is said to be of type (g, n) if it has n boundary cycles and $2 - 2g = \#V - \#E + n$.

Definition (graph)

A *graph* is quadruple (V, F, ∂, ι) consisting of a vertex-set V, a flag-set F, a boundary map $\partial : F \to V$ and an involution $\iota : F \to F$

A two-element orbit of ι is called an *edge*, any fixpoint of ι an *outer flag*. For each vertex $v \in V$ the set $\partial^{-1}(v)$ is the flag-set of the *star* $*_v$ at v. Each graph decomposes into stars.

Definition (ribbon graph)

- A *ribbon graph* is a graph equipped with a cyclic flag-ordering for each vertex-star, i.e. equipped with a permutation $\sigma_0 : F \to F$ whose orbits correspond to the vertex-stars.
- A *boundary cycle* of a ribbon graph is an orbit of $\sigma_\infty = \sigma_0\iota$.
- A ribbon graph is said to be of type (g, n) if it has n boundary cycles and $2 - 2g = \#V - \#E + n$.

Definition (graph)

A *graph* is quadruple (V, F, ∂, ι) consisting of a vertex-set V, a flag-set F, a boundary map $\partial : F \to V$ and an involution $\iota : F \to F$

A two-element orbit of ι is called an *edge*, any fixpoint of ι an *outer flag*. For each vertex $v \in V$ the set $\partial^{-1}(v)$ is the flag-set of the *star* $*_v$ at v. Each graph decomposes into stars.

Definition (ribbon graph)

- A *ribbon graph* is a graph equipped with a cyclic flag-ordering for each vertex-star, i.e. equipped with a permutation $\sigma_0 : F \to F$ whose orbits correspond to the vertex-stars.
- A *boundary cycle* of a ribbon graph is an orbit of $\sigma_\infty = \sigma_0\iota$.
- A ribbon graph is said to be of type (g, n) if it has n boundary cycles and $2 - 2g = \#V - \#E + n$.

Definition (graph)

A *graph* is quadruple (V, F, ∂, ι) consisting of a vertex-set V, a flag-set F, a boundary map $\partial : F \to V$ and an involution $\iota : F \to F$

A two-element orbit of ι is called an *edge*, any fixpoint of ι an *outer flag*. For each vertex $v \in V$ the set $\partial^{-1}(v)$ is the flag-set of the *star* $*_v$ at v. Each graph decomposes into stars.

Definition (ribbon graph)

- A *ribbon graph* is a graph equipped with a cyclic flag-ordering for each vertex-star, i.e. equipped with a permutation $\sigma_0 : F \to F$ whose orbits correspond to the vertex-stars.
- A *boundary cycle* of a ribbon graph is an orbit of $\sigma_\infty = \sigma_0\iota$.
- A ribbon graph is said to be of type (g, n) if it has n boundary cycles and $2 - 2g = \#V - \#E + n$.

Definition (graph)

A *graph* is quadruple (V, F, ∂, ι) consisting of a vertex-set V, a flag-set F, a boundary map $\partial : F \to V$ and an involution $\iota : F \to F$

A two-element orbit of ι is called an *edge*, any fixpoint of ι an *outer flag*. For each vertex $v \in V$ the set $\partial^{-1}(v)$ is the flag-set of the *star* $*_v$ at v. Each graph decomposes into stars.

Definition (ribbon graph)

- A *ribbon graph* is a graph equipped with a cyclic flag-ordering for each vertex-star, i.e. equipped with a permutation $\sigma_0 : F \to F$ whose orbits correspond to the vertex-stars.
- A *boundary cycle* of a ribbon graph is an orbit of $\sigma_\infty = \sigma_0\iota$.
- A ribbon graph is said to be of type (g, n) if it has n boundary cycles and $2 - 2g = \#V - \#E + n$.

Definition (graph)

A *graph* is quadruple (V, F, ∂, ι) consisting of a vertex-set V, a flag-set F, a boundary map $\partial : F \to V$ and an involution $\iota : F \to F$

A two-element orbit of ι is called an *edge*, any fixpoint of ι an *outer flag*. For each vertex $v \in V$ the set $\partial^{-1}(v)$ is the flag-set of the *star* $*_v$ at v. Each graph decomposes into stars.

Definition (ribbon graph)

- A *ribbon graph* is a graph equipped with a cyclic flag-ordering for each vertex-star, i.e. equipped with a permutation $\sigma_0 : F \to F$ whose orbits correspond to the vertex-stars.
- A *boundary cycle* of a ribbon graph is an orbit of $\sigma_\infty = \sigma_0\iota$.
- A ribbon graph is said to be of type (g, n) if it has n boundary cycles and $2 - 2g = \#V - \#E + n$.

Definition (graph)

A *graph* is quadruple (V, F, ∂, ι) consisting of a vertex-set V, a flag-set F, a boundary map $\partial : F \to V$ and an involution $\iota : F \to F$

A two-element orbit of ι is called an *edge*, any fixpoint of ι an *outer flag*. For each vertex $v \in V$ the set $\partial^{-1}(v)$ is the flag-set of the *star* $*_v$ at v. Each graph decomposes into stars.

Definition (ribbon graph)

- A *ribbon graph* is a graph equipped with a cyclic flag-ordering for each vertex-star, i.e. equipped with a permutation $\sigma_0 : F \to F$ whose orbits correspond to the vertex-stars.
- A *boundary cycle* of a ribbon graph is an orbit of $\sigma_\infty = \sigma_0\iota$.
- A ribbon graph is said to be of type (g, n) if it has n boundary cycles and $2 - 2g = \#V - \#E + n$.

Proposition (unflagged ribbon graphs)

For a finite graph G without outer flags TFAE

- G comes equipped with a ribbon structure of type (g, n);
- $|G|$ embeds into a closed oriented surface S of genus g such that $S - |G|$ is a disjoint union of n discs.

Corollary (planar graphs)

A graph is planar (resp. a planar tree) if and only if it carries a ribbon structure of type $(0, n)$ (resp. $(0, 1)$).

Remark (flagged ribbon graphs - doubling construction)

- The outer flags of a ribbon graph G form a polycyclic set.
- Gluing $G \cup G^{op}$ along outer flags yields an *unflagged* ribbon graph with an "orientation-reversing" involution.
- This yields an equivalence between *flagged* ribbon graphs and certain *involutive* unflagged ribbon graphs.

Proposition (unflagged ribbon graphs)

For a finite graph G without outer flags TFAE

- G comes equipped with a ribbon structure of type (g, n);
- $|G|$ embeds into a closed oriented surface S of genus g such that $S - |G|$ is a disjoint union of n discs.

Corollary (planar graphs)

A graph is planar (resp. a planar tree) if and only if it carries a ribbon structure of type $(0, n)$ (resp. $(0, 1)$).

Remark (flagged ribbon graphs - doubling construction)

- The outer flags of a ribbon graph G form a polycyclic set.
- Gluing $G \cup G^{op}$ along outer flags yields an *unflagged* ribbon graph with an "orientation-reversing" involution.
- This yields an equivalence between *flagged* ribbon graphs and certain *involutive* unflagged ribbon graphs.

Proposition (unflagged ribbon graphs)

For a finite graph G without outer flags TFAE

- G comes equipped with a ribbon structure of type (g, n);
- $|G|$ embeds into a closed oriented surface S of genus g such that $S - |G|$ is a disjoint union of n discs.

