

OPERADS OF NATURAL OPERATIONS I: LATTICE PATHS, BRACES AND HOCHSCHILD COCHAINS

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ABSTRACT. In this first paper of a series we study various operads of natural operations on Hochschild cochains and relationships between them.

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1. INTRODUCTION

This paper continues the efforts of [14, 3, 2] in which we studied operads naturally acting on Hochschild cochains of an associative or symmetric Frobenius algebra. A general approach to the operads of natural operations in algebraic categories was set up in [14] and the first breakthrough in computing the homotopy type of such an operad has been achieved in [3]. In [2], the same problem was approached from a combinatorial point of view, and a machinery which produces operads acting on the Hochschild cochain complex in a general categorical setting was introduced.

However, some special instances of the construction of [2] are important in applications and have specific features not present in general. In this first paper of a series entitled ‘Operads of Natural Operations’ we begin a detailed study of these special cases.

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It is very natural to start with the classical Hochschild cochain complex of an associative algebra. This is, by far, the most studied case. It seems to us, however, that a systematic treatment is missing despite its long history and a vast amount of literature available. One of the motivations for this paper was our wish to relate various approaches in literature and to provide a uniform combinatorial language for this purpose.

We achieve this goal by first describing the lattice path operad \mathcal{L} and its condensation in the differential graded setting in section 2. This description leads to a careful treatment of (higher) brace operations and their relationship with lattice paths in section 3.

The lattice path operad comes equipped with a filtration by complexity [2]. The second filtration stage $\mathcal{L}_{(2)}$ is the most important for understanding natural operations on the Hochschild cochains. In section 4 we give an alternative description of $\mathcal{L}_{(2)}$ in terms of trees, closely related to the operad of natural operations from [14]. Finally, in section 5 we study various suboperads generated by brace operations. The main result is that all these operads have the same homotopy type, namely that of a chain model of the little disks operad. For sake of completeness we add a brief appendix containing an overview of some categorical constructions used in this paper.

Convention. If not stated otherwise, by an *operad* we mean a classical symmetric (i.e. with the symmetric groups acting on its components) operad in an appropriate symmetric monoidal category which will be obvious from the context. The same convention is applied to coloured operads, substitutes, multitensors and functor-operads recalled in the appendix.

2. THE LATTICE PATH OPERAD

As usual, for a non-negative integer m , $[m]$ denotes the ordinal $0 < \dots < m$. We will use the same symbol also for the category with objects $0, \dots, m$ and the unique morphism $i \rightarrow j$ if and only if $i \leq j$. The *tensor product* $[m] \otimes [n]$ is the category freely generated by the (m, n) -grid which is, by definition, the oriented graph with vertices (i, j) , $0 \leq i \leq m$, $0 \leq j \leq n$, and one oriented edge $(i', j') \rightarrow (i'', j'')$ if and only if $(i'', j'') = (i' + 1, j')$ or $(i'', j'') = (i', j' + 1)$.

Let us recall, closely following [2], the *lattice path operad* and its basic properties. For non-negative integers k_1, \dots, k_n, l and $n \in \mathbb{N}$ put

$$\mathcal{L}(k_1, \dots, k_n; l) := \text{Cat}_{*,*}([l+1], [k_1+1] \otimes \dots \otimes [k_n+1])$$

where \otimes is the tensor product recalled above and $\text{Cat}_{*,*}([l+1], [k_1+1] \otimes \dots \otimes [k_n+1])$ the set of functors φ that preserve the extremal points, by which we mean that

$$(1) \quad \varphi(0) = (0, \dots, 0) \quad \text{and} \quad \varphi(l+1) = (k_1+1, \dots, k_n+1).$$

A functor $\varphi \in \mathcal{L}(k_1, \dots, k_n; l)$ is given by a chain of $l+1$ morphisms $\varphi(0) \rightarrow \varphi(1) \rightarrow \dots \rightarrow \varphi(l+1)$ in $[k_1+1] \otimes \dots \otimes [k_n+1]$ with $\varphi(0)$ and $\varphi(l+1)$ fulfilling (1). Each morphism $\varphi(i) \rightarrow \varphi(i+1)$ is determined by a finite oriented edge-path in the (k_1+1, \dots, k_n+1) -grid.

[June 17, 2009]

2.1. Marked lattice paths. We will use a slight modification of the terminology of [2]. For non-negative integers $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$ denote by $\mathcal{Q}(k_1, \dots, k_n)$ the integral hypercube

$$\mathcal{Q}(k_1, \dots, k_n) := [k_1 + 1] \times \dots \times [k_n + 1] \subset \mathbb{Z}^n.$$

A *lattice path* is a sequence $p = (x_1, \dots, x_N)$ of $N := k_1 + \dots + k_n + n + 1$ points of $\mathcal{Q}(k_1, \dots, k_n)$ such that x_{a+1} is, for each $0 \leq a < N$, given by increasing exactly one coordinate of x_a by 1. A *marking* of p is a function $\mu : p \rightarrow \mathbb{Z}_{\geq 0}$ that assigns to each point x_a of p a non-negative number $\mu_a := \mu(x_a)$ such that $\sum_{a=1}^N \mu_a = l$.

We can describe functors in $\mathcal{L}(k_1, \dots, k_n; l)$ as marked lattice paths (p, μ) in the hypercube $\mathcal{Q}(k_1, \dots, k_n)$. The marking $\mu_a = \mu(x_a)$ represents the number of *internal* points of $[l + 1]$ that are mapped by φ to the a th lattice point x_a of p . We call lattice points marked by 0 *unmarked* points so the set of marked points equals $\varphi(\{1, \dots, l\})$. For example, the marked lattice path



represents a functor $\varphi \in \mathcal{L}(3, 2; 8)$ with $\varphi(0) = (0, 0)$, $\varphi(1) = \varphi(2) = \varphi(3) = (1, 0)$, $\varphi(4) = (2, 0)$, $\varphi(5) = \varphi(6) = (3, 1)$ and $\varphi(7) = \varphi(8) = \varphi(9) = (4, 3)$. The path has 4 angles, 2 internal points, 4 unmarked points and 1 unmarked internal point.

2.2. Definition. Let $p \in \mathcal{L}(k_1, \dots, k_n; l)$ be a lattice path. A point of p at which p changes its direction is an *angle* of p . An *internal point* of p is a point that is not an angle nor an extremal point of p . We denote by $Angl(p)$ (resp. $Int(p)$) the set of all angles (resp. internal points) of p .

Following again [2] closely, we denote, for $1 \leq i < j \leq n$, by p_{ij} the projection of the path $p \in \mathcal{L}(k_1, \dots, k_n; l)$ to the face $[k_i + 1] \times [k_j + 1]$ of $\mathcal{Q}(k_1, \dots, k_n)$; let $c_{ij} := \#Angl(p_{ij})$ be the number of its angles. The maximum $c(p) := \max\{c_{ij}\}$ is called the *complexity* of p . Let us finally denote by $\mathcal{L}_{(c)}(k_1, \dots, k_n; l) \subset \mathcal{L}(k_1, \dots, k_n; l)$ the subset of marked lattice paths of complexity $\leq c$. The case $c = 2$ is particularly interesting, because $\mathcal{L}_{(2)}(k_1, \dots, k_n; l)$ is, by [2, Proposition 2.14], isomorphic to the space of unlabeled $(l; k_1, \dots, k_n)$ -trees recalled on page 18. For convenience of the reader we recall this isomorphism on page 18.

As shown in [2], the sets $\mathcal{L}(k_1, \dots, k_n; l)$ together with the subsets $\mathcal{L}_{(c)}(k_1, \dots, k_n; l)$, $c \geq 0$, form a $\mathbb{Z}_{\geq 0}$ -colored operad \mathcal{L} and its sub-operads $\mathcal{L}_{(c)}$. To simplify formulations, we will allow $c = \infty$, putting $\mathcal{L}_{(\infty)} := \mathcal{L}$.

2.3. Convention. Since we aim to work in the category of abelian groups, we will make no notational difference between the sets $\mathcal{L}_{(c)}(k_1, \dots, k_n; l)$ and their linear spans.

The underlying category of the colored operad \mathcal{L} (which coincides with the underlying category of $\mathcal{L}_{(c)}$ for any $c \geq 0$) is, by definition, the category whose objects are non-negative integers and morphism $n \rightarrow m$ are elements of $\mathcal{L}(n, m)$, i.e. non-decreasing maps $\varphi : [m+1] \rightarrow [n+1]$ preserving the endpoints. By Joyal's duality [12], this category is isomorphic to the (skeletal) category Δ of finite ordered sets, i.e. $\mathcal{L}(n, m) = \Delta(n, m)$. The operadic composition makes the collection $\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; \bullet)$ (with $c = \infty$ allowed) a functor $(\Delta^{\text{op}})^{\times n} \times \Delta \rightarrow \text{Abel}$, i.e. n -times simplicial 1-time cosimplicial Abelian group.

Morphisms in the category Δ are generated by the cofaces $d_i : [m-1] \rightarrow [m]$ given by the non-decreasing map that misses i , and the codegeneracies $s_i : [m+1] \rightarrow [m]$ given by the non-decreasing map that hits i twice. In both cases, $0 \leq i \leq m$. Let us inspect how these generating maps act on the pieces of the operad $\mathcal{L}_{(c)}$.

2.4. Simplicial structures. We describe the induced r th ($1 \leq r \leq n$) simplicial maps

$$\partial_i^r : \mathcal{L}_{(c)}(k_1, \dots, k_{i-1}, m, k_{r+1}, \dots, k_n; l) \rightarrow \mathcal{L}_{(c)}(k_1, \dots, k_{i-1}, m-1, k_{r+1}, \dots, k_n; l),$$

where $m \geq 1$, $0 \leq i \leq m$, and

$$\sigma_i^r : \mathcal{L}_{(c)}(k_1, \dots, k_{i-1}, m, k_{r+1}, \dots, k_n; l) \rightarrow \mathcal{L}_{(c)}(k_1, \dots, k_{i-1}, m+1, k_{r+1}, \dots, k_n; l),$$

where $0 \leq i \leq m$. To this end, we define, for each $m \geq 1$ and $0 \leq i \leq m$, the epimorphism of the hypercubes

$$D_i^r : \mathcal{Q}(k_1, \dots, k_{i-1}, m, k_{r+1}, \dots, k_n) \twoheadrightarrow \mathcal{Q}(k_1, \dots, k_{i-1}, m-1, k_{r+1}, \dots, k_n)$$

by

$$D_i^r(a_1, \dots, a_r, \dots, a_n) := \begin{cases} (a_1, \dots, a_r, \dots, a_n), & \text{if } a_r \leq i, \text{ and} \\ (a_1, \dots, a_r - 1, \dots, a_n), & \text{if } a_r > i, \end{cases}$$

where $(a_1, \dots, a_r, \dots, a_n) \in \mathcal{Q}(k_1, \dots, k_{i-1}, m, k_{r+1}, \dots, k_n)$ is an arbitrary point. In a similar fashion, the monomorphism

$$S_i^r : \mathcal{Q}(k_1, \dots, k_{i-1}, m, k_{r+1}, \dots, k_n) \hookrightarrow \mathcal{Q}(k_1, \dots, k_{i-1}, m+1, k_{r+1}, \dots, k_n)$$

is, for $0 \leq i \leq m$, given by

$$S_i^r(a_1, \dots, a_r, \dots, a_n) := \begin{cases} (a_1, \dots, a_r, \dots, a_n), & \text{if } a_r \leq i, \text{ and} \\ (a_1, \dots, a_r + 1, \dots, a_n), & \text{if } a_r > i. \end{cases}$$

Let (p, μ) be a marked lattice path in $\mathcal{Q}(k_1, \dots, k_{i-1}, m, k_{r+1}, \dots, k_n)$ representing a functor $\varphi \in \mathcal{L}_{(c)}(k_1, \dots, k_{i-1}, m, k_{r+1}, \dots, k_n; l)$. Then $\partial_i^r(\varphi)$ is represented by the marked path $(\partial_i^r(p), \partial_i^r(\mu))$, where $\partial_i^r(p)$ is the image $D_i^r(p)$ of p in $\mathcal{Q}(k_1, \dots, k_{i-1}, m-1, k_{r+1}, \dots, k_n, l)$. The marking $\partial_i^r(\mu)$ is given by $\partial_i^r(\mu)(D_i^r(x)) := \sum_{\tilde{x}} \mu(\tilde{x})$, with the sum taken over all $\tilde{x} \in p$ such that $D_i^r(\tilde{x}) = D_i^r(x)$. A less formal description of this marking is the following.

