Feynman categories, derived modular envelopes and moduli spaces

Clemens Berger[1]

University of Nice

Topology Feest Utrecht
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[1]joint with Ralph Kaufmann (Purdue University)

Definition (moduli space for oriented surfaces/ribbon graphs)

- $\mathcal{M}_{g,n}$ moduli space of *hyperbolic metrics* on a surface $S_{g,n}$ of *genus* g with n *punctures* where $\chi(S_{g,n}) < 0$ and $n > 0$.
- $\mathcal{M}_G$ moduli space of *admissible metrics* on ribbon graph G.

Theorem (Mumford, Strebel, Penner, Kontsevich, ...)

$\mathcal{M}_{g,n} \simeq \bigcup_G \mathcal{M}_G$ where the metric ribbon graphs G are of type (g,n) and at least trivalent.

Proposition (Igusa)

$\bigcup_G \mathcal{M}_G \simeq |\mathrm{nerve}(\mathrm{rb}_{g,n})|$ where the *ribbon category* $\mathrm{rb}_{g,n}$ is generated by orientation preserving edge contractions between ribbon graphs of type (g,n).

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$\mathcal{M}_{g,s}^{p_1,\dots,p_\nu}$ moduli space of hyperbolic metrics on a surface $S_{g,s}^{p_1,\dots,p_\nu}$ of *genus* g with s *punctures* and ν *cyclic boundary components* containing $p_i > 0$ *marked points* respectively.

Theorem (Penner, Igusa, B-K)

$\mathcal{M}_{g,s}^{p_1,\dots,p_\nu} \simeq \bigcup_G \mathcal{M}_G \simeq |\mathrm{rb}_{g,s}^{p_1,\dots,p_\nu}|$ where the *flagged* ribbon graphs G are of type $(g, s; p_1, \dots, p_\nu)$ and at least trivalent.

Proof sketch (via doubling construction).

(bordered R. surface with $\chi < 0$)$\leftrightsquigarrow$(*involutive* hyperbolic surface)
(flagged ribbon graph with $\chi < 0$)$\leftrightsquigarrow$(*involutive* ribbon graph)
involution = orientation-*reversing* with *separating* fixpoint set $\square$

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Remark (dual point of view: Harer, Kaufmann-Penner)

- $\mathrm{nerve}(\mathrm{rb}_{g,n}) \cong (\text{quasi-filling arc systems on } S_{g,n})^{\mathrm{op}}$
- $\mathrm{nerve}(\mathrm{rb}_{g,s}^{p_1,\ldots,p_\nu}) \cong (\text{quasi-filling arc systems on } S_{g,s}^{p_1,\ldots,p_\nu})^{\mathrm{op}}$

Purpose of the talk

- define surface-modular operads (cf. Markl)
- show that the functor

$$J : (\text{planar-cyclic operads}) \longrightarrow (\text{surface-modular operads})$$

induces homotopy equivalences

$$\mathbb{L}J(\mathbb{1})(g, s; p_1, \ldots, p_\nu) \simeq \mathcal{M}_{g,s}^{p_1,\ldots,p_\nu}$$

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Proposition (May-Thomason, Elmendorf-Mandell, Hermida)

Each coloured operad $\mathcal{O}(i_1, \dots, i_k; i)$ induces a symmetric monoidal category $\mathfrak{F}_{\mathcal{O}}$ having as objects ordered sequences of colours and as morphisms ordered sequences of operations.

Remark (framed symmetric monoidal categories)

$\mathfrak{F}_{\mathcal{O}}$ contains the invertible unary operations of $\mathcal{O}$ as subgroupoid $\mathcal{V}_{\mathcal{O}}$ such that $(\mathcal{V}_{\mathcal{O}})^{\otimes} \simeq \mathrm{Iso}(\mathfrak{F}_{\mathcal{O}})$ (we call $\mathcal{V}_{\mathcal{O}}$ a *framing* of $\mathfrak{F}_{\mathcal{O}}$).

Proposition (Getzler, B-K, Batanin-Kock-Weber)

Coloured operads are *coreflective* inside framed sym. monoidal categories. The essential image consists of *Feynman categories*.