Corollary (planar graphs)

A graph is planar (resp. a planar tree) if and only if it carries a ribbon structure of type $(0, n)$ (resp. $(0, 1)$).

Remark (flagged ribbon graphs - doubling construction)

- The outer flags of a ribbon graph G form a polycyclic set.
- Gluing $G \cup G^{op}$ along outer flags yields an *unflagged* ribbon graph with an "orientation-reversing" involution.
- This yields an equivalence between *flagged* ribbon graphs and certain *involutive* unflagged ribbon graphs.

Proposition (unflagged ribbon graphs)

For a finite graph G without outer flags TFAE

- G comes equipped with a ribbon structure of type (g, n);
- $|G|$ embeds into a closed oriented surface S of genus g such that $S - |G|$ is a disjoint union of n discs.

Corollary (planar graphs)

A graph is planar (resp. a planar tree) if and only if it carries a ribbon structure of type $(0, n)$ (resp. $(0, 1)$).

Remark (flagged ribbon graphs - doubling construction)

- The outer flags of a ribbon graph G form a polycyclic set.
- Gluing $G \cup G^{op}$ along outer flags yields an *unflagged* ribbon graph with an "orientation-reversing" involution.
- This yields an equivalence between *flagged* ribbon graphs and certain *involutive* unflagged ribbon graphs.

Proposition (unflagged ribbon graphs)

For a finite graph G without outer flags TFAE

- G comes equipped with a ribbon structure of type (g, n);
- $|G|$ embeds into a closed oriented surface S of genus g such that $S - |G|$ is a disjoint union of n discs.

Corollary (planar graphs)

A graph is planar (resp. a planar tree) if and only if it carries a ribbon structure of type $(0, n)$ (resp. $(0, 1)$).

Remark (flagged ribbon graphs - doubling construction)

- The outer flags of a ribbon graph G form a polycyclic set.
- Gluing $G \cup G^{op}$ along outer flags yields an *unflagged* ribbon graph with an “orientation-reversing” involution.
- This yields an equivalence between *flagged* ribbon graphs and certain *involutive* unflagged ribbon graphs.

Proposition (unflagged ribbon graphs)

For a finite graph G without outer flags TFAE

- G comes equipped with a ribbon structure of type (g, n);
- $|G|$ embeds into a closed oriented surface S of genus g such that $S - |G|$ is a disjoint union of n discs.

Corollary (planar graphs)

A graph is planar (resp. a planar tree) if and only if it carries a ribbon structure of type $(0, n)$ (resp. $(0, 1)$).

Remark (flagged ribbon graphs - doubling construction)

- The outer flags of a ribbon graph G form a polycyclic set.
- Gluing $G \cup G^{op}$ along outer flags yields an *unflagged* ribbon graph with an "orientation-reversing" involution.
- This yields an equivalence between *flagged* ribbon graphs and certain *involutive* unflagged ribbon graphs.

Proposition (unflagged ribbon graphs)

For a finite graph G without outer flags TFAE

- G comes equipped with a ribbon structure of type (g, n);
- $|G|$ embeds into a closed oriented surface S of genus g such that $S - |G|$ is a disjoint union of n discs.

Corollary (planar graphs)

A graph is planar (resp. a planar tree) if and only if it carries a ribbon structure of type $(0, n)$ (resp. $(0, 1)$).

Remark (flagged ribbon graphs - doubling construction)

- The outer flags of a ribbon graph G form a polycyclic set.
- Gluing $G \cup G^{op}$ along outer flags yields an *unflagged* ribbon graph with an "orientation-reversing" involution.
- This yields an equivalence between *flagged* ribbon graphs and certain *involutive* unflagged ribbon graphs.

Definition (Borisov-Manin graph morphism)

A morphism $G = (V, F, \partial, \iota) \to G' = (V', F', \partial', \iota')$ consists of a triple $(\phi_V, \phi^F, \gamma_\phi)$ where

- $\phi_V : V \to V'$ surjection;
- $\phi^F : F' \to F$ outer flag preserving injection;
- γ_ϕ fixpoint-free involution on $F_{out} \backslash \phi^F(F'_{out})$

such that $\partial' = \phi_V \partial \phi^F$ and $\phi^F \iota' = \iota \phi^F$.

A morphism is called a *grafting* if ϕ_V and ϕ^F are bijections. A morphism is called a *virtual contraction* if there is no grafting.

Lemma (unique factorisation system)

Each Borisov-Manin graph morphism factors essentially uniquely as a grafting followed by a virtual contraction.

Definition (Borisov-Manin graph morphism)

A morphism $G = (V, F, \partial, \iota) \to G' = (V', F', \partial', \iota')$ consists of a triple $(\phi_V, \phi^F, \gamma_\phi)$ where

- $\phi_V : V \to V'$ surjection;
- $\phi^F : F' \to F$ outer flag preserving injection;
- γ_ϕ fixpoint-free involution on $F_{out} \backslash \phi^F(F'_{out})$

such that $\partial' = \phi_V \partial \phi^F$ and $\phi^F \iota' = \iota \phi^F$.

A morphism is called a *grafting* if ϕ_V and ϕ^F are bijections. A morphism is called a *virtual contraction* if there is no grafting.

Lemma (unique factorisation system)

Each Borisov-Manin graph morphism factors essentially uniquely as a grafting followed by a virtual contraction.

Definition (Borisov-Manin graph morphism)

A morphism $G = (V, F, \partial, \iota) \to G' = (V', F', \partial', \iota')$ consists of a triple $(\phi_V, \phi^F, \gamma_\phi)$ where

- $\phi_V : V \to V'$ surjection;
- $\phi^F : F' \to F$ outer flag preserving injection;
- γ_ϕ fixpoint-free involution on $F_{out} \backslash \phi^F(F'_{out})$

such that $\partial' = \phi_V \partial \phi^F$ and $\phi^F \iota' = \iota \phi^F$.

A morphism is called a *grafting* if ϕ_V and ϕ^F are bijections. A morphism is called a *virtual contraction* if there is no grafting.

Lemma (unique factorisation system)

Each Borisov-Manin graph morphism factors essentially uniquely as a grafting followed by a virtual contraction.

Definition (Borisov-Manin graph morphism)

A morphism $G = (V, F, \partial, \iota) \to G' = (V', F', \partial', \iota')$ consists of a triple $(\phi_V, \phi^F, \gamma_\phi)$ where

- $\phi_V : V \to V'$ surjection;
- $\phi^F : F' \to F$ outer flag preserving injection;
- γ_ϕ fixpoint-free involution on $F_{out} \backslash \phi^F(F'_{out})$

such that $\partial' = \phi_V \partial \phi^F$ and $\phi^F \iota' = \iota \phi^F$.

A morphism is called a *grafting* if ϕ_V and ϕ^F are bijections. A morphism is called a *virtual contraction* if there is no grafting.

Lemma (unique factorisation system)

Each Borisov-Manin graph morphism factors essentially uniquely as a grafting followed by a virtual contraction.

Definition (Borisov-Manin graph morphism)

A morphism $G = (V, F, \partial, \iota) \to G' = (V', F', \partial', \iota')$ consists of a triple $(\phi_V, \phi^F, \gamma_\phi)$ where

- $\phi_V : V \to V'$ surjection;
- $\phi^F : F' \to F$ outer flag preserving injection;
- γ_ϕ fixpoint-free involution on $F_{out} \backslash \phi^F(F'_{out})$

such that $\partial' = \phi_V \partial \phi^F$ and $\phi^F \iota' = \iota \phi^F$.