There are precisely two different points of p , say x' and x'' , such that $D_i^r(x') = D_i^r(x'')$; let us call the remaining points of p regular. The marking of $D_i^r(x)$ is the same as the marking of x if [June 17, 2009]

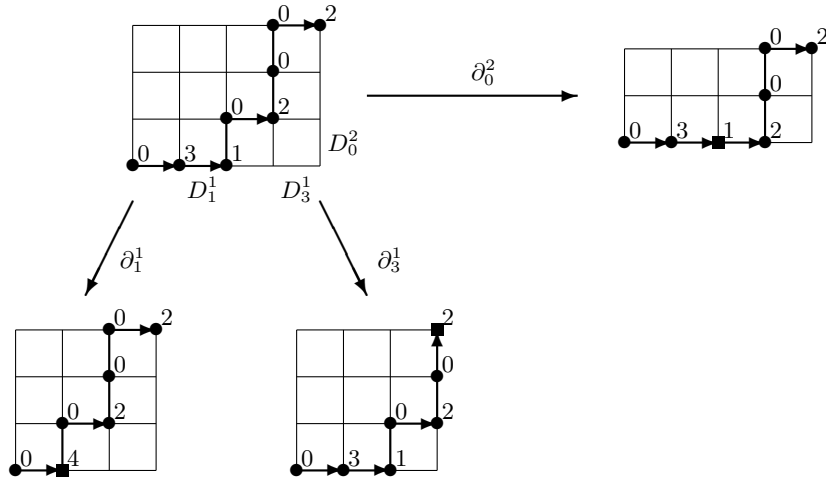


FIGURE 1. The simplicial boundaries acting on the element of (2).

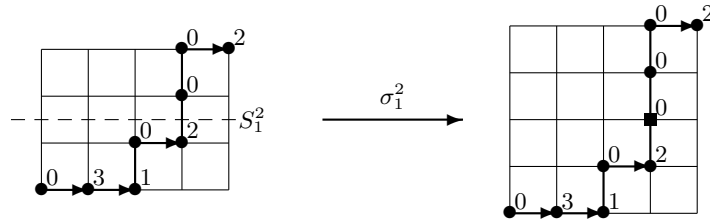


FIGURE 2. The operator σ_1^2 acting on the element of (2).

x is regular. If x' and x'' are the two non-regular points, then the marking of the common value $D_i^r(x') = D_i^r(x'')$ is $\mu(x') + \mu(x'')$. See Figure 1 in which the operator ∂_1^1 contracts the column denoted D_1^1 and decorates the point \blacksquare obtained by identifying the point $(1,0)$ marked 3 with the point $(2,0)$ marked 1 by $3 + 1 = 4$. The remaining operators act in the similar fashion.

To define the marked lattice path $(\sigma_i^r(p), \sigma_i^r(\mu))$ representing the degeneracy $\sigma_i^r(\varphi)$, we need to observe that the image $S_i^r(p)$ is not a lattice path in $\mathcal{Q}(k_1, \dots, k_{i-1}, m-1, k_{r+1}, \dots, k_n)$, but that it can be made one by adding a unique ‘missing’ lattice point \hat{x} . The resulting lattice path is $\sigma_i^r(p)$. The marking $\sigma_i^r(\mu)$ is given by $\sigma_i^r(\mu)(S_i^t(x)) := \mu(x)$ for $x \in p$ while $\sigma_i^r(\mu)(\hat{x}) := 0$, i.e. the newly added point \hat{x} is unmarked. See Figure 2 in which the new point \hat{x} is denoted \blacksquare . Observe that \hat{x} is always an internal point.

2.5. The cosimplicial structure. We describe, for $l \geq 1$ and $0 \leq i \leq l$, the boundaries

$$\delta^i : \mathcal{L}_{(c)}(k_1, \dots, k_n; l-1) \rightarrow \mathcal{L}_{(c)}(k_1, \dots, k_n; l)$$

and, for $0 \leq i \leq l$, the degeneracies

$$s^i : \mathcal{L}_{(c)}(k_1, \dots, k_n, l+1) \rightarrow \mathcal{L}_{(c)}(k_1, \dots, k_n; l),$$

of the induced cosimplicial structure. Let (p, μ) be a marked edge path in $\mathcal{Q}(k_1, \dots, k_n; l \mp 1)$ representing a functor $\varphi \in \mathcal{L}_{(c)}(k_1, \dots, k_n; l \mp 1)$. Neither δ^i nor s^i changes the underlying path, so $\delta^i(\varphi)$ is represented by $(p, \delta^i(\mu))$ and $s^i(\varphi)$ by $(p, s^i(\mu))$.

Let $\hat{x} := \varphi(i)$. Then the markings $\delta^i(\mu)$ and $s^i(\mu)$ are defined by $\delta^i(\mu)(x) = s^i(\mu)(x) = \mu(x)$ for $x \neq \hat{x}$, while $\delta^i(\mu)(\hat{x}) := \mu(\hat{x}) + 1$ and $s^i(\mu)(\hat{x}) := \mu(\hat{x}) - 1$.

3. WEAK EQUIVALENCES

3.1. Un-normalized totalizations. For an n -simplicial cosimplicial abelian group, i.e. a functor $X : \Delta^{\text{op} \times n} \times \Delta \rightarrow \text{Abel}$, denote by $X_*^\bullet = \underline{\text{Tot}}(X(\bullet_1, \dots, \bullet_n; \bullet))$ the simplicial totalization. It is a cosimplicial dg-abelian group with components

$$(3) \quad X_*^\bullet := \bigoplus_{* = -(k_1 + \dots + k_n)} X(k_1, \dots, k_n; \bullet)$$

bearing the degree +1 differential $\partial = \partial^1 + \dots + \partial^n$, where each ∂^r is induced from the boundaries of the r th simplicial structure in the standard manner. We also denote by $|X|^* = \overline{\text{Tot}}(\underline{\text{Tot}}(X(\bullet_1, \dots, \bullet_n; \bullet)))$ the cosimplicial totalization of the cosimplicial dg-abelian group X_*^\bullet . It is a dg-abelian group with components

$$|X|^* = \prod_{* = l - (k_1 + \dots + k_n)} X(k_1, \dots, k_n; l) = \prod_{l \geq 0} \bigoplus_{l - * = k_1 + \dots + k_n} X(k_1, \dots, k_n; l)$$

and the degree +1 differential $d = \delta + \partial$, where ∂ is as above and δ is the standard alternating sum of the cosimplicial boundary operators.

According to appendix A, the dg-abelian groups $|\mathcal{L}_{(c)}|(n) := |\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; \bullet)|$ are the result of condensation and, therefore, assemble, for each $c \geq 0$, into a dg-operad $|\mathcal{L}_{(c)}| = \{|\mathcal{L}_{(c)}|(n)\}_{n \geq 0}$. Observe that $|\mathcal{L}_{(2)}|$ is isomorphic to the Tamarkin-Tsygan operad \mathcal{T} recalled on page 18.¹

Let us denote, for each $n, c \geq 0$, by $\mathcal{B}r_{(c)}(n)$ the simplicial totalization of the n -times simplicial abelian group $\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; 0)$, that is,

$$\mathcal{B}r_{(c)}^*(n) := \bigoplus_{* = -(k_1 + \dots + k_n)} \mathcal{L}_{(c)}(k_1, \dots, k_n; 0),$$

with the induced differential $\partial = \partial^1 + \dots + \partial^n$. Elements of $\mathcal{B}r_{(c)}(n)$ are represented by marked lattice paths $(p, 0)$ with the trivial marking $\mu = 0$ (all points of p are unmarked). Since the trivial marking bears no information, we will discard it from the notation. The *whiskering* $w : \mathcal{B}r_{(c)}(n) \rightarrow |\mathcal{L}_{(c)}|(n)$ is defined as

$$(4) \quad w(p) := \prod_{s \geq 0} w_s(p),$$

¹Whenever we refer to Sections 4 and/or 5 we shall keep in mind that Convention 4.2 is used there.

where $w_s(p) \in |\mathcal{L}_{(c)}|(n)$ is the sum of all marked paths obtained from p by inserting precisely s distinct internal lattice points marked 1. Notice that all terms in (4) appear with the $+1$ sign.

For $p' \in \mathcal{L}_{(c)}(a_1, \dots, a_n; 0)$, $p'' \in \mathcal{L}_{(c)}(b_1, \dots, b_m; 0)$ and $1 \leq i \leq n$ define

$$(5) \quad p' \circ_i p'' := p' \circ_i w_{a_i}(p'') \in \bigoplus_{b'_1 + \dots + b'_m = b_1 + \dots + b_m + a_i} \mathcal{L}_{(c)}(a_1, \dots, a_{i-1}, b'_1, \dots, b'_m, a_{i+1}, \dots, a_n; 0)$$

where $w_{a_i}(p'')$ is the whiskering of the lattice path p'' by a_i points and \circ_i in the right hand side is the operadic composition in the colored operad $\mathcal{L}_{(c)}$. By linearity, (5) extends to the operation $\circ_i : \mathcal{B}r_{(c)}(m) \otimes \mathcal{B}r_{(c)}(n) \rightarrow \mathcal{B}r_{(c)}(m+n-1)$.

3.2. Proposition. *Operations \circ_i defined above make the collection $\mathcal{B}r_{(c)} = \{\mathcal{B}r_{(c)}(n)\}_{n \geq 1}$ a dg-operad. The map $w : \mathcal{B}r_{(c)} \hookrightarrow |\mathcal{L}_{(c)}|$ is an inclusion of dg-operads.*

Proof. A direct verification. □

So the operad structure of $\mathcal{B}r_{(c)}$ is induced by the operad structure of $|\mathcal{L}_{(c)}|$ and the whiskering map. Notice that $\mathcal{B}r_{(2)}$ is the brace operad $\mathcal{B}r$ recalled on page 20 and the map $w : \mathcal{B}r_{(2)} \rightarrow |\mathcal{L}_{(2)}|$ the whiskering defined in (15). Proposition 3.2 therefore generalizes Proposition 5.7.

3.3. Normalized totalizations. Let $X(\bullet_1, \dots, \bullet_n; \bullet)$ be an n -simplicial cosimplicial abelian group as in 3.1. We will need also the more traditional n -simplicial *normalized* totalization, or simplicial normalization for short, denoted $\overline{X}_*^\bullet = \underline{\text{Nor}}(X(\bullet_1, \dots, \bullet_n; \bullet))$, obtained from the un-normalized totalization (3) by modding out the images of simplicial degeneracies. We then denote by $|\overline{X}|^* = \overline{\text{Nor}}(\underline{\text{Nor}}(X(\bullet_1, \dots, \bullet_n; \bullet)))$ the normalized cosimplicial totalization of the cosimplicial dg-abelian group \overline{X}_*^\bullet . It is the intersection of the kernels of cosimplicial degeneracies in the un-normalized cosimplicial totalization of \overline{X}_*^\bullet . As argued in [2], the n -simplicial cosimplicial normalization $|\overline{\mathcal{L}}_{(c)}|$ of the lattice path operad $\mathcal{L}_{(c)}$ is a dg-operad.

Let us denote, for each $n, c \geq 0$, by $\overline{\mathcal{B}r}_{(c)}(n) = \underline{\text{Nor}}(\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; 0))$ the simplicial normalization of the n -simplicial abelian group $\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; 0)$, with the induced differential. The explicit description of the simplicial structure in 2.4 makes it obvious that elements of $\overline{\mathcal{B}r}_{(c)}(n)$ are represented by (unmarked) lattice paths with no internal points.