A morphism is called a *grafting* if ϕ_V and ϕ^F are bijections. A morphism is called a *virtual contraction* if there is no grafting.

Lemma (unique factorisation system)

Each Borisov-Manin graph morphism factors essentially uniquely as a grafting followed by a virtual contraction.

Definition (Borisov-Manin graph morphism)

A morphism $G = (V, F, \partial, \iota) \to G' = (V', F', \partial', \iota')$ consists of a triple $(\phi_V, \phi^F, \gamma_\phi)$ where

- $\phi_V : V \to V'$ surjection;
- $\phi^F : F' \to F$ outer flag preserving injection;
- γ_ϕ fixpoint-free involution on $F_{out} \backslash \phi^F(F'_{out})$

such that $\partial' = \phi_V \partial \phi^F$ and $\phi^F \iota' = \iota \phi^F$.

A morphism is called a *grafting* if ϕ_V and ϕ^F are bijections. A morphism is called a *virtual contraction* if there is no grafting.

Lemma (unique factorisation system)

Each Borisov-Manin graph morphism factors essentially uniquely as a grafting followed by a virtual contraction.

Definition (Borisov-Manin graph morphism)

A morphism $G = (V, F, \partial, \iota) \to G' = (V', F', \partial', \iota')$ consists of a triple $(\phi_V, \phi^F, \gamma_\phi)$ where

- $\phi_V : V \to V'$ surjection;
- $\phi^F : F' \to F$ outer flag preserving injection;
- γ_ϕ fixpoint-free involution on $F_{out} \backslash \phi^F(F'_{out})$

such that $\partial' = \phi_V \partial \phi^F$ and $\phi^F \iota' = \iota \phi^F$.

A morphism is called a *grafting* if ϕ_V and ϕ^F are bijections. A morphism is called a *virtual contraction* if there is no grafting.

Lemma (unique factorisation system)

Each Borisov-Manin graph morphism factors essentially uniquely as a grafting followed by a virtual contraction.

Definition (Borisov-Manin graph morphism)

A morphism $G = (V, F, \partial, \iota) \to G' = (V', F', \partial', \iota')$ consists of a triple $(\phi_V, \phi^F, \gamma_\phi)$ where

- $\phi_V : V \to V'$ surjection;
- $\phi^F : F' \to F$ outer flag preserving injection;
- γ_ϕ fixpoint-free involution on $F_{out} \backslash \phi^F(F'_{out})$

such that $\partial' = \phi_V \partial \phi^F$ and $\phi^F \iota' = \iota \phi^F$.

A morphism is called a *grafting* if ϕ_V and ϕ^F are bijections. A morphism is called a *virtual contraction* if there is no grafting.

Lemma (unique factorisation system)

Each Borisov-Manin graph morphism factors essentially uniquely as a grafting followed by a virtual contraction.

Definition (Feynman category $\mathfrak{F}_{agg}$ of aggregates)

The objects of $\mathfrak{F}_{agg}$ are finite coproducts of corollas.
The morphisms of $\mathfrak{F}_{agg}$ are Borisov-Manin graph morphisms.

where $\Gamma(\psi\phi)$ is the graph obtained by *inserting* the n formal components of $\Gamma(\phi)$ into the n vertices of $\Gamma(\psi)$.

Remark (insertional classes $\mathfrak{C}$ of graphs vs Feynman subcategories)

Each $\mathfrak{C}$ induces a Feynman subcategory $\mathfrak{F}_{\mathfrak{C}} \subset \mathfrak{F}_{agg}$ and vice-versa.
For $\mathfrak{C}$ =(connected graphs), we get a Feynman category $\mathfrak{F}_{ctd}$ where the formal components of $\Gamma(\phi)$ coincide with its path components !

Definition (Feynman category $\mathfrak{F}_{agg}$ of aggregates)

The objects of $\mathfrak{F}_{agg}$ are finite coproducts of corollas.
The morphisms of $\mathfrak{F}_{agg}$ are Borisov-Manin graph morphisms.

$$\begin{array}{ccccccccc} & & & & \Gamma(\psi\phi) & & & & \\ & & & \overset{gr}{\nearrow} & & \overset{vc}{\searrow} & & & \\ & & \Gamma(\phi) & & & & \Gamma(\psi) & & \\ & \overset{gr}{\nearrow} & & \overset{vc}{\searrow} & & \overset{gr}{\nearrow} & & \overset{vc}{\searrow} & \\ \star_{v_1}\sqcup\cdots\sqcup\star_{v_m} & & \xrightarrow{\phi} & & \star_{w_1}\sqcup\cdots\sqcup\star_{w_n} & & \xrightarrow{\psi} & & \star_{u_1}\sqcup\cdots\sqcup\star_{u_p}\end{array}$$

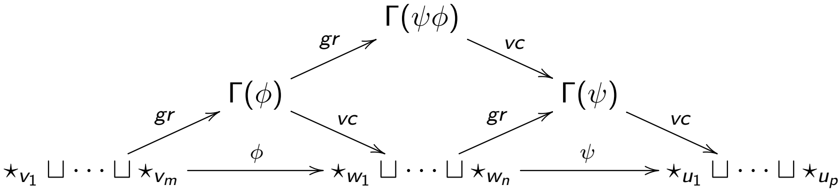
where $\Gamma(\psi\phi)$ is the graph obtained by *inserting* the n formal components of $\Gamma(\phi)$ into the n vertices of $\Gamma(\psi)$.

Remark (insertional classes $\mathfrak{C}$ of graphs vs Feynman subcategories)

Each $\mathfrak{C}$ induces a Feynman subcategory $\mathfrak{F}_{\mathfrak{C}} \subset \mathfrak{F}_{agg}$ and vice-versa.
For $\mathfrak{C} =$(connected graphs), we get a Feynman category $\mathfrak{F}_{ctd}$ where the formal components of $\Gamma(\phi)$ coincide with its path components !

Definition (Feynman category $\mathfrak{F}_{agg}$ of aggregates)

The objects of $\mathfrak{F}_{agg}$ are finite coproducts of corollas.
The morphisms of $\mathfrak{F}_{agg}$ are Borisov-Manin graph morphisms.



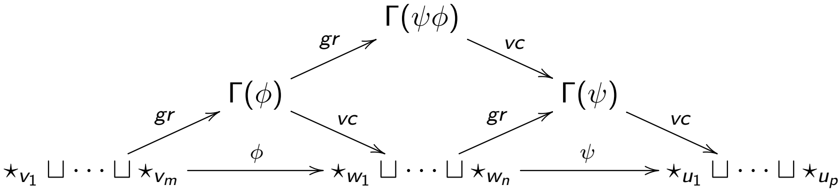
where $\Gamma(\psi\phi)$ is the graph obtained by *inserting* the n formal components of $\Gamma(\phi)$ into the n vertices of $\Gamma(\psi)$.

Remark (insertional classes $\mathfrak{C}$ of graphs vs Feynman subcategories)

Each $\mathfrak{C}$ induces a Feynman subcategory $\mathfrak{F}_{\mathfrak{C}} \subset \mathfrak{F}_{agg}$ and vice-versa.
For $\mathfrak{C}$ =(connected graphs), we get a Feynman category $\mathfrak{F}_{ctd}$ where the formal components of $\Gamma(\phi)$ coincide with its path components !

Definition (Feynman category $\mathfrak{F}_{agg}$ of aggregates)

The objects of $\mathfrak{F}_{agg}$ are finite coproducts of corollas.
The morphisms of $\mathfrak{F}_{agg}$ are Borisov-Manin graph morphisms.



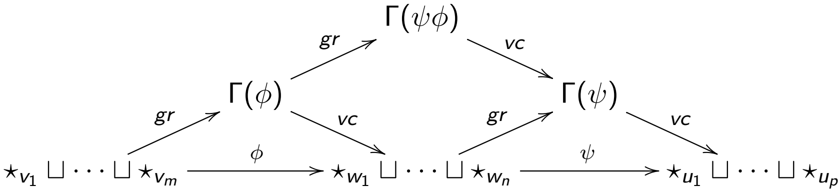
where $\Gamma(\psi\phi)$ is the graph obtained by *inserting* the n formal components of $\Gamma(\phi)$ into the n vertices of $\Gamma(\psi)$.

Remark (insertional classes $\mathfrak{C}$ of graphs vs Feynman subcategories)

Each $\mathfrak{C}$ induces a Feynman subcategory $\mathfrak{F}_{\mathfrak{C}} \subset \mathfrak{F}_{agg}$ and vice-versa.
For $\mathfrak{C}$ =(connected graphs), we get a Feynman category $\mathfrak{F}_{ctd}$ where the formal components of $\Gamma(\phi)$ coincide with its path components !

Definition (Feynman category $\mathfrak{F}_{agg}$ of aggregates)

The objects of $\mathfrak{F}_{agg}$ are finite coproducts of corollas.
The morphisms of $\mathfrak{F}_{agg}$ are Borisov-Manin graph morphisms.



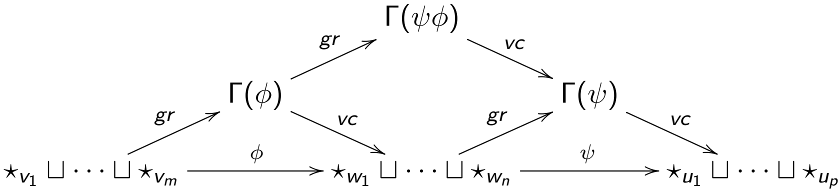
where $\Gamma(\psi\phi)$ is the graph obtained by *inserting* the n formal components of $\Gamma(\phi)$ into the n vertices of $\Gamma(\psi)$.

Remark (insertional classes $\mathfrak{C}$ of graphs vs Feynman subcategories)

Each $\mathfrak{C}$ induces a Feynman subcategory $\mathfrak{F}_{\mathfrak{C}} \subset \mathfrak{F}_{agg}$ and vice-versa.
For $\mathfrak{C} =$(connected graphs), we get a Feynman category $\mathfrak{F}_{ctd}$ where the formal components of $\Gamma(\phi)$ coincide with its path components !

Definition (Feynman category $\mathfrak{F}_{agg}$ of aggregates)

The objects of $\mathfrak{F}_{agg}$ are finite coproducts of corollas.
The morphisms of $\mathfrak{F}_{agg}$ are Borisov-Manin graph morphisms.



where $\Gamma(\psi\phi)$ is the graph obtained by *inserting* the n formal components of $\Gamma(\phi)$ into the n vertices of $\Gamma(\psi)$.

Remark (insertional classes $\mathfrak{C}$ of graphs vs Feynman subcategories)

Each $\mathfrak{C}$ induces a Feynman subcategory $\mathfrak{F}_{\mathfrak{C}} \subset \mathfrak{F}_{agg}$ and vice-versa.
For $\mathfrak{C} =$(connected graphs), we get a Feynman category $\mathfrak{F}_{ctd}$ where the formal components of $\Gamma(\phi)$ coincide with its path components !

Proposition (Street-Walters, K-B)

Each (Feynman) functor factors essentially uniquely as a connected (Feynman) functor followed by a covering.

Definition

Define $\mathfrak{F}_{cyc} = \mathfrak{F}_{(trees)}$ and $\mathfrak{F}_{mod}$ by the connected/covering factorisation $\mathfrak{F}_{(trees)} \xrightarrow{connected} \mathfrak{F}_{mod} \xrightarrow{covering} \mathfrak{F}_{(ctd\ graphs)}$.

Proposition (Getzler-Kapranov)

Symmetric monoidal functors out of $\mathfrak{F}_{cyc}$ (resp. $\mathfrak{F}_{mod}$) are non-unital cyclic (resp. modular) operads.

Proposition (K-Lucas, K-B)

For each $P \in \mathrm{Func}_{\otimes}(\mathfrak{F}, \mathrm{Sets})$ there is a covering $\mathfrak{F}_{\mathrm{dec}(P)} \to \mathfrak{F}$ inducing an equivalence $\mathrm{Func}_{\otimes}(\mathfrak{F}, \mathrm{Sets})/P \simeq \mathrm{Func}_{\otimes}(\mathfrak{F}_{\mathrm{dec}(P)}, \mathrm{Sets})$.

Proposition (Street-Walters, K-B)

Each (Feynman) functor factors essentially uniquely as a connected (Feynman) functor followed by a covering.

Definition

Define $\mathfrak{F}_{cyc} = \mathfrak{F}_{(trees)}$ and $\mathfrak{F}_{mod}$ by the connected/covering factorisation $\mathfrak{F}_{(trees)} \xrightarrow{connected} \mathfrak{F}_{mod} \xrightarrow{covering} \mathfrak{F}_{(ctd\ graphs)}$.

Proposition (Getzler-Kapranov)

Symmetric monoidal functors out of $\mathfrak{F}_{cyc}$ (resp. $\mathfrak{F}_{mod}$) are non-unital cyclic (resp. modular) operads.

Proposition (K-Lucas, K-B)

For each $P \in \mathrm{Func}_{\otimes}(\mathfrak{F}, \mathrm{Sets})$ there is a covering $\mathfrak{F}_{\mathrm{dec}(P)} \to \mathfrak{F}$ inducing an equivalence $\mathrm{Func}_{\otimes}(\mathfrak{F}, \mathrm{Sets})/P \simeq \mathrm{Func}_{\otimes}(\mathfrak{F}_{\mathrm{dec}(P)}, \mathrm{Sets})$.

Proposition (Street-Walters, K-B)

Each (Feynman) functor factors essentially uniquely as a connected (Feynman) functor followed by a covering.

Definition

Define $\mathfrak{F}_{cyc} = \mathfrak{F}_{(trees)}$ and $\mathfrak{F}_{mod}$ by the connected/covering factorisation $\mathfrak{F}_{(trees)} \xrightarrow{connected} \mathfrak{F}_{mod} \xrightarrow{covering} \mathfrak{F}_{(ctd\ graphs)}$.

Proposition (Getzler-Kapranov)

Symmetric monoidal functors out of $\mathfrak{F}_{cyc}$ (resp. $\mathfrak{F}_{mod}$) are non-unital cyclic (resp. modular) operads.

Proposition (K-Lucas, K-B)

For each $P \in \mathrm{Func}_{\otimes}(\mathfrak{F}, \mathrm{Sets})$ there is a covering $\mathfrak{F}_{\mathrm{dec}(P)} \to \mathfrak{F}$ inducing an equivalence $\mathrm{Func}_{\otimes}(\mathfrak{F}, \mathrm{Sets})/P \simeq \mathrm{Func}_{\otimes}(\mathfrak{F}_{\mathrm{dec}(P)}, \mathrm{Sets})$.