One defines the operadic composition on the collection $\overline{\mathcal{B}r}_{(c)} = \{\overline{\mathcal{B}r}_{(c)}(n)\}_{n \geq 0}$ and the whiskering $w : \overline{\mathcal{B}r}_{(c)} \hookrightarrow |\overline{\mathcal{L}}_{(c)}|$ by the same formulas as in the un-normalized case. The operad $\overline{\mathcal{B}r}_{(2)}$ is the normalized brace operad $\widehat{\mathcal{B}r}$ recalled on page 20. We leave as an exercise to verify that $\overline{\mathcal{B}r}_{(1)}$ is the operad for associative algebras and $\overline{\mathcal{B}r}_{(0)}$ the trivial unital operad.

3.4. Proposition. *The operads $\overline{\mathcal{B}r}_{(c)}$ equal the suboperads $F_c \mathcal{X}$ of the surjection operad \mathcal{X} introduced in [4, 1.6.2], resp. the suboperads \mathcal{S}_c of the sequence operad \mathcal{S} introduced in [17, Definition 3.2].*

Proof. We rely on the terminology of [4, 1.6.2]. A non-degenerate surjection $u : \{1, \dots, m\} \rightarrow \{1, \dots, n\}$, $m \geq n$, in $F_c\mathcal{X}(n)$ induces a lattice path φ_u representing an element of $\overline{\mathcal{B}r}_{(c)}(n)$ as follows. For $1 \leq i \leq n$ denote by $d_i \in \mathbb{Z}^{\times n}$ the vector $(0, \dots, 1, \dots, 0)$ with 1 at the i th position, and $k_i := \#u^{-1}(i) - 1$. Then φ_u is the path in the grid $[k_1 + 1] \otimes \cdots \otimes [k_n + 1]$ that starts at the ‘lower left corner’ $(0, \dots, 0)$, advances by $d_{u(1)}$, then by $d_{u(2)}$, etc., and finally by $d_{u(m)}$. It is obvious that the correspondence $u \mapsto \varphi_u$ is one-to-one. \square

The following statement follows from [4, Section 1.2].

3.5. Proposition. *The whiskering $w : \overline{\mathcal{B}r}_{(c)} \hookrightarrow |\overline{\mathcal{L}}_{(c)}|$ is an inclusion of dg-operads.*

We will need also the following statement.

3.6. Proposition. *The natural projection $\pi : \mathcal{B}r_{(c)} \rightarrow \overline{\mathcal{B}r}_{(c)}$ to the normalization is an epimorphism of dg-operads for each $c \geq 0$.*

Proof. It is almost obvious that the operadic composition in $\mathcal{B}r_{(c)}$ preserves the number of internal points, that is, if p' (resp. p'') is a lattice path with a' (resp. a'') internal points, then $p' \circ_i p''$ is, for each i for which this expression makes sense, a linear combination of lattice paths with $a' + a''$ internal points. This implies that the degenerate subspace $Dgn(\mathcal{B}r_{(c)})$ of $\mathcal{B}r_{(c)}$ which is the sub-collection spanned by lattice paths with at least one internal point, form a dg-operadic ideal in $\mathcal{B}r_{(c)}$, so the projection $\pi : \mathcal{B}r_{(c)} \rightarrow \mathcal{B}r_{(c)}/Dgn(\mathcal{B}r_{(c)}) = \overline{\mathcal{B}r}_{(c)}$ is an operad map. The fact that π commutes with the differentials follows from the standard properties of the simplicial normalizations. \square

3.7. Semi-normalizations. For each $n, c \geq 0$, one may also consider the collection $|\dot{\mathcal{L}}_{(c)}| := \{|\dot{\mathcal{L}}_{(c)}|(n)\}_{n \geq 0}$ defined by

$$|\dot{\mathcal{L}}_{(c)}|(n) := \overline{\text{Tot}}(\underline{\text{Nor}}(\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; \bullet)))$$

i.e. as the n -simplicial normalization followed by the un-normalized cosimplicial totalization.

We emphasize that, for $c \geq 3$, the collection $|\dot{\mathcal{L}}_{(c)}|$ has no natural dg-operad structure although it will still play an important auxiliary role in this section. We, however, have

3.8. Proposition. *For $c \leq 2$, the sub-collection $|\dot{\mathcal{L}}_{(c)}|$ of $|\mathcal{L}_{(c)}|$ is closed under the dg-operad structure of $|\mathcal{L}_{(c)}|$. In particular, for $c \leq 2$, $|\dot{\mathcal{L}}_{(c)}|$ is a dg-operad.*

Proof. The proof uses the fact that $|\dot{\mathcal{L}}_{(2)}|$ is the normalized Tamarkin-Tsygan operad $\widehat{\mathcal{T}}$ which is a suboperad of $\mathcal{T} = |\mathcal{L}_{(2)}|$, see 4.7. This proves the proposition for $c = 2$. For $c = 0, 1$ the claim is obvious. \square

Another particular feature of the $c \leq 2$ -case is:

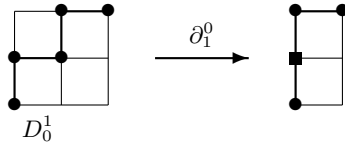
[June 17, 2009]

3.9. Proposition. *For $c \leq 2$, there is a natural dg-operad inclusion $\overline{\mathcal{B}r}_{(c)} \hookrightarrow \mathcal{B}r_{(c)}$ which is a right inverse of the normalization projection $\pi : \mathcal{B}r_{(c)} \rightarrow \overline{\mathcal{B}r}_{(c)}$.*

Proof. The natural inclusion $\overline{\mathcal{B}r}_{(c)} \hookrightarrow \mathcal{B}r_{(c)}$ is given, for any $c \geq 0$, by representing elements of $\overline{\mathcal{B}r}_{(c)}$ as linear combinations of lattice paths without internal points. We need to prove that, for $c \leq 2$, $\overline{\mathcal{B}r}_{(c)}$ interpreted as a subcollection of $\mathcal{B}r_{(c)}$ via the above inclusion, is a dg-suboperad, and that the induced operad structure agrees with the one of $\overline{\mathcal{B}r}_{(c)}$.

For $c = 2$ this follows from the fact that $\overline{\mathcal{B}r}_{(2)} = \widehat{\mathcal{B}r}$ is a dg-suboperad of $\mathcal{B}r_{(2)} = \mathcal{B}r$, see page 20. For $c = 0, 1$, the proposition is obvious. □

It follows from the property stated in the proof of Proposition 3.6 that the sub-collection $\overline{\mathcal{B}r}_{(c)}$ is closed under the operad structure of $\mathcal{B}r_{(c)}$ for an arbitrary $c \geq 0$. Unfortunately, the differential may create internal points if $c \geq 3$ as shown in the following picture where the piece ∂_1^0 of the differential creates the internal point ■ :



The above observation is so interesting that we formulate it as

3.10. Proposition. *The collection $\overline{\mathcal{B}r}_{(c)}$ is a (non-dg) suboperad of $\mathcal{B}r_{(c)}$ for any $c \geq 3$. It is dg-closed only if $c \leq 2$.*

The main theorem of this section reads:

3.11. Theorem. *For each $c \geq 0$, there is the following chain of weak equivalences of dg-operads:*

$$|\mathcal{L}_{(c)}| \xleftarrow{w} \mathcal{B}r_{(c)} \xrightarrow{\pi} \overline{\mathcal{B}r}_{(c)} \xrightarrow{w} |\overline{\mathcal{L}}_{(c)}|,$$

in which the maps w are the whiskerings of Propositions 3.2 and 3.5, and π is the normalization projection of Proposition 3.6.

Proof. The map $\pi : \mathcal{B}r_{(c)}(n) \rightarrow \overline{\mathcal{B}r}_{(c)}(n)$ is a homology isomorphism for each $n, c \geq 0$ because it is the normalization map of an n -simplicial abelian group, so π is a weak equivalence of dg operads. Let us analyze the un-normalized whiskering $w : \mathcal{B}r_{(c)}(n) \hookrightarrow |\mathcal{L}_{(c)}|(n)$.

The n th piece of the dg-operad $|\mathcal{L}_{(c)}|$ can be organized into the bicomplex of Figure 3 in which the l th column $\mathcal{L}_{(c)}(n)_*^l$, $l \geq 0$, is the simplicial totalization $\text{Tot}(\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; l))$ and the horizontal differentials are induced from the cosimplicial structure. The dg-abelian group $|\mathcal{L}_{(c)}|(n)$ is then the corresponding Tot^Π -total complex (see [19, Section 5.6] for the terminology).

The dg-abelian group $\mathcal{B}r_{(c)}(n)$ appears as the leftmost column of Figure 3, so one has the projection $\text{proj} : |\mathcal{L}_{(c)}|(n) \rightarrow \mathcal{B}r_{(c)}(n)$ of dg-abelian groups which is the identity on the leftmost

$$\begin{array}{ccccccc}
& & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{L}_{(c)}(n)_0^0 & \longrightarrow & \mathcal{L}_{(c)}(n)_0^1 & \longrightarrow & \mathcal{L}_{(c)}(n)_0^2 & \longrightarrow & \cdots \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{L}_{(c)}(n)_1^0 & \longrightarrow & \mathcal{L}_{(c)}(n)_1^1 & \longrightarrow & \mathcal{L}_{(c)}(n)_1^2 & \longrightarrow & \cdots \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{L}_{(c)}(n)_2^0 & \longrightarrow & \mathcal{L}_{(c)}(n)_2^1 & \longrightarrow & \mathcal{L}_{(c)}(n)_2^2 & \longrightarrow & \cdots \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{L}_{(c)}(n)_3^0 & \longrightarrow & \mathcal{L}_{(c)}(n)_3^1 & \longrightarrow & \mathcal{L}_{(c)}(n)_3^2 & \longrightarrow & \cdots \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & \vdots & & \vdots & & \vdots & &
\end{array}$$

FIGURE 3. The structure of the dg-operad $|\mathcal{L}_{(c)}|$.

column and sends the remaining columns to 0. Since clearly $proj \circ w = \text{id}$, it is enough to prove that $proj$ is a homology isomorphism.

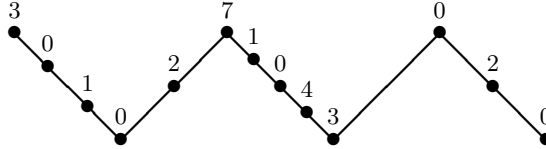
We interpret $proj : |\mathcal{L}_{(c)}|(n) \rightarrow \mathcal{B}r_{(c)}(n)$ as a map of bicomplexes, with $\mathcal{B}r_{(c)}(n)$ consisting of one column, and prove that $proj$ induces an isomorphism of the E^2 -terms of the spectral sequences induced by the column filtrations. These filtrations are complete and exhaustive, thus the Eilenberg-Moore comparison theorem [19, Theorem 5.5.1] implies that $proj$ is a homology isomorphism.

Let (E_{**}^0, d^0) be the 0th term of column spectral sequence for $|\mathcal{L}_{(c)}|(n)$. This means that $(E_{l,*}^0, d^0) = (\underline{\text{Tot}}(\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; l))_*, \partial)$, the l th column of the bicomplex in Figure 3 with the simplicial differential. To calculate $E_{l,*}^1 := H_*(E_{l,*}^0, d^0)$, we recall the explicit description of the simplicial structures given in 2.4 and observe that the vertical differential $d^0 = \partial$ does not increase the number of angles of lattice paths.

We therefore have, for each $l \geq 0$, another spectral sequence $(\overline{E}_{**}^r, \overline{d}^r)$ induced by the filtration of $\mathcal{L}_{(c)}(n)_*^l$ by the number of angles. The piece \overline{E}_{uv}^0 of the initial sheet of this spectral sequence is spanned by marked paths $(p, \mu) \in \mathcal{L}_{(c)}(k_1, \dots, k_n; l)$ with u angles and $v = k_1 + \dots + k_n$. Since $(\overline{E}_{**}^r, \overline{d}^r)$ is a first-quadrant spectral sequence, no convergence problems occur. One easily sees that, as dg-abelian groups,

$$(6) \quad (\overline{E}_{**}^0, \overline{d}^0) \cong \bigoplus_{p \in \overline{\mathcal{B}r}_{(c)}(n)} \underbrace{\{\mathbf{B}_* \otimes_{\mathbb{Z}[x]} \cdots \otimes_{\mathbb{Z}[x]} \mathbf{B}_*\}}_{\# \text{Angl}(p) + 1 \text{ factors}}$$

where $\mathbf{B}_* = \mathbf{B}_*(\mathbb{Z}[x], \mathbb{Z}[x], \mathbb{Z}[x])$ is the un-normalized two-sided bar construction of the polynomial algebra $\mathbb{Z}[x]$. The subscript l in (6) denotes the l -homogeneous part with respect to the grading induced by the number of instances of x . The factors of the direct sum are indexed by unmarked paths with no internal points representing a basis of $\overline{\mathcal{B}r}_{(c)}(n)$. The isomorphism (6) is best explained by looking at the marked path



with 4 angles which is represented, via the isomorphism (6), by the element

$$x^3 \otimes [x^0|x^1] \otimes x^0 \otimes [x^2] \otimes x^7 \otimes [x^1|x^0|x^4] \otimes x^3 \otimes [] \otimes x^0 \otimes [x^2] \otimes x^0$$

in $\mathbf{B}_2 \otimes_{\mathbb{Z}[x]} \mathbf{B}_1 \otimes_{\mathbb{Z}[x]} \mathbf{B}_3 \otimes_{\mathbb{Z}[x]} \mathbf{B}_0 \otimes_{\mathbb{Z}[x]} \mathbf{B}_1$. It is a standard result of homological algebra that the summands of (6) are acyclic in positive dimensions, with the homology spanned by cycles of the form

$$(7) \quad x^l \otimes [] \otimes \cdots \otimes [].$$

At this point we need to observe that the differential ∂ decreases the number of angles of lattice paths p with no internal points representing elements of $\overline{\mathcal{B}r}_{(c)}(n)$ by one. Indeed, it is easy to see that a simplicial boundary operator described in 2.4 may either decrease the number of angles of p by 1 or by 2. When it decreases it by 2 it creates an internal point, so the contributions of all simplicial boundaries that decrease the number of angles by 2 sum up to 0, by the standard property of the simplicial normalization. We conclude that $(\bigoplus_{*=u+v} \overline{E}_{uv}^1, \overline{d}^1) \cong (\overline{\mathcal{B}r}_{(c)*}(n), \partial)$ as dg-abelian groups and that $(\overline{E}_{**}^r, \overline{d}^r)$ collapses at this level.

Let us return to the column spectral sequence (E_{**}^r, d^r) for the bicomplex in Figure 3. It follows from the above calculation that the l th column E_{l*}^1 of the first term (E_{**}^1, d^1) equals $H_*(\overline{\mathcal{B}r}_{(c)}(n))$ for each $l \geq 0$. It remains to describe the differential $d^1 : E_{l*}^1 \rightarrow E_{(l+1)*}^1$. To this end, one needs to observe that the expressions (7) representing elements of $E_{l*}^1 = H_*(\overline{\mathcal{B}r}_{(c)}(n))$ correspond to marked lattice paths without internal points, whose only marked point is the initial one, marked by l . From the description of the cosimplicial structure given in 2.5 one easily obtains that

$$d^1 : E_{l*}^1 \rightarrow E_{(l+1)*}^1 = \begin{cases} 0, & \text{if } l \text{ is even and} \\ \text{id}, & \text{if } l \text{ is odd.} \end{cases}$$

We conclude that $E_{**}^2 := H_*(E_{**}^1, d^1)$ is concentrated at the leftmost column which equals $H_*(\overline{\mathcal{B}r}_{(c)}(n))$ and that, from the obvious degree reasons, the column spectral sequence collapses at this stage. Since we already know that the projection $\mathcal{B}r_{(c)} \xrightarrow{\pi} \overline{\mathcal{B}r}_{(c)}$ is a weak equivalence i.e., in particular, that $H_*(\mathcal{B}r_{(c)}(n)) \cong H_*(\overline{\mathcal{B}r}_{(c)}(n))$, the above facts imply that $proj : |\mathcal{L}_{(c)}|(n) \rightarrow \mathcal{B}r_{(c)}(n)$ induces an isomorphism of the E^2 -terms of the column spectral sequences, so it is a homology isomorphism and w is a homology isomorphism, too.

Let us finally prove that the normalized whiskering $w : \overline{\mathcal{B}r}_{(c)}(n) \hookrightarrow |\overline{\mathcal{L}}_{(c)}|(n)$ is a weak equivalence. We have the composition

$$(8) \quad \overline{\mathcal{B}r}_{(c)}(n) \xrightarrow{w} |\overline{\mathcal{L}}_{(c)}|(n) \xrightarrow{i} |\dot{\mathcal{L}}_{(c)}|(n)$$

in which the obvious inclusion i is a homology isomorphism by a simple lemma formulated below. As in the un-normalized case, the dg-abelian group $\overline{\mathcal{B}r}_{(c)}(n)$ is the first column of the semi-normalized version of the bicomplex in Figure 3, so there is a natural projection $proj : |\dot{\mathcal{L}}_{(c)}|(n) \rightarrow \overline{\mathcal{B}r}_{(c)}(n)$. This $proj$ is a homology isomorphism by the same arguments as in the un-normalized case, only using in (6) the normalized bar construction instead. The proof is finished by observing that $proj$ is the left inverse of the composition (8). \square

In the proof of Theorem 3.11 we used the following

3.12. Lemma. *The inclusion $i : |\overline{\mathcal{L}}_{(c)}|(n) \hookrightarrow |\dot{\mathcal{L}}_{(c)}|(n)$ is a homology isomorphism for each $n, c \geq 0$.*

Proof. The lemma follows from the fact that $|\overline{\mathcal{L}}_{(c)}|(n)$ is the cosimplicial normalization of the dg-cosimplicial group $|\dot{\mathcal{L}}_{(c)}|(n)$. \square

In the following two sections, we consider several operads and related spaces of trees. To simplify the navigation, we give a glossary of notation.

\mathcal{B} ,	“big” operad of all natural operations,	page 14
$\widehat{\mathcal{B}}$,	non-unital big operad,	page 17
\mathcal{T} ,	Tamarkin-Tsygan operad of \mathcal{B} ,	page 17
$\widehat{\mathcal{T}}$,	non-unital Tamarkin-Tsygan operad,	page 18
$\mathcal{B}r$,	brace operad,	page 20
$\widehat{\mathcal{B}r}$,	non-unital brace operad,	page 20
Tr_{k_1, \dots, k_n}^l ,	set of $(l; k_1, \dots, k_n)$ -trees,	page 15
A_{k_1, \dots, k_n} ,	set of amputated (k_1, \dots, k_n) -trees,	page 20
$\widehat{A}_{k_1, \dots, k_n}$,	set of amputated (k_1, \dots, k_n) -trees without stubs,	page 20

The operads mentioned in the list and their maps are organized in Figure 9.

4. OPERADS OF NATURAL OPERATIONS

In the previous sections we studied versions of the lattice path operad and its suboperads. We only briefly mentioned that some of these operads act on the Hochschild cochain complex of an associative algebra. The present and the following section will be devoted to this action. It turns out that, in order to retain some nice features of the constructions in the previous section, namely the ‘whiskering’ formula (4) without signs, one one hand, and to have simple rules for [June 17, 2009]

the signs in formulas for natural operations on the other hand, one needs to use the ‘operadic’ degree convention, recalled in the next subsection.

4.1. Classical vs. operadic. There are two conventions in defining the Hochschild cohomology of an associative algebra A . The *classical one* used for instance in the 1963 paper [10] is based on the chain complex $C_{cl}^*(A; A) = \bigoplus_{n \geq 0} C_{cl}^n(A; A)$, where $C_{cl}^n(A; A) := \text{Lin}(A^{\otimes n}, A)$ (the subscript *cl* refers to “classical”). Another appropriate name would be the *(co)simplicial* convention, because $C_{cl}^*(A; A)$ is a natural cosimplicial abelian group. With this convention, the cup product \cup is a degree 0 operation and the Gerstenhaber bracket $[-, -]$ has degree -1 , see [10, Section 7] for the ‘classical’ definitions of these operations.

On the other hand, it is typical for this part of mathematics that signs are difficult to handle. A systematic way to control them is the *Koszul sign rule* requiring that whenever we interchange two “things” of odd degrees, we multiply the sign by -1 . This, however, needs the definition of the Hochschild cohomology as the operadic cohomology [9] of associative algebras. Now the underlying chain complex is

$$(9) \quad C^*(A; A) := \text{Lin}((\downarrow A)^{\otimes *}, \downarrow A) = \downarrow C_{cl}^*(A; A),$$

where \downarrow denotes the desuspension of a (graded) vector space. In particular, $\downarrow A$ is A placed in degree -1 . More explicitly, $C^*(A; A) = \bigoplus_{n \geq -1} C^n(A; A)$, where $C^n(A; A) := \text{Lin}(A^{\otimes n+1}, A)$. With this convention, the cup product has degree $+1$ and the Gerstenhaber bracket degree 0.

Depending on the choice of the convention, there are two definitions of the ‘big’ operad of natural operations, see 4.3 below. The *classical* one introduces \mathcal{B}_{cl} as a certain suboperad of the endomorphism operad $\mathcal{E}nd_{C_{cl}^*(A; A)}$ of the graded vector space $C_{cl}^*(A; A)$, and the *operadic* one introduces \mathcal{B} as a suboperad of the endomorphism operad $\mathcal{E}nd_{C^*(A; A)}$. The difference between \mathcal{B}_{cl} and \mathcal{B} is merely conventional; the operad \mathcal{B}_{cl} is the operadic suspension $\mathbf{s}\mathcal{B}$ of the operad \mathcal{B} [15, Definition II.3.15] while, of course, $\mathcal{E}nd_{C_{cl}^*(A; A)} \cong \mathbf{s}\mathcal{E}nd_{C^*(A; A)}$. This is summarized in the commutative diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\text{incl}} & \mathcal{E}nd_{C^*(A; A)} \\ \mathbf{s} \downarrow \cong & & \cong \downarrow \mathbf{s} \\ \mathcal{B}_{cl} & \xrightarrow{\text{incl}} & \mathcal{E}nd_{C_{cl}^*(A; A)} \end{array}$$

in which the vertical maps are the operadic suspensions and the horizontal ones the inclusions.

4.2. Convention. In sections 4 and 5 we accept the *operadic* convention because we want to rely on the Koszul sign rule. As explained above, the operads \mathcal{B} and \mathcal{B}_{cl} differ only by the regrading and sign factors.

4.3. The big operad. Let us recall the dg-operad $\mathcal{B} = \{\mathcal{B}(n)\}_{n \geq 0}$ of *all natural multilinear operations* on the (operadic) Hochschild cochain complex (9) of a ‘generic’ associative algebra A with coefficients in itself introduced in [14] (but notice that we are using here the operadic degree convention, see 4.2, while [14] uses the classical one). A moment’s reflection shows that any thinkable natural operation must be a linear combination of compositions of the following ‘elementary’ operations:

(a) The insertion $\circ_i : C^k(A; A) \otimes C^l(A; A) \rightarrow C^{k+l}(A; A)$ given, for $k, l \geq -1$ and $0 \leq i \leq k$, by the formula

$$\circ_i(f, g)(a_0, \dots, a_{k+l}) := (-1)^{il} f(a_0, \dots, a_{i-1}, g(a_i, \dots, a_{i+l}), a_{i+l+1}, \dots, a_{k+l}),$$

for $a_1, \dots, a_{k+l-1} \in A$ – the sign is dictated by the Koszul rule!