Proposition (Street-Walters, K-B)

Each (Feynman) functor factors essentially uniquely as a connected (Feynman) functor followed by a covering.

Definition

Define $\mathfrak{F}_{cyc} = \mathfrak{F}_{(trees)}$ and $\mathfrak{F}_{mod}$ by the connected/covering factorisation $\mathfrak{F}_{(trees)} \overset{connected}{\longrightarrow} \mathfrak{F}_{mod} \overset{covering}{\longrightarrow} \mathfrak{F}_{(ctd\ graphs)}$.

Proposition (Getzler-Kapranov)

Symmetric monoidal functors out of $\mathfrak{F}_{cyc}$ (resp. $\mathfrak{F}_{mod}$) are non-unital cyclic (resp. modular) operads.

Proposition (K-Lucas, K-B)

For each $P \in \mathrm{Func}_{\otimes}(\mathfrak{F}, \mathrm{Sets})$ there is a covering $\mathfrak{F}_{\mathrm{dec}(P)} \to \mathfrak{F}$ inducing an equivalence $\mathrm{Func}_{\otimes}(\mathfrak{F}, \mathrm{Sets})/P \simeq \mathrm{Func}_{\otimes}(\mathfrak{F}_{\mathrm{dec}(P)}, \mathrm{Sets})$.

Proposition (Street-Walters, K-B)

Each (Feynman) functor factors essentially uniquely as a connected (Feynman) functor followed by a covering.

Definition

Define $\mathfrak{F}_{cyc} = \mathfrak{F}_{(trees)}$ and $\mathfrak{F}_{mod}$ by the connected/covering factorisation $\mathfrak{F}_{(trees)} \overset{connected}{\longrightarrow} \mathfrak{F}_{mod} \overset{covering}{\longrightarrow} \mathfrak{F}_{(ctd\ graphs)}$.

Proposition (Getzler-Kapranov)

Symmetric monoidal functors out of $\mathfrak{F}_{cyc}$ (resp. $\mathfrak{F}_{mod}$) are non-unital cyclic (resp. modular) operads.

Proposition (K-Lucas, K-B)

For each $P \in \mathrm{Func}_{\otimes}(\mathfrak{F}, \mathrm{Sets})$ there is a covering $\mathfrak{F}_{\mathrm{dec}(P)} \to \mathfrak{F}$ inducing an equivalence $\mathrm{Func}_{\otimes}(\mathfrak{F}, \mathrm{Sets})/P \simeq \mathrm{Func}_{\otimes}(\mathfrak{F}_{\mathrm{dec}(P)}, \mathrm{Sets})$.

$$\begin{array}{ccccc}
\mathfrak{F}_{\textit{plan-cyc}} & \xrightarrow[\textit{connected}]{J} & \mathfrak{F}_{\textit{surf-mod}} & & \\
\textit{covering} \downarrow \mathrm{CycAss} & & \textit{covering} \downarrow j_!\mathrm{CycAss} & & \\
\mathfrak{F}_{\textit{cyc}} & \xrightarrow[\textit{connected}]{j} & \mathfrak{F}_{\textit{mod}} & \xrightarrow[\textit{covering}]{k} & \mathfrak{F}_{\textit{ctd}}
\end{array}$$

Example (Computation of $j_!\mathrm{CycAss}$, cf. Markl, Doubek, B-K)

$(j_!\mathrm{CycAss})(\star_{\gamma,\nu})$
$=\mathrm{colim}_{j(-)\downarrow\star_{\gamma,\nu}}\mathrm{CycAss}(-)$
$=$(equ. cl. of ribbon graphs with γ loops and ν outer flags)
$=$(topological types $(g, n, S_1, \ldots, S_k)$ of bordered oriented surf.)

Corollary (B-K)

The morphisms of the Feynman category $\mathfrak{F}_{\textit{surf-mod}}$ are genus/puncture-labelled *polycyclic* graphs.

$$\begin{array}{ccccc}
\mathfrak{F}_{\mathit{plan\text{-}cyc}} & \xrightarrow[\mathit{connected}]{J} & \mathfrak{F}_{\mathit{surf\text{-}mod}} & & \\
\mathit{covering}\ \big\downarrow\ \mathrm{CycAss} & & \mathit{covering}\ \big\downarrow\ j_!\mathrm{CycAss} & & \\
\mathfrak{F}_{\mathit{cyc}} & \xrightarrow[\mathit{connected}]{j} & \mathfrak{F}_{\mathit{mod}} & \xrightarrow[\mathit{covering}]{k} & \mathfrak{F}_{\mathit{ctd}}
\end{array}$$

Example (Computation of $j_!\mathrm{CycAss}$, cf. Markl, Doubek, B-K)

$(j_!\mathrm{CycAss})(\star_{\gamma,\nu})$
$=\mathrm{colim}_{j(-)\downarrow\star_{\gamma,\nu}}\mathrm{CycAss}(-)$
$=$(equ. cl. of ribbon graphs with γ loops and ν outer flags)
$=$(topological types $(g, n, S_1, \ldots, S_k)$ of bordered oriented surf.)

Corollary (B-K)

The morphisms of the Feynman category $\mathfrak{F}_{\mathit{surf\text{-}mod}}$ are genus/puncture-labelled *polycyclic* graphs.

$$\begin{array}{ccccc}
\mathfrak{F}_{plan\text{-}cyc} & \xrightarrow[\text{connected}]{J} & \mathfrak{F}_{surf\text{-}mod} & & \\
\text{covering} \Big\downarrow \mathrm{CycAss} & & \text{covering} \Big\downarrow j_!\mathrm{CycAss} & & \\
\mathfrak{F}_{cyc} & \xrightarrow[\text{connected}]{j} & \mathfrak{F}_{mod} & \xrightarrow[\text{covering}]{k} & \mathfrak{F}_{ctd}
\end{array}$$

Example (Computation of $j_!\mathrm{CycAss}$, cf. Markl, Doubek, B-K)

$(j_!\mathrm{CycAss})(\star_{\gamma,\nu})$
$=\mathrm{colim}_{j(-)\downarrow\star_{\gamma,\nu}}\mathrm{CycAss}(-)$
=(equ. cl. of ribbon graphs with γ loops and ν outer flags)
=(topological types $(g, n, S_1, \ldots, S_k)$ of bordered oriented surf.)

Corollary (B-K)

The morphisms of the Feynman category $\mathfrak{F}_{surf\text{-}mod}$ are genus/puncture-labelled *polycyclic* graphs.

$$
\begin{array}{ccccc}
\mathfrak{F}_{plan\text{-}cyc} & \xrightarrow[\text{connected}]{J} & \mathfrak{F}_{surf\text{-}mod} & & \\
{\scriptstyle covering}\big\downarrow{\scriptstyle \mathrm{CycAss}} & & {\scriptstyle covering}\big\downarrow{\scriptstyle j_!\mathrm{CycAss}} & & \\
\mathfrak{F}_{cyc} & \xrightarrow[\text{connected}]{j} & \mathfrak{F}_{mod} & \xrightarrow[\text{covering}]{k} & \mathfrak{F}_{ctd}
\end{array}
$$

Example (Computation of $j_!\mathrm{CycAss}$, cf. Markl, Doubek, B-K)

$(j_!\mathrm{CycAss})(\star_{\gamma,\nu})$
$=\mathrm{colim}_{j(-)\downarrow\star_{\gamma,\nu}}\mathrm{CycAss}(-)$
=(equ. cl. of ribbon graphs with γ loops and ν outer flags)
=(topological types $(g, n, S_1, \ldots, S_k)$ of bordered oriented surf.)