(b) Let $\mu : A \otimes A \rightarrow A$ be the associative product, $\text{id} : A \rightarrow A$ the identity map and $1 \in A$ the unit. Then elementary operations are also the ‘constants’ $\mu \in C^1(A; A)$, $\text{id} \in C^0(A; A)$ and $1 \in C^{-1}(A; A)$.

(c) The assignment $f \mapsto \text{sgn}(\sigma) \cdot f\sigma$ permuting the inputs of a cochain $f \in C^k(A; A)$ according to a permutation Σ_{k+1} and multiplying by the signum of σ is an elementary operation.

Let B_{k_1, \dots, k_n}^l denote, for $l, k_1, \dots, k_n \geq 0$, the vector space of all natural, in the above sense, operations

$$(10) \quad O : C^{k_1-1}(A; A) \otimes \dots \otimes C^{k_n-1}(A; A) \rightarrow C^{l-1}(A; A).$$

Notice the regrading in the above equation. It guarantees that the super- and subscripts of B_{k_1, \dots, k_n}^l will all be non-negative integers. Moreover, with this definition the spaces B_{k_1, \dots, k_n}^l agree with the ones introduced in [3].

Recall that the Hochschild differential $d_H : C^{n-1}(A; A) \rightarrow C^n(A; A)$ is, for $n \geq 0$, given by the formula

$$\begin{aligned} d_H f(a_0 \otimes \dots \otimes a_n) &:= (-1)^{n+1} a_0 f(a_1 \otimes \dots \otimes a_n) + f(a_0 \otimes \dots \otimes a_{n-1}) a_n \\ &\quad + \sum_{i=0}^{n-1} (-1)^{i+n} f(a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n), \end{aligned}$$

for $a_i \in A$. Apparently, d_H is a natural operation belonging to B_n^{n+1} . Therefore, if $O \in B_{k_1, \dots, k_n}^l$ is as in (10), one may define $\delta O \in B_{k_1, \dots, k_n}^{l+1}$ and, for $1 \leq i \leq k$, also $\partial_i O \in B_{k_1, \dots, k_{i-1}, k_i-1, k_{i+1}, \dots, k_n}^l$ by

$$(11) \quad \begin{aligned} \delta O(f_1, \dots, f_n) &:= d_H O(f_1, \dots, f_n) \text{ and} \\ \partial_i O(f_1, \dots, f_n) &:= (-1)^{k_i + \dots + k_n + l + n + i} \cdot O(f_1, \dots, f_{i-1}, d_H f_i, f_{i+1}, \dots, f_n). \end{aligned}$$

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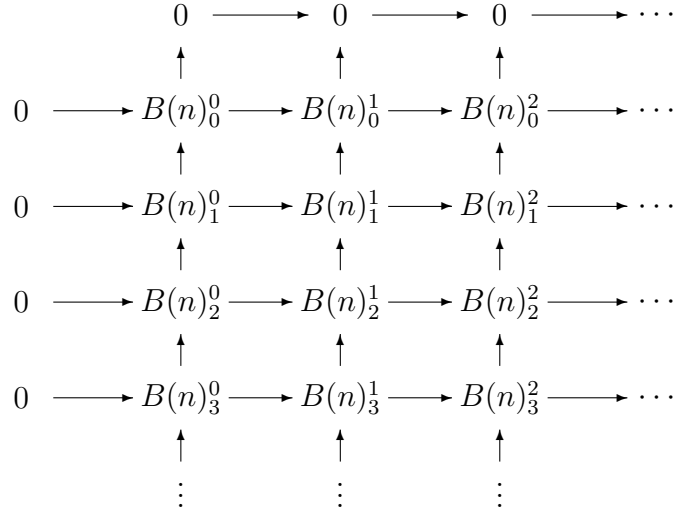


FIGURE 4. The structure of the big operad \mathcal{B} . In the above diagram, $B(n)_k^m := \prod_{k_1+\dots+k_n=k} B_{k_1,\dots,k_n}^m$. The vertical arrows are the simplicial differentials ∂ and the horizontal arrows are the cosimplicial differentials δ .

The sign in the second line of the above display equals $(-1)^{\deg(f_1)+\dots+\deg(f_{i-1})} \cdot (-1)^{\deg(O)}$ as dictated by the Koszul rule. We define the arity n piece of the big operad as

$$\mathcal{B}^*(n) := \prod_{l-(k_1+\dots+k_n)+n-1=*} B_{k_1,\dots,k_n}^l,$$

with the degree $+1$ differential $d : \mathcal{B}^* \rightarrow \mathcal{B}^{*+1}$ defined by $d := (\partial_1 + \dots + \partial_n) - \delta$. It is evident that the collection $\mathcal{B} = \{\mathcal{B}^*(n)\}_{n \geq 0}$, with the operadic composition inherited from the inclusion $\mathcal{B} \subset \mathcal{E}nd_{C^*(A;A)}$, is a dg operad.

The structure of the operad \mathcal{B} is visualized in Figure 4. We emphasize that the degree m -piece of $\mathcal{B}(n)$ is the direct *product*, not the direct sum, of elements on the diagonal $p + q + n - 1 = m$. Elements of B_{k_1,\dots,k_n}^l can be represented by linear combinations of $(l; k_1, \dots, k_n)$ -trees in the sense of the following definition in which, as usual, the *arity* of a vertex of a rooted tree is the number of its input edges and the *legs* are the input edges of a tree, see [15, II.1.5] for the terminology.

4.4. Definition. Let l, k_1, \dots, k_n be non-negative integers. An $(l; k_1, \dots, k_n)$ -tree is a planar rooted tree with legs labeled by $1, \dots, l$ and three types of vertices:

- (a) ‘white’ vertices of arities k_1, \dots, k_n labeled by $1, \dots, n$,
- (b) ‘black’ vertices of arities ≥ 2 and
- (c) ‘special’ black vertices of arity 0 (no input edges).

We moreover require that there are no edges connecting two black vertices or a black vertex with a special vertex. For $n = 0$ we allow also the exceptional tree \mid with no vertices.

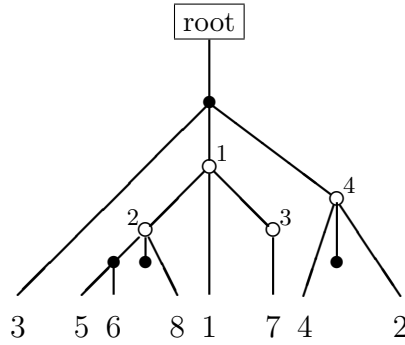


FIGURE 5. An $(8; 3, 3, 1, 3)$ -tree representing an operation in $\mathcal{B}_{3,3,1,3}^8$. It has 4 white vertices, 2 black vertices and 2 stubs. We use the convention that directed edges point upwards so the root is always on the top.

We call an edge whose initial vertex is special a *stub* (also called, in [13], a *tail*). It follows from definition that the terminal vertex of a stub is white. An example of an $(l; k_1, \dots, k_n)$ -tree is given in Figure 5.

An $(l; k_1, \dots, k_n)$ -tree T as in Definition 4.4 determines the natural operation $O_T \in B_{k_1, \dots, k_n}^l$ given by decorating, for each $1 \leq i \leq n$, the i th white vertex by $f_i \in C^{k_i-1}(A; A)$, the black vertices by the iterated multiplication, the special vertices by the unit $1 \in A$, and performing the composition along the tree. When evaluating on concrete elements, we apply the Koszul sign rule and use the ‘desuspended’ degrees, that is $f : A^{\otimes n} \rightarrow A$ is assigned degree $n - 1$ and $a \in A$ degree -1 , see 4.2. For instance, the tree in Figure 5 represents the operation

$$O(f_1, f_2, f_3, f_4)(a_1, \dots, a_8) := -a_3 f_1(f_2(a_5 a_6, 1, a_8), a_1, f_3(a_7)), f_4(a_4, 1, a_2)), \quad a_1, \dots, a_8 \in A,$$

where, as usual, we omit the symbol for the iteration of the associative multiplication μ . The minus sign in the right hand side follows from the Koszul rule explained above. The exceptional $(1;)$ -tree \mid represents the identity $\text{id} \in C^0(A; A)$. The following proposition is folklore.

4.5. Proposition. *Let us denote $\text{Tr}_{k_1, \dots, k_n}^l$ the set of all $(l; k_1, \dots, k_n)$ -trees as in Definition 4.4. The correspondence $O \mapsto O_T$ defines, for each non-negative integers l, k_1, \dots, k_n , an isomorphism*

$$\text{Span}(\text{Tr}_{k_1, \dots, k_n}^l) \cong B_{k_1, \dots, k_n}^l.$$

The groups B_{k_1, \dots, k_n}^l form a coloured operad B and the big operad \mathcal{B} is the condensation of B in the sense recalled in the appendix.

Proof. It is clear that each natural operation in B_{k_1, \dots, k_n}^l can be represented by a linear combination of $(l; k_1, \dots, k_n)$ -trees. The difficult part is to prove that, when a linear combination of $(l; k_1, \dots, k_n)$ -trees represents a trivial operation, it is itself trivial. This can be done by induction on the number of internal edges. We will not give this elementary but tedious proof here.


For the last claim see the Appendix. □

The correspondence of Proposition 4.9 identifies the pieces of \mathcal{B} with spaces spanned by trees. It is easy to show that the differential (resp. its vertical and horizontal components, see Figure 6) and the operad structure translate via this correspondence to trees as follows.

The component ∂_i , $1 \leq i \leq n$, of the vertical differential $\partial = \partial_1 + \dots + \partial_n$ replaces the white vertex of an $(l; k_1, \dots, k_n)$ -tree T labelled i with $k_i \geq 1$ inputs by the linear combination

$$(12) \quad \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + \dots + (-1)^{k_i+1} \sum_{1 \leq s \leq k_i-1} \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array}$$

in which the white vertex has $k_i - 1$ inputs and retains the label i . The result is then multiplied by the overall sign in the second line of (11). In the summation of (12), the black binary vertex is inserted into the s th input of the white vertex. If the i th white vertex of T has no inputs then $\partial_i(T) = 0$.

The horizontal differential replaces an $(l; k_1, \dots, k_n)$ -tree symbolized by the triangle  with l inputs by the linear combination

$$\begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} + \dots + (-1)^l \sum_{1 \leq s \leq l} \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array}$$

The \circ_i -operations are the following vertex insertions. Let T' be an $(l'; k'_1, \dots, k'_n)$ -tree and T'' be an $(l''; k''_1, \dots, k''_m)$ -tree. Then $T' \circ_i T'' = 0$ if $l'' \neq k'_i$ while, for $l'' = k'_i$, $T' \circ_i T''$ is the tree obtained by replacing the white vertex of T' labelled i by T'' .

It may happen that the replacements above create edges connecting black vertices. In that case the replacement is followed by collapsing these edges. The symmetric group acts by relabeling the white vertices.

4.6. Convention. From now on, we will make no distinction between natural operations and the corresponding linear combinations of trees.

4.7. Variant. An important suboperad of \mathcal{B} is the suboperad $\widehat{\mathcal{B}}$ generated by trees *without* stubs. The operad $\widehat{\mathcal{B}}$ is the operad of all natural multilinear operations on the Hochschild complex of a *non-unital* associative algebra. It is generated by natural operations (a)–(c) above but without the unit $1 \in C^1(A; A)$ in (b). Let us denote by $\widehat{B}_{k_1, \dots, k_n}^l$ the space of all operations (10) of this restricted type. An important feature of the operad $\widehat{\mathcal{B}}$ is that it is, in a certain sense, bounded. Indeed, one may easily prove that $\widehat{B}_{k_1, \dots, k_n}^l = 0$ if $k_1 + \dots + k_n - l \geq n$, see Figure 6.

4.8. Tamarkin-Tsygan operad. There is also a suboperad \mathcal{T} of \mathcal{B} generated by elementary operations of types (a) and (b) only, without the use of permutations in (c). Its arity- n piece equals

$$\mathcal{T}^*(n) := \prod_{l-(k_1+\dots+k_n)+n-1=*} T_{k_1, \dots, k_n}^l,$$

$$\begin{array}{ccccccc}
& & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \widehat{B}(n)_0^0 & \longrightarrow & \widehat{B}(n)_0^1 & \longrightarrow & \widehat{B}(n)_0^2 & \longrightarrow & \cdots \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \widehat{B}(n)_1^0 & \longrightarrow & \widehat{B}(n)_1^1 & \longrightarrow & \widehat{B}(n)_1^2 & \longrightarrow & \cdots \\
& & \vdots & & \vdots & & \vdots & & \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \widehat{B}(n)_{n-1}^0 & \longrightarrow & \widehat{B}(n)_{n-1}^1 & \longrightarrow & \widehat{B}(n)_{n-1}^2 & \longrightarrow & \cdots \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & 0 & \longrightarrow & \widehat{B}(n)_n^1 & \longrightarrow & \widehat{B}(n)_n^2 & \longrightarrow & \cdots \\
& & & & \uparrow & & \uparrow & & \\
& & & & 0 & \longrightarrow & \widehat{B}(n)_{n+1}^2 & \longrightarrow & \cdots \\
& & & & \uparrow & & & & \\
& & & & 0 & \longrightarrow & \cdots & &
\end{array}$$

FIGURE 6. The structure of the normalized operad $\widehat{\mathcal{B}}$. In the diagram, $\widehat{B}(n)_k^m := \prod_{k_1+\dots+k_n=k} \widehat{B}_{k_1,\dots,k_n}^m$.

where operations in T_{k_1,\dots,k_n}^l are represented by linear combinations of *unlabeled* $(l; k_1, \dots, k_n)$ -trees, that is, planar trees as in Definition 4.4 but without the labels of the legs. The inclusion $T_{k_1,\dots,k_n}^l \hookrightarrow B_{k_1,\dots,k_n}^l$ is realized by labeling the legs of an unlabeled tree from the left to the right in the orientation given by the planar embedding. The groups T_{k_1,\dots,k_n}^l form a coloured operad T and the inclusion above is the inclusion of operads $T \hookrightarrow B$.

The operad \mathcal{T} is the condensation of T and this is a chain version of the operad considered in [18, Section 3]. There is, finally, the operad $\widehat{\mathcal{T}} := \widehat{\mathcal{B}} \cap \mathcal{T}$ generated by unlabeled trees without stubs. It is clear from its definition that $\widehat{\mathcal{T}}$ is bounded in the same way as $\widehat{\mathcal{B}}$. It turns out that all these operads have the same homotopy type, see Theorem 5.9 below.

For the convenience of the reader we formulate the following obvious proposition.

4.9. Proposition. *The isomorphism of Proposition 4.5 induced by the correspondence $T \mapsto O_T$ restricts, for each $n \geq 0$, $l, k_1, \dots, k_n \geq 0$, into an isomorphism between*

- the span of the set of all $(l; k_1, \dots, k_n)$ -trees without stubs and the space $\widehat{B}_{k_1,\dots,k_n}^l$,
- the span of the set of all unlabeled $(l; k_1, \dots, k_n)$ -trees and the space T_{k_1,\dots,k_n}^l ,
- the span of the set of all unlabeled $(l; k_1, \dots, k_n)$ -trees without stubs and the space $\widehat{T}_{k_1,\dots,k_n}^l := T_{k_1,\dots,k_n}^l \cap \widehat{B}_{k_1,\dots,k_n}^l$.

We close this section by recalling the isomorphism between the set of unlabeled $(l; k_1, \dots, k_n)$ -trees and $\mathcal{L}_{(2)}(k_1, \dots, k_n; l)$ constructed in the proof of [2, Proposition 2.14]. Let T be an unlabeled

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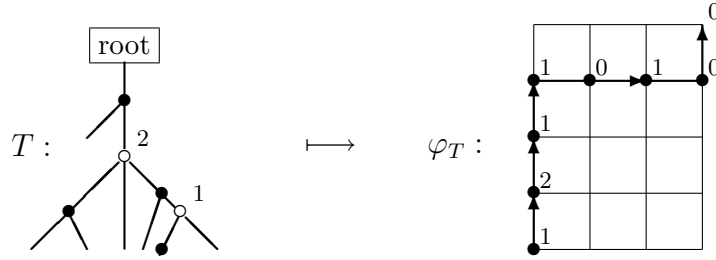


FIGURE 7. An unlabeled $(6; 2, 2)$ -tree T and the corresponding lattice path $\varphi_T \in \mathcal{L}_{(c)}(2, 2; 6)$.

$(l; k_1, \dots, k_n)$ -tree. We run around T counterclockwise via the unique edge-path that begins and ends at the root and goes through each edge of T exactly twice (in opposite directions). The lattice path $\varphi_T : [l + 1] \rightarrow [k_1 + 1] \otimes \dots \otimes [k_n + 1]$ corresponding to T starts at the ‘lower left’ corner with coordinates $(0, \dots, 0)$ and advances according the following rules:

- when the edge-path hits the white vertex labeled i , $1 \leq i \leq n$, we advance φ_T in the direction of the vector $d_i := (0, \dots, 1, \dots, 0)$ (1 at the i th place),
- when the edge-path hits the leg, we do not move but increase the marking of our position by one.

As proved in [2, Proposition 2.14], this correspondence induces an isomorphism of coloured operads \mathcal{T} and $\mathcal{L}_{(2)}$, and hence, the isomorphism between \mathcal{T} and $|\mathcal{L}_{(2)}|$. The construction $T \mapsto \varphi_T$ is illustrated in Figure 7.

More conceptually the difference between \mathcal{T} and \mathcal{B} can be explained as follows.

Let \mathcal{O} and \mathcal{O}_1 be the categories of operads and of nonsymmetric operads in the category of chain complexes Ch correspondingly. Let $Des_1 : \mathcal{O} \rightarrow \mathcal{O}_1$ be the classical forgetful functor and let \mathcal{M} be the *nonsymmetric* operad for monoids.

4.10. Definition. The category of multiplicative nonsymmetric operads [11] is the comma-category $\mathcal{M}/\mathcal{O}_1$. The category of multiplicative operads is the comma-category \mathcal{M}/Des_1 .

The description in [5, 1.5.6] of the colored operad whose algebras are symmetric operads, readily implies the following proposition which illuminates the main result of [3].

4.11. Proposition. *The category of algebras over the colored operad T is isomorphic to the category of multiplicative nonsymmetric operads. The category of algebras of the coloured operad B is isomorphic to the category of multiplicative operads. Under this identification, the inclusion $T \hookrightarrow B$ induces the forgetful functor from multiplicative operads to nonsymmetric multiplicative operads.*

5. OPERADS OF BRACES

There is another very important suboperad $\mathcal{B}r$ of \mathcal{B} generated by braces, cup-products and the unit introduced, under the notation \mathcal{H} , in [16, Section 1]. Let us recall its definition. The operad $\mathcal{B}r$ is the suboperad of the big operad \mathcal{B} generated by the following operations.

(a) The *cup product* $-\cup- : C^*(A; A) \otimes C^*(A; A) \rightarrow C^*(A; A)$ defined $f \cup g := \mu(f, g)$.

(b) The constant $1 \in C^{-1}(A; A)$.

(c) The *braces* $-\{-, \dots, -\} : C^*(A; A)^{\otimes n} \rightarrow C^*(A; A)$, $n \geq 2$, given by

$$(13) \quad f\{g_2, \dots, g_n\} := \sum f(\text{id}, \dots, \text{id}, g_2, \text{id}, \dots, \text{id}, g_n, \text{id}, \dots, \text{id}),$$

where id is the identity map of A and the summation runs over all possible substitutions of g_2, \dots, g_n (in that order) into f .

Notice that, for $f \in C^k(A; A)$ and $g \in C^l(A; A)$, the cup product $f \cup g \in C^{k+l+1}(A; A)$ evaluated at $a_0, \dots, a_{k+l+1} \in A$ equals

$$(14) \quad (f \cup g)(a_0, \dots, a_{k+l+1}) = (-1)^{(k+1)l} f(a_0, \dots, a_k) g(a_{k+1}, \dots, a_{k+l+1}),$$

with the sign dictated by the Koszul rule. This formula differs from the original one [10, Section 7] due to a different degree convention used here, see 4.2. We leave as an exercise to write a similar explicit formula for the brace.

The brace operad has also its non-unital version $\widehat{\mathcal{B}r} := \widehat{\mathcal{B}} \cap \mathcal{B}r$ generated by elementary operations (a) and (c). One can verify that both $\mathcal{B}r$ and $\widehat{\mathcal{B}r}$ are indeed dg-suboperads of \mathcal{B} , see [16].

The operad $\mathcal{B}r$ and its suboperad $\widehat{\mathcal{B}r}$ also admit a description in terms of trees, which is however not as straightforward as the tree description of the operad \mathcal{B} and its suboperads $\widehat{\mathcal{B}}$, \mathcal{T} and $\widehat{\mathcal{T}}$ given in Proposition 4.9.

5.1. Definition. Let k_1, \dots, k_n be non-negative integers. An *amputated* (k_1, \dots, k_n) -tree is an $(0; k_1, \dots, k_n)$ -tree in the sense of Definition 4.4. We denote by A_{k_1, \dots, k_n} the (finite) set of all amputated (k_1, \dots, k_n) -trees and by $\widehat{A}_{k_1, \dots, k_n}$ its subset consisting of amputated (k_1, \dots, k_n) -trees without stubs.

5.2. Proposition. For each $n \geq 0$ and $d \leq n - 1$, there is a natural isomorphism

$$w : \text{Span}(\{A_{k_1, \dots, k_n}; n - 1 - (k_1 + \dots + k_n) = d\}) \cong \mathcal{B}r^d(n)$$

which restricts into the isomorphism (denoted by the same symbol)

$$w : \text{Span}(\{\widehat{A}_{k_1, \dots, k_n}; n - 1 - (k_1 + \dots + k_n) = d\}) \cong \widehat{\mathcal{B}r}^d(n).$$

From reasons apparent below we call the map w defined by formula (15) below the *whiskering*. Before we give its definition, we illustrate the notion of amputated trees in the following:

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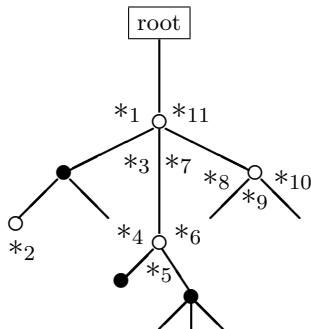


FIGURE 8. Angles of a tree symbolized by $*_1, \dots, *_{11}$. Their linear order, indicated by the subscripts, is given by walking around the tree counterclockwise, starting at the root. Unlike [13, Section 5.2], black vertices do not have angles. The labels of white vertices are not shown.

gives the brace (13). In particular, \circlearrowleft_2^1 gives Gerstenhaber's \circ -product and $\circlearrowleft_2^1 - \circlearrowright_1^2$ the Gerstenhaber bracket.

We are going to define operations ∂ and \circ_i acting on amputated trees that translate, via the whiskering (15), into the dg-operad structure of $\mathcal{B}r$. For an amputated (k_1, \dots, k_n) -tree S as in Definition 5.1 denote $\partial(S) := \partial_1(S) + \dots + \partial_n(S)$, where $\partial_i(S)$ is, for $k_i \geq 1$, the linear combination of amputated trees obtained by replacing the i th white vertex of S by (12) followed by the contraction of edges connecting black vertices if necessary. For $k_i = 0$ we put $\partial_i(S) = 0$.