Corollary (B-K)

The morphisms of the Feynman category $\mathfrak{F}_{surf\text{-}mod}$ are genus/puncture-labelled *polycyclic* graphs.

$$\begin{array}{ccccc}
\mathfrak{F}_{plan\text{-}cyc} & \xrightarrow[\text{connected}]{J} & \mathfrak{F}_{surf\text{-}mod} & & \\
\text{covering}\ \Big\downarrow\ \mathrm{CycAss} & & \text{covering}\ \Big\downarrow\ j_!\mathrm{CycAss} & & \\
\mathfrak{F}_{cyc} & \xrightarrow[\text{connected}]{j} & \mathfrak{F}_{mod} & \xrightarrow[\text{covering}]{k} & \mathfrak{F}_{ctd}
\end{array}$$

Example (Computation of $j_!\mathrm{CycAss}$, cf. Markl, Doubek, B-K)

$(j_!\mathrm{CycAss})(\star_{\gamma,\nu})$
$=\mathrm{colim}_{j(-)\downarrow\star_{\gamma,\nu}}\mathrm{CycAss}(-)$
$=$(equ. cl. of ribbon graphs with γ loops and ν outer flags)
$=$(topological types $(g, n, S_1, \ldots, S_k)$ of bordered oriented surf.)

Corollary (B-K)

The morphisms of the Feynman category $\mathfrak{F}_{surf\text{-}mod}$ are genus/puncture-labelled *polycyclic* graphs.

$$\begin{array}{ccccc}
\mathfrak{F}_{\mathit{plan\text{-}cyc}} & \xrightarrow[\mathit{connected}]{J} & \mathfrak{F}_{\mathit{surf\text{-}mod}} & & \\
\mathit{covering}\ \Big\downarrow\ \mathrm{CycAss} & & \mathit{covering}\ \Big\downarrow\ j_!\mathrm{CycAss} & & \\
\mathfrak{F}_{\mathit{cyc}} & \xrightarrow[\mathit{connected}]{j} & \mathfrak{F}_{\mathit{mod}} & \xrightarrow[\mathit{covering}]{k} & \mathfrak{F}_{\mathit{ctd}}
\end{array}$$

Example (Computation of $j_!\mathrm{CycAss}$, cf. Markl, Doubek, B-K)

$(j_!\mathrm{CycAss})(\star_{\gamma,\nu})$
$=\mathrm{colim}_{j(-)\downarrow\star_{\gamma,\nu}}\mathrm{CycAss}(-)$
$=$(equ. cl. of ribbon graphs with γ loops and ν outer flags)
$=$(topological types $(g, n, S_1, \dots, S_k)$ of bordered oriented surf.)

Corollary (B-K)

The morphisms of the Feynman category $\mathfrak{F}_{\mathit{surf\text{-}mod}}$ are genus/puncture-labelled *polycyclic* graphs.

Definition (K-Ward)

A Feynman category $\mathfrak{F}$ is *cubical* if there is a degree function $\deg : Mor(\mathfrak{F}) \to \mathbb{N}_0$ such that

- $\deg(\phi \otimes \psi) = \deg(\phi) + \deg(\psi)$
- $\deg(\phi \circ \psi) = \deg(\phi) + \deg(\psi)$
- Degree 0 morphisms are invertible
- Each degree n morphism factors (up to iso) in $n!$ ways into degree 1 morphisms "compatibly with composition"

Remark

The Feynman categories $\mathfrak{F}_{ctd}, \mathfrak{F}_{cyc}, \mathfrak{F}_{mod}, \mathfrak{F}_{plan\text{-}cyc}, \mathfrak{F}_{surf\text{-}mod}$ are all cubical. The degree of ϕ is the number of edges of the associated graph $\Gamma(\phi)$. However, $\mathfrak{F}_{agg}$ is *not* cubical.

Definition (K-Ward)

A Feynman category $\mathfrak{F}$ is *cubical* if there is a degree function $\deg : Mor(\mathfrak{F}) \to \mathbb{N}_0$ such that

- $\deg(\phi \otimes \psi) = \deg(\phi) + \deg(\psi)$
- $\deg(\phi \circ \psi) = \deg(\phi) + \deg(\psi)$
- Degree 0 morphisms are invertible
- Each degree n morphism factors (up to iso) in $n!$ ways into degree 1 morphisms "compatibly with composition"

Remark

The Feynman categories $\mathfrak{F}_{ctd}, \mathfrak{F}_{cyc}, \mathfrak{F}_{mod}, \mathfrak{F}_{plan\text{-}cyc}, \mathfrak{F}_{surf\text{-}mod}$ are all cubical. The degree of ϕ is the number of edges of the associated graph $\Gamma(\phi)$. However, $\mathfrak{F}_{agg}$ is *not* cubical.

Definition (K-Ward)

A Feynman category $\mathfrak{F}$ is *cubical* if there is a degree function $\deg : Mor(\mathfrak{F}) \to \mathbb{N}_0$ such that

- $\deg(\phi \otimes \psi) = \deg(\phi) + \deg(\psi)$
- $\deg(\phi \circ \psi) = \deg(\phi) + \deg(\psi)$
- Degree 0 morphisms are invertible
- Each degree n morphism factors (up to iso) in $n!$ ways into degree 1 morphisms "compatibly with composition"

Remark

The Feynman categories $\mathfrak{F}_{ctd}, \mathfrak{F}_{cyc}, \mathfrak{F}_{mod}, \mathfrak{F}_{plan\text{-}cyc}, \mathfrak{F}_{surf\text{-}mod}$ are all cubical. The degree of ϕ is the number of edges of the associated graph $\Gamma(\phi)$. However, $\mathfrak{F}_{agg}$ is *not* cubical.

Definition (K-Ward)

A Feynman category $\mathfrak{F}$ is *cubical* if there is a degree function $\deg : Mor(\mathfrak{F}) \to \mathbb{N}_0$ such that

- $\deg(\phi \otimes \psi) = \deg(\phi) + \deg(\psi)$
- $\deg(\phi \circ \psi) = \deg(\phi) + \deg(\psi)$
- Degree 0 morphisms are invertible
- Each degree n morphism factors (up to iso) in $n!$ ways into degree 1 morphisms "compatibly with composition"

Remark

The Feynman categories $\mathfrak{F}_{ctd}, \mathfrak{F}_{cyc}, \mathfrak{F}_{mod}, \mathfrak{F}_{plan\text{-}cyc}, \mathfrak{F}_{surf\text{-}mod}$ are all cubical. The degree of ϕ is the number of edges of the associated graph $\Gamma(\phi)$. However, $\mathfrak{F}_{agg}$ is *not* cubical.

Definition (K-Ward)

A Feynman category $\mathfrak{F}$ is *cubical* if there is a degree function $\deg : Mor(\mathfrak{F}) \to \mathbb{N}_0$ such that

- $\deg(\phi \otimes \psi) = \deg(\phi) + \deg(\psi)$
- $\deg(\phi \circ \psi) = \deg(\phi) + \deg(\psi)$
- Degree 0 morphisms are invertible
- Each degree n morphism factors (up to iso) in $n!$ ways into degree 1 morphisms "compatibly with composition"

Remark

The Feynman categories $\mathfrak{F}_{ctd}, \mathfrak{F}_{cyc}, \mathfrak{F}_{mod}, \mathfrak{F}_{plan\text{-}cyc}, \mathfrak{F}_{surf\text{-}mod}$ are all cubical. The degree of ϕ is the number of edges of the associated graph $\Gamma(\phi)$. However, $\mathfrak{F}_{agg}$ is *not* cubical.

Definition (K-Ward)

A Feynman category $\mathfrak{F}$ is *cubical* if there is a degree function $\deg : Mor(\mathfrak{F}) \to \mathbb{N}_0$ such that

- $\deg(\phi \otimes \psi) = \deg(\phi) + \deg(\psi)$
- $\deg(\phi \circ \psi) = \deg(\phi) + \deg(\psi)$
- Degree 0 morphisms are invertible
- Each degree n morphism factors (up to iso) in $n!$ ways into degree 1 morphisms "compatibly with composition"

Remark

The Feynman categories $\mathfrak{F}_{ctd}, \mathfrak{F}_{cyc}, \mathfrak{F}_{mod}, \mathfrak{F}_{plan\text{-}cyc}, \mathfrak{F}_{surf\text{-}mod}$ are all cubical. The degree of ϕ is the number of edges of the associated graph $\Gamma(\phi)$. However, $\mathfrak{F}_{agg}$ is *not* cubical.

Definition (K-Ward)

A Feynman category $\mathfrak{F}$ is *cubical* if there is a degree function $\deg : Mor(\mathfrak{F}) \to \mathbb{N}_0$ such that

- $\deg(\phi \otimes \psi) = \deg(\phi) + \deg(\psi)$
- $\deg(\phi \circ \psi) = \deg(\phi) + \deg(\psi)$
- Degree 0 morphisms are invertible
- Each degree n morphism factors (up to iso) in $n!$ ways into degree 1 morphisms "compatibly with composition"

Remark

The Feynman categories $\mathfrak{F}_{ctd}, \mathfrak{F}_{cyc}, \mathfrak{F}_{mod}, \mathfrak{F}_{plan\text{-}cyc}, \mathfrak{F}_{surf\text{-}mod}$ are all cubical. The degree of ϕ is the number of edges of the associated graph $\Gamma(\phi)$. However, $\mathfrak{F}_{agg}$ is *not* cubical.

Definition (W-construction for $P \in \mathrm{Func}_{\otimes}(\mathfrak{F}, s\mathrm{Sets})$, $\mathfrak{F}$ cubical)

$$(W_{\mathfrak{F}}P)(B) = \Delta[1]^{\deg(-)} \otimes_{\mathfrak{F}\downarrow B} P \circ dom(-)$$

where $\Delta[1]^{\deg(-)}$ are cube embeddings via 0-face.

Proposition (K-Ward, cf. Boardman-Vogt, B-Moerdijk for $\mathfrak{F}_{sym}$)

For any cubical Feynman category $\mathfrak{F}$, the category $\mathrm{Func}_{\otimes}(\mathfrak{F}, s\mathrm{Sets})$ admits a *transferred model structure*. If P has an underying cofibrant $\mathcal{V}$-collection then $W_{\mathfrak{F}}P$ is *cofibrant* in $\mathrm{Func}_{\otimes}(\mathfrak{F}, s\mathrm{Sets})$.

Lemma (relative W-construction for cubical $f : \mathfrak{F} \to \mathfrak{F}'$)

$$(f_!(W_{\mathfrak{F}}P))(B) = \Delta[1]^{\deg(-)} \otimes_{f(-)\downarrow B} P \circ dom(-)$$

Definition (W-construction for $P \in \mathrm{Func}_{\otimes}(\mathfrak{F}, sSets)$, $\mathfrak{F}$ cubical)

$$(W_{\mathfrak{F}}P)(B) = \Delta[1]^{\deg(-)} \otimes_{\mathfrak{F}\downarrow B} P \circ dom(-)$$

where $\Delta[1]^{\deg(-)}$ are cube embeddings via 0-face.

Proposition (K-Ward, cf. Boardman-Vogt, B-Moerdijk for $\mathfrak{F}_{sym}$)

For any cubical Feynman category $\mathfrak{F}$, the category $\mathrm{Func}_{\otimes}(\mathfrak{F}, sSets)$ admits a *transferred model structure*. If P has an underying cofibrant $\mathcal{V}$-collection then $W_{\mathfrak{F}}P$ is *cofibrant* in $\mathrm{Func}_{\otimes}(\mathfrak{F}, sSets)$.

Lemma (relative W-construction for cubical $f : \mathfrak{F} \to \mathfrak{F}'$)

$$(f_!(W_{\mathfrak{F}}P))(B) = \Delta[1]^{\deg(-)} \otimes_{f(-)\downarrow B} P \circ dom(-)$$

Definition (W-construction for $P \in \mathrm{Func}_{\otimes}(\mathfrak{F}, s\mathrm{Sets})$, $\mathfrak{F}$ cubical)

$$(W_{\mathfrak{F}}P)(B) = \Delta[1]^{\deg(-)} \otimes_{\mathfrak{F}\downarrow B} P \circ dom(-)$$

where $\Delta[1]^{\deg(-)}$ are cube embeddings via 0-face.

Proposition (K-Ward, cf. Boardman-Vogt, B-Moerdijk for $\mathfrak{F}_{sym}$)

For any cubical Feynman category $\mathfrak{F}$, the category $\mathrm{Func}_{\otimes}(\mathfrak{F}, s\mathrm{Sets})$ admits a *transferred model structure*. If P has an underying cofibrant $\mathcal{V}$-collection then $W_{\mathfrak{F}}P$ is *cofibrant* in $\mathrm{Func}_{\otimes}(\mathfrak{F}, s\mathrm{Sets})$.

Lemma (relative W-construction for cubical $f : \mathfrak{F} \to \mathfrak{F}'$)

$$(f_!(W_{\mathfrak{F}}P))(B) = \Delta[1]^{\deg(-)} \otimes_{f(-)\downarrow B} P \circ dom(-)$$

Definition (W-construction for $P \in \mathrm{Func}_{\otimes}(\mathfrak{F}, s\mathrm{Sets})$, $\mathfrak{F}$ cubical)

$$(W_{\mathfrak{F}}P)(B) = \Delta[1]^{\deg(-)} \otimes_{\mathfrak{F}\downarrow B} P \circ dom(-)$$

where $\Delta[1]^{\deg(-)}$ are cube embeddings via 0-face.

Proposition (K-Ward, cf. Boardman-Vogt, B-Moerdijk for $\mathfrak{F}_{sym}$)

For any cubical Feynman category $\mathfrak{F}$, the category $\mathrm{Func}_{\otimes}(\mathfrak{F}, s\mathrm{Sets})$ admits a *transferred model structure*. If P has an underying cofibrant $\mathcal{V}$-collection then $W_{\mathfrak{F}}P$ is *cofibrant* in $\mathrm{Func}_{\otimes}(\mathfrak{F}, s\mathrm{Sets})$.