The description of the \circ_i -operations is more delicate. Following [13, Section 5.2], define the *set of angles* of an amputated (k_1, \dots, k_n) -tree S to be the disjoint union

$$Angl(S) := \bigsqcup_{1 \leq i \leq n} \{0, \dots, k_i\}.$$

Angles come with a natural linear order whose definition is clear from Figure 8 borrowed from [13]. Now, for an amputated (k'_1, \dots, k'_n) -tree S' , an amputated (k''_1, \dots, k''_m) -tree S'' and $1 \leq i \leq n$, define $S' \circ_i S''$ to be the linear combination

$$S' \circ_i S'' := \sum_{\beta} (S' \circ_i S'')_{\beta},$$

where the sum runs over all (non-strictly) monotonic maps $\beta : In(w'_i) \rightarrow Angl(S'')$ from the set of incoming edges of the vertex w'_i of S' labelled i , to the set of angles of S'' . In the sum, $(S' \circ_i S'')_{\beta}$ is the tree obtained by removing the vertex w'_i from S' and replacing it by S'' , with the incoming edges of w'_i glued into the angles of S'' following β . An important particular case is $k'_i = 0$ when w'_i has no input edges. Then $S' \circ_i S''$ is defined as the tree obtained by amputating w'_i from S' and grafting the root of S'' at the place of w'_i .

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We call the operation \circ_i the *whiskered insertion*. A similar operation defines in [6] the structure of the operad for pre-Lie algebras. As observed in [13], the whiskering of Proposition 5.2 can also be expressed as the product

$$(16) \quad w(S) = \prod_{d \geq 0} \left(\begin{array}{c} \text{1} \\ \diagup \quad \diagdown \\ \dots \\ \text{d-times} \end{array} \right) \circ_1 S.$$

The following proposition can be verified directly.

5.7. Proposition. *With ∂ and \circ_i as defined above, the whiskering of Proposition 5.2 satisfies*

$$w(\partial S) = \partial(w(S)) \text{ and } w(S' \circ_i S'') = w(S') \circ_i w(S''),$$

for all amputated trees S, S', S'' and for all i for which the second equation makes sense.

5.8. Example. We show how the classical calculations of [10] can be concisely performed in the language of amputated trees (but recall that we are using a different sign and degree convention, see 4.2). Let us start by calculating the differentials of trees representing the cap product, the circle product and the Gerstenhaber bracket. By definition, one has

$$(17) \quad \partial \left(\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \end{array} \right) = 0.$$

Since (12) replaces $\begin{array}{c} \bullet \\ | \\ \circ \end{array}$ by $\begin{array}{c} \bullet \\ \diagdown \\ \circ \end{array} + \begin{array}{c} \bullet \\ \diagup \\ \circ \end{array}$, one gets

$$(18) \quad \partial \left(\begin{array}{c} \bullet \\ | \\ \begin{array}{c} \circ \quad \circ \\ | \\ \circ \quad \circ \end{array} \end{array} \right) = \begin{array}{c} \bullet \\ \diagdown \\ \begin{array}{c} \circ \quad \circ \\ | \\ \circ \quad \circ \end{array} \end{array} + \begin{array}{c} \bullet \\ \diagup \\ \begin{array}{c} \circ \quad \circ \\ | \\ \circ \quad \circ \end{array} \end{array}$$

which implies that

$$(19) \quad \partial \left(\begin{array}{c} \bullet \\ | \\ \begin{array}{c} \circ \quad \circ \\ | \\ \circ \quad \circ \end{array} \end{array} - \begin{array}{c} \bullet \\ | \\ \begin{array}{c} \circ \quad \circ \\ | \\ \circ \quad \circ \end{array} \end{array} \right) = \begin{array}{c} \bullet \\ \diagdown \\ \begin{array}{c} \circ \quad \circ \\ | \\ \circ \quad \circ \end{array} \end{array} + \begin{array}{c} \bullet \\ \diagup \\ \begin{array}{c} \circ \quad \circ \\ | \\ \circ \quad \circ \end{array} \end{array} - \begin{array}{c} \bullet \\ \diagdown \\ \begin{array}{c} \circ \quad \circ \\ | \\ \circ \quad \circ \end{array} \end{array} - \begin{array}{c} \bullet \\ \diagup \\ \begin{array}{c} \circ \quad \circ \\ | \\ \circ \quad \circ \end{array} \end{array} = 0.$$

We want to interpret these equations in terms of operations. To save the space, let us agree that in the rest of this example f will be an element of $C^m(A; A)$, g an element of $C^n(A; A)$ and h an element of $C^k(A; A)$, $m, n, k \geq -1$ arbitrary. By Proposition 5.7, (17) means that the differential of $-\cup-$ considered as an element of $\mathcal{B}(2)$ is zero, $d(-\cup-) = 0$, which, by the definition (11) of the differential in \mathcal{B} means that

$$d_H(f \cup g) + d_H f \cup g + (-1)^{m-1} f \cup d_H g = 0.$$

We recognize [10, Eqn. (20)] saying that $-\cup-$ is a chain operation. Since $\begin{array}{c} \bullet \\ | \\ \begin{array}{c} \circ \quad \circ \\ | \\ \circ \quad \circ \end{array} \end{array}$ represents the \circ -product, (18) means that

$$d_H f \circ g + (-1)^{m-1} f \circ d_H g - d_H(f \circ g) = f \cup g + (-1)^{(m-1)(n-1)} g \cup f,$$

which is the graded commutativity² of the cup product up to the homotopy $- \circ -$ proved in [10, Theorem 3]. The meaning of (19) is that

$$d_H[f, g] + [d_H f, g] + (-1)^{m-1}[f, d_H g] = 0,$$

so the bracket $[-, -]$ is a chain operation.

Let us investigate the compatibility between the cup product and the bracket. Since, in $\mathcal{B}(3)$, $[- \cup -, -] = [-, -] \circ_1 (- \cup -)$, the description of the \circ_i -operations in terms of amputated trees gives that $[f \cup g, h]$ is represented by

$$\begin{array}{c} \text{f} \quad \text{g} \\ \diagdown \quad \diagup \\ \circ \quad \circ \\ | \quad | \\ \text{h} \end{array} + \begin{array}{c} \text{f} \quad \text{g} \\ \diagdown \quad \diagup \\ \circ \quad \circ \\ | \quad | \\ \text{h} \quad \circ \end{array} - \begin{array}{c} \text{h} \\ | \\ \circ \\ \diagup \quad \diagdown \\ \text{f} \quad \text{g} \end{array}$$

where we, for ease of reading, replaced the labels of white vertices by the corresponding cochains. Similarly, since $- \cup [-, -] = (- \cup -) \circ_2 [-, -]$ in $\mathcal{B}(3)$, $f \cup [g, h]$ is represented by

$$\begin{array}{c} \text{f} \quad \text{g} \\ \diagdown \quad \diagup \\ \circ \quad \circ \\ | \quad | \\ \text{h} \quad \circ \end{array} - \begin{array}{c} \text{f} \quad \text{h} \\ \diagdown \quad \diagup \\ \circ \quad \circ \\ | \quad | \\ \text{g} \quad \circ \end{array}$$

and, by the same reason, $[f, h] \cup g$ is represented by

$$\begin{array}{c} \text{f} \quad \text{g} \\ \diagdown \quad \diagup \\ \circ \quad \circ \\ | \quad | \\ \text{h} \end{array} - \begin{array}{c} \text{h} \quad \text{g} \\ \diagdown \quad \diagup \\ \circ \quad \circ \\ | \quad | \\ \text{f} \end{array}.$$

Combining the above, one concludes that the expression $[f \cup g, h] - f \cup [g, h] - [f, h] \cup g$ is represented by

$$(20) \quad \begin{array}{c} \text{f} \quad \text{h} \\ \diagdown \quad \diagup \\ \circ \quad \circ \\ | \quad | \\ \text{g} \quad \circ \end{array} + \begin{array}{c} \text{h} \quad \text{g} \\ \diagdown \quad \diagup \\ \circ \quad \circ \\ | \quad | \\ \text{f} \quad \circ \end{array} - \begin{array}{c} \text{h} \\ | \\ \circ \\ \diagup \quad \diagdown \\ \text{f} \quad \text{g} \end{array}.$$

Because, by (12), ∂ replaces $\begin{array}{c} \text{h} \\ | \\ \circ \\ \diagup \quad \diagdown \end{array}$ by

$$\begin{array}{c} \text{h} \\ | \\ \circ \\ \diagup \quad \diagdown \end{array} + \begin{array}{c} \text{h} \\ | \\ \circ \\ \diagdown \quad \diagup \end{array} - \begin{array}{c} \text{h} \\ | \\ \circ \\ \diagup \quad \diagdown \end{array},$$

the expression in (20) equals

$$\partial \left(\begin{array}{c} \text{h} \\ | \\ \circ \\ \diagup \quad \diagdown \\ \text{f} \quad \text{g} \end{array} \right).$$

The meaning of the above calculations is that the bracket and the cup product are compatible up to the homotopy given by the brace $-\{-, -\}$.

²Since we use the convention in which the cup product has degree +1, its commutativity is the *antisymmetry*.

Proof of Proposition 5.2. It follows from Proposition 5.7 that the image of w contains $\mathcal{B}r$. Indeed, $Im(w)$ is a suboperad of \mathcal{B} which, by Example 5.6, contains the generators of $\mathcal{B}r$, i.e. the cup product, brackets and 1. The map w is clearly a monomorphism, since each amputated (k_1, \dots, k_n) -tree S equals the amputated part (i.e. the component belonging to $\prod B_{k_1, \dots, k_n}^0$) of its whiskering $w(S)$.

Therefore it remains to prove that $Im(w) \subset \mathcal{B}r$ or, more specifically, that $w(S) \in \mathcal{B}r(n)$ for each amputated (k_1, \dots, k_n) -tree S and $n \geq 0$. We need to show that each such S is build up, by iterated whiskered insertions, from the ‘atoms’

$$(21) \quad \circlearrowleft, \bullet, \cup := \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \end{array} \quad \text{and} \quad br_d := \begin{array}{c} \circ \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \\ \circ \quad \circ \end{array}, \quad d \geq 1,$$

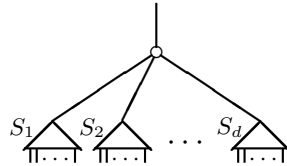
representing the generators of $\mathcal{B}r$.

The first step is to get rid of the stubs. If S has $s \geq 1$ stubs, we denote by \bar{S} the tree S with each stub replaced by \circlearrowleft . Let us label these new white vertices of \bar{S} by $k+1, \dots, k+s$. Then clearly

$$S = \pm(\cdots((\bar{S} \circ_{k+1} \bullet) \circ_{k+2} \bullet) \cdots) \circ_{k+s} \bullet.$$

The sign in the above expression, not important for our purposes, is a consequence of the Koszul sign rule, since \bullet represents $1 \in A$ placed in degree -1 . So we may suppose that S has no stubs and proceed by induction on the number of internal edges. Assume that S has e internal edges. If $e \leq 1$ then S is either \circlearrowleft or br_1 , so we may assume that $e \geq 2$. We distinguish two cases.

Case 1. The root vertex (i.e. the vertex adjacent to the root edge) is white; assume it has $d \geq 1$ input edges. The tree then looks as:

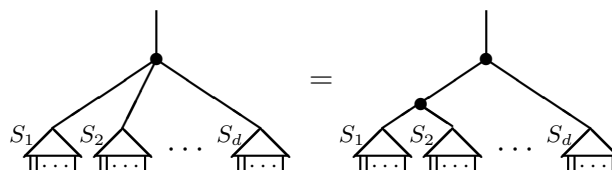


where S_1, \dots, S_d are suitable amputated trees. It is then clear that S can be obtained from

$$(\cdots((br_d \circ_1 S_1) \circ_2 S_2) \cdots) \circ_d S_d,$$

where br_d is the tree in (21), by relabeling the white vertices and changing the sign if necessary. Clearly, each S_1, \dots, S_d has less than e internal edges, and the induction goes on.

Case 2. The root vertex is black, with $d \geq 2$ inputs. If $d = 2$, we argue as in Case 1, only using \cup instead of br_2 . If $d \geq 3$, we can the equality



and argue as if $d = 2$. This finishes the proof. □

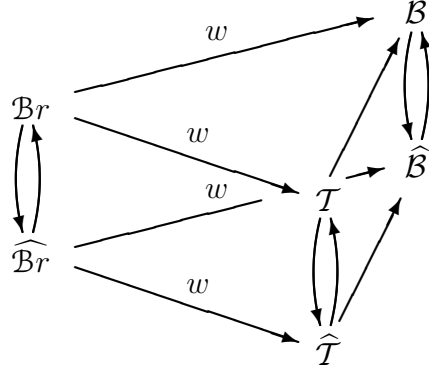


FIGURE 9. Operads of natural operations and their maps.

The operads introduced above can be organized into the diagram shown in Figure 9, in which the maps w are the whiskerings (15), the down-pointing arrows the projections and the unnamed arrows the inclusions.

5.9. Theorem. *All maps in Figure 9 are homology equivalences. In particular, the operads \mathcal{B} , $\widehat{\mathcal{B}}$, \mathcal{T} , $\widehat{\mathcal{T}}$, $\mathcal{B}r$ and $\widehat{\mathcal{B}r}$ have the same homotopy type.*

Proof. The fact that the whiskerings $w : \mathcal{B}r \rightarrow \mathcal{T}$, $w : \widehat{\mathcal{B}r} \rightarrow \widehat{\mathcal{T}}$ and the inclusion $\widehat{\mathcal{B}r} \rightarrow \mathcal{B}r$ are weak equivalences follows from Theorem 3.11, Proposition 3.9, Lemma 3.12 and the identifications $\mathcal{T} \cong |\mathcal{L}_{(2)}|$, $\widehat{\mathcal{T}} \cong |\dot{\mathcal{L}}_{(2)}|$, $\mathcal{B}r \cong \mathcal{B}r_{(2)}$ and $\widehat{\mathcal{B}} \cong \overline{\mathcal{B}r}_{(2)}$. The fact that the inclusions $\mathcal{T} \hookrightarrow \mathcal{B}$ and $\widehat{\mathcal{T}} \hookrightarrow \widehat{\mathcal{B}}$ are weak equivalences was proved in [3]. \square

APPENDIX A. SUBSTITUTES, CONVOLUTION AND CONDENSATION

In this appendix we briefly remind the reader of some categorical definitions and constructions we use in the paper. Most of the material is contained in [7],[8],[17] and [2].

Let V be a symmetric monoidal closed category. Let A be a small V -category and let $[A, V]$ be the V -category of V -functors from A to V . The enriched Hom -functor $Nat_A(F, G)$ is given by the end:

$$Nat_A(F, G) := \int_{X \in A} V(F(X), G(X)).$$

We also define the tensor product of the V -functors $F : A^{op} \rightarrow V$ and $G : A \rightarrow V$ by the coend

$$F \otimes_A G := \int^{X \in A} F(X) \otimes G(X).$$

A.1. Definition. A V -substitute (P, A) is a small V -category A together with a sequence of V -functors:

$$P_n : \underbrace{A^{op} \otimes \cdots \otimes A^{op}}_{n\text{-times}} \otimes A \rightarrow V, \quad n \geq 0,$$

$$P_n(X_1, \dots, X_n; X) = P_{X_1, \dots, X_n}^X$$

equipped with

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- a V -natural family of substitution operations

$$\mu : P_{X_1, \dots, X_n}^X \otimes P_{X_{11}, \dots, X_{1m_1}}^{X_1} \otimes \cdots \otimes P_{X_{n1}, \dots, X_{nm_n}}^{X_n} \rightarrow P_{X_{11}, \dots, X_{nm_n}}^X$$

- a V -natural family of morphisms (unit of substitute)

$$\eta : A(X, Y) \rightarrow P_1(X; Y) = P_X^Y$$

- for each permutation $\sigma \in S_n$ a V -natural family of isomorphisms

$$\gamma_\sigma : P_{X_1, \dots, X_n}^X \rightarrow P_{X_{\sigma(1)}, \dots, X_{\sigma(n)}}^X,$$

satisfying some associativity, unitality and equivariancy conditions [8].

Observe that P_1 is a V -monad on A in the bicategory of V -bimodules (V -profunctors or V -distributors). The Kleisli category of this monad is called *the underlying category of P* .

The concept of substitute generalizes operads and symmetric lax-monoidal categories. Indeed, any coloured operad P in V with the set of colours S is naturally a substitute $(P, U(P))$ with $U(P)$ equal the V -category with the set of objects S and the object of morphisms $U(P)(X, Y) = P(X; Y) \in V$. The substitution operation in the coloured operad P makes the assignment $P_n(X_1, \dots, X_n; X) = P_{X_1, \dots, X_n}^X$ a functor

$$P_n : \underbrace{U(P)^{op} \otimes \cdots \otimes U(P)^{op}}_{n\text{-times}} \otimes U(P) \rightarrow V, \quad n \geq 0,$$

and the sequence of these functors form a substitute. The category $U(P)$ is the underlying category of this substitute also called the underlying category of the coloured operad P . In fact, a substitute is a coloured operad P together with a small V -category A and a V -functor $\eta : A \rightarrow U(P)$ [7, Prop. 6.3].

A.2. Definition. [1, 8] A *symmetric lax-monoidal structure* or a *multitensor* on a V -category C is a sequence of V -functors

$$E_n : \underbrace{C \otimes \cdots \otimes C}_{n\text{-times}} \rightarrow C$$

equipped with

- a family of V -natural transformations:

$$\mu : E_n(E_{m_1}, \dots, E_{m_k}) \rightarrow E_{m_1 + \dots + m_k};$$

- A V -natural transformation (unit)

$$Id \rightarrow E_1;$$

- an action of symmetric group

$$\gamma_\sigma : E_n(X_1, \dots, X_n) \rightarrow E_n(X_{\sigma^{-1}(1)}, \dots, X_{\sigma^{-1}(n)}),$$

satisfying some natural associativity, unitarity and equivariance conditions.

A.3. Definition. [17] A multitensor is called a *functor-operad* if its unit is an isomorphism.

McClure and Smith observed in [17] that functor-operads can be used to define operads. Their observation works also for multitensors. Let $\delta \in C$ be an object of C then the coendomorphism operad of δ with respect to a multitensor E is given by a collection of objects in V

$$Coend^E(\delta)(n) = C(\delta, E_n(\delta, \dots, \delta)).$$

Substitudes and multitensors are related by the convolution operation [7, 8].

A.4. Definition. Let (P, A) be a substitute. We define a multitensor E^P on $C = [A, V]$ as follows:

$$(22) \quad E_n^P(\phi_1, \dots, \phi_n)(X) = P_{-, \dots, -}^X \otimes_A \phi_1(-) \otimes_A \dots \otimes_A \phi_n(-).$$

A special case of this construction is when A is equal to the underlying category of P . In this case the convolution operation produces a functor-operad.

Let (P, A) be a substitute and let $\delta : A \rightarrow V$ be a V -functor.

A.5. Definition. A δ -*condensation* of the substitute (P, A) is the operad $C^{(P,A)}(\delta) = Coend^{E^P}(\delta)$. So, as a collection it is given by

$$C^{(P,A)}(\delta)(n) = Nat_A(\delta, E_n^P(\delta, \dots, \delta)).$$

The operad $C^{(P,A)}(\delta)$ naturally acts on the objects of the form

$$Tot_\delta(\phi) = Nat_A(\delta, \phi)$$

for an arbitrary V -functor $\phi : A \rightarrow V$ (δ -totalization of ϕ) [17, 2].

Let $i : B \rightarrow A$ and $\delta : B \rightarrow V$ be two V -functors. Let $Lan_i(\delta)$ be a (V -enriched) left Kan extension of δ along i . Then

$$Tot_{Lan_i(\delta)}(\phi) = Nat_A(Lan_i(\delta), \phi) = Nat_B(\delta, i^*(\phi)) = Tot_\delta(i^*(\phi)),$$

where i^* is the restriction functor induced by i .

There is a similar formula which expresses the condensation with respect to $Lan_i(\delta)$. Let (P, A) be a substitute and let $i_{*, \dots, *}(P)$ be a sequence of functors

$$\begin{aligned} i_{*, \dots, *}(P)_n &: B^{op} \otimes \dots \otimes B^{op} \otimes A \rightarrow V, \\ i_{*, \dots, *}(P)_{B_1, \dots, B_n}^A &= P_{i(B_1), \dots, i(B_n)}^A \quad . \end{aligned}$$

We define a sequence of functors

$$E_n^{i_{*, \dots, *}(P)} : [B, V] \otimes \dots \otimes [B, V] \rightarrow [A, V]$$

by the formula similar to formula (22). We also define i^*P as the substitute (i^*P, B) obtained from P by restricting P_n along i .

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A.6. Proposition. *For the functors $\phi_1, \dots, \phi_n \in [B, V]$ the following V -natural isomorphisms hold:*

$$E_n^P(\text{Lan}_i(\phi_1), \dots, \text{Lan}_i(\phi_n)) = E_n^{i^*, \dots, i^* (P)}(\phi_1, \dots, \phi_n).$$

In particular,

$$C^{(P,A)}(\text{Lan}_i(\delta))(n) = \text{Tot}_\delta(i^* E_n^{i^*, \dots, i^* (P)}(\delta, \dots, \delta)) = \text{Tot}_\delta(E_n^{i^* (P)}(\delta, \dots, \delta)) = C^{(i^* (P), B)}(\delta)(n).$$

This result allows to see many of the operads in this paper as the result of δ -condensation of some substitutes. For us V will be the category of chain complexes Ch . Our category A will be the category of nonempty ordinals Δ (linearized) or the crossed interval category $(IS)^{op}$ [3] (also linearized). B can be Δ or its subcategory of injective order preserving maps Δ_{in} . These categories are related by identity on objects injective on morphisms functors:

$$\Delta_{in} \xrightarrow{i} \Delta \xrightarrow{j} (IS)^{op}.$$

Let $\delta : \Delta \rightarrow Ch$ be the cosimplicial chain complex of normalized chains on standard simplices. It is classical that the totalization of a cosimplicial chain complex X^\bullet with respect to δ is the normalized cosimplicial totalization $\overline{\text{Nor}}(X^\bullet)$ and the tensor product $X_\bullet \otimes_\Delta \delta$ for a simplicial chain complex X_\bullet is the normalized simplicial realization $\underline{\text{Nor}}(X_\bullet)$. Hence, the condensation of the lattice path operad $\mathcal{L}_{(c)}$ with respect to δ is precisely the n -simplicial cosimplicial normalization

$$|\overline{\mathcal{L}_{(c)}}| = \overline{\text{Nor}}(\underline{\text{Nor}}(\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; \bullet))) = C^{(\mathcal{L}_{(c)}, \Delta)}(\delta).$$

Proposition A.6 shows that the condensation of the lattice path operad $\mathcal{L}_{(c)}$ with respect to $\text{Lan}_i(i^*(\delta))$ is the unnormalized n -simplicial cosimplicial totalization

$$|\mathcal{L}_{(c)}| = \overline{\text{Tot}}(\underline{\text{Tot}}(\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; \bullet))) = C^{(i^*(\mathcal{L}_{(c)}), \Delta_{in})}(i^*(\delta)) = C^{(\mathcal{L}_{(c)}, \Delta)}(\text{Lan}_i(i^*(\delta)))$$

Analogously, for the "big" operad of natural operations on the Hochschild cochains we use the condensation with respect to $\text{Lan}_j(\delta)$ for the normalized version and with respect to $\text{Lan}_{j_i}(i^*\delta)$ for the unnormalized version that is

$$\mathcal{B} = C^{(B, (IS)^{op})}(\text{Lan}_i(i^*(\delta))).$$

In [2] similar calculations were applied to the cyclic version of the lattice path operad.

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