Lemma (relative W-construction for cubical $f : \mathfrak{F} \to \mathfrak{F}'$)

$$(f_!(W_{\mathfrak{F}}P))(B) = \Delta[1]^{\deg(-)} \otimes_{f(-)\downarrow B} P \circ dom(-)$$

Example (cubically subdivided convex polytopes)

- $W_{\mathfrak{F}_{sym}}(\mathrm{Ass})$(rooted corolla)=associahedron
- $W_{\mathfrak{F}_{cyc}}(\mathrm{CycAss})$(corolla)=cyclohedron

Theorem (B-K)

$|J_!(W_{\mathfrak{F}_{plan\text{-}cyc}}\mathbb{1})(g,n,S_\bullet)| \simeq \mathcal{MRG}_{g,n,S_\bullet}$

Definition (Igusa)

The category $\mathrm{rb}_{g,n,S_\bullet}$ has at least trivalent ribbon graphs of type $(g,n,S_\bullet)$ as objects, and ribbon contractions as morphisms.

Theorem (Igusa, B-K)

$\mathcal{MRG}_{g,n,S_\bullet} \simeq |nerve(\mathrm{rb}_{g,n,S_\bullet})|$

Thank you !

Example (cubically subdivided convex polytopes)

- $W_{\mathfrak{F}_{sym}}(\mathrm{Ass})$(rooted corolla)=associahedron
- $W_{\mathfrak{F}_{cyc}}(\mathrm{CycAss})$(corolla)=cyclohedron

Theorem (B-K)

$|J_!(W_{\mathfrak{F}_{plan\text{-}cyc}}\mathbb{1})(g,n,S_\bullet)| \simeq \mathcal{MRG}_{g,n,S_\bullet}$

Definition (Igusa)

The category $\mathrm{rb}_{g,n,S_\bullet}$ has at least trivalent ribbon graphs of type $(g,n,S_\bullet)$ as objects, and ribbon contractions as morphisms.

Theorem (Igusa, B-K)

$\mathcal{MRG}_{g,n,S_\bullet} \simeq |nerve(\mathrm{rb}_{g,n,S_\bullet})|$

Thank you !

Example (cubically subdivided convex polytopes)

- $W_{\mathfrak{F}_{sym}}(\mathrm{Ass})$(rooted corolla)=associahedron
- $W_{\mathfrak{F}_{cyc}}(\mathrm{CycAss})$(corolla)=cyclohedron

Theorem (B-K)

$|J_!(W_{\mathfrak{F}_{plan\text{-}cyc}}\mathbb{1})(g,n,S_\bullet)| \simeq \mathcal{MRG}_{g,n,S_\bullet}$

Definition (Igusa)

The category $\mathrm{rb}_{g,n,S_\bullet}$ has at least trivalent ribbon graphs of type $(g,n,S_\bullet)$ as objects, and ribbon contractions as morphisms.

Theorem (Igusa, B-K)

$\mathcal{MRG}_{g,n,S_\bullet} \simeq |nerve(\mathrm{rb}_{g,n,S_\bullet})|$

Thank you !

Example (cubically subdivided convex polytopes)

- $W_{\mathfrak{F}_{sym}}(\mathrm{Ass})$(rooted corolla)=associahedron
- $W_{\mathfrak{F}_{cyc}}(\mathrm{CycAss})$(corolla)=cyclohedron

Theorem (B-K)

$$|J_!(W_{\mathfrak{F}_{plan\text{-}cyc}}\mathbb{1})(g,n,S_\bullet)| \simeq \mathcal{MRG}_{g,n,S_\bullet}$$

Definition (Igusa)

The category $\mathrm{rb}_{g,n,S_\bullet}$ has at least trivalent ribbon graphs of type $(g,n,S_\bullet)$ as objects, and ribbon contractions as morphisms.

Theorem (Igusa, B-K)

$$\mathcal{MRG}_{g,n,S_\bullet} \simeq |nerve(\mathrm{rb}_{g,n,S_\bullet})|$$

Thank you !

Example (cubically subdivided convex polytopes)

- $W_{\mathfrak{F}_{sym}}(\mathrm{Ass})$(rooted corolla)=associahedron
- $W_{\mathfrak{F}_{cyc}}(\mathrm{CycAss})$(corolla)=cyclohedron

Theorem (B-K)

$|J_!(W_{\mathfrak{F}_{plan\text{-}cyc}}\mathbb{1})(g,n,S_\bullet)| \simeq \mathcal{MRG}_{g,n,S_\bullet}$

Definition (Igusa)

The category $\mathrm{rb}_{g,n,S_\bullet}$ has at least trivalent ribbon graphs of type $(g,n,S_\bullet)$ as objects, and ribbon contractions as morphisms.

Theorem (Igusa, B-K)

$\mathcal{MRG}_{g,n,S_\bullet} \simeq |nerve(\mathrm{rb}_{g,n,S_\bullet})|$

Thank you !

Example (cubically subdivided convex polytopes)

- $W_{\mathfrak{F}_{sym}}(\mathrm{Ass})$(rooted corolla)=associahedron
- $W_{\mathfrak{F}_{cyc}}(\mathrm{CycAss})$(corolla)=cyclohedron

Theorem (B-K)

$$|J_!(W_{\mathfrak{F}_{plan\text{-}cyc}}\mathbb{1})(g,n,S_\bullet)| \simeq \mathcal{MRG}_{g,n,S_\bullet}$$

Definition (Igusa)

The category $\mathrm{rb}_{g,n,S_\bullet}$ has at least trivalent ribbon graphs of type $(g,n,S_\bullet)$ as objects, and ribbon contractions as morphisms.

Theorem (Igusa, B-K)

$$\mathcal{MRG}_{g,n,S_\bullet} \simeq |nerve(\mathrm{rb}_{g,n,S_\bullet})|$$

Thank you !

Example (cubically subdivided convex polytopes)

- $W_{\mathfrak{F}_{sym}}(\mathrm{Ass})$(rooted corolla)=associahedron
- $W_{\mathfrak{F}_{cyc}}(\mathrm{CycAss})$(corolla)=cyclohedron

Theorem (B-K)

$|J_!(W_{\mathfrak{F}_{plan\text{-}cyc}}\mathbb{1})(g,n,S_\bullet)| \simeq \mathcal{MRG}_{g,n,S_\bullet}$

Definition (Igusa)

The category $\mathrm{rb}_{g,n,S_\bullet}$ has at least trivalent ribbon graphs of type $(g,n,S_\bullet)$ as objects, and ribbon contractions as morphisms.

Theorem (Igusa, B-K)

$\mathcal{MRG}_{g,n,S_\bullet} \simeq |nerve(\mathrm{rb}_{g,n,S_\bullet})|$

Thank you !

Example (cubically subdivided convex polytopes)

- $W_{\mathfrak{F}_{sym}}(\mathrm{Ass})$(rooted corolla)=associahedron
- $W_{\mathfrak{F}_{cyc}}(\mathrm{CycAss})$(corolla)=cyclohedron

Theorem (B-K)

$$|J_!(W_{\mathfrak{F}_{plan\text{-}cyc}}\mathbb{1})(g,n,S_\bullet)| \simeq \mathcal{MRG}_{g,n,S_\bullet}$$

Definition (Igusa)

The category $\mathrm{rb}_{g,n,S_\bullet}$ has at least trivalent ribbon graphs of type $(g,n,S_\bullet)$ as objects, and ribbon contractions as morphisms.

Theorem (Igusa, B-K)

$$\mathcal{MRG}_{g,n,S_\bullet} \simeq |nerve(\mathrm{rb}_{g,n,S_\bullet})|$$

Thank you !