Definition (Kaufmann-Ward)

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Lemma ($\mathcal{O}$-algebra=$\mathfrak{F}_{\mathcal{O}}$-operad)

Any $\mathcal{O}$-algebra extends to a strong sym. mon. functor $\mathfrak{F}_{\mathcal{O}} \to \mathrm{Sets}$.

Proposition (Kaufmann-Ward)

Any Feynman functor $j : \mathfrak{F} \to \mathfrak{F}'$ induces an adjunction

$$j_! : \mathfrak{F}\text{-operads} \longrightarrow \mathfrak{F}'\text{-operads} : j^*$$

such that the left adjoint is given by pointwise left Kan extension

$$(j_! P)(A') = \mathrm{colim}_{j(-)\downarrow A'} P(-).$$

Proposition (B-K, cf. Street-Walters' comprehensive factorisation)

Any Feynman functor $j : \mathfrak{F} \to \mathfrak{F}'$ factors essentially uniquely as a *connected* Feynman functor followed by a *covering* where j is connected (resp. a covering) iff $j_!(1) = 1$ (resp. $\mathfrak{F} \cong \mathrm{el}_{\mathfrak{F}'}(j_!(1))$).

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$$j_! : \mathfrak{F}\text{-operads} \longrightarrow \mathfrak{F}'\text{-operads} : j^*$$

such that the left adjoint is given by pointwise left Kan extension

$$(j_! P)(A') = \mathrm{colim}_{j(-)\downarrow A'} P(-).$$

Proposition (B-K, cf. Street-Walters' comprehensive factorisation)

Any Feynman functor $j : \mathfrak{F} \to \mathfrak{F}'$ factors essentially uniquely as a *connected* Feynman functor followed by a *covering* where j is connected (resp. a covering) iff $j_!(1) = 1$ (resp. $\mathfrak{F} \cong \mathrm{el}_{\mathfrak{F}'}(j_!(1))$).

Lemma (Ginzburg-Kapranov, B-Moerdijk, Kontsevich-Soibelman)

There is a coloured operad $\mathcal{S}$ whose algebras are symmetric operads. Its associated Feynman category $\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{sym}$ has

- as objects disjoint unions of *rooted* corollas
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Lemma (Getzler-Kapranov)

The Feynman category $\mathfrak{F}_{cyc}$ for cyclic operads has

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Lemma (Borisov-Manin, Kaufmann-Ward)

There are Feynman functors $\mathfrak{F}_{sym} \to \mathfrak{F}_{cyc} \to \mathfrak{F}_{ctd}$ where $\mathfrak{F}_{ctd}$ has

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Proposition (Getzler-Kapranov)

The Feynman functor $h : \mathfrak{F}_{cyc} \to \mathfrak{F}_{ctd}$ factors as connected functor $j : \mathfrak{F}_{cyc} \to \mathfrak{F}_{mod}$ followed by a covering $k : \mathfrak{F}_{mod} \to \mathfrak{F}_{ctd}$ where $\mathfrak{F}_{mod}$ is the Feynman category for modular operads.

Corollary (B-K)

The $\mathfrak{F}_{ctd}$-operad $h_!(1)$ is "genus-labeling" and $j_!(1) = 1$.

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$$\begin{array}{ccccc} \mathfrak{F}_{non\text{-}sym} & \xrightarrow{I} & \mathfrak{F}_{plan\text{-}cyc} & \xrightarrow{J} & \mathfrak{F}_{surf\text{-}mod} \\ \downarrow{\scriptstyle p(\tau_{assoc})} & & \downarrow{\scriptstyle p(\tau_{planar})} & & \downarrow{\scriptstyle p(\tau_{ribbon})} \\ \mathfrak{F}_{sym} & \xrightarrow[i]{} & \mathfrak{F}_{cyc} & \xrightarrow[j]{} & \mathfrak{F}_{mod} \\ & & & \searrow^{h} & \downarrow{\scriptstyle k=p(\tau_{genus})} \\ & & & & \mathfrak{F}_{ctd} \end{array}$$

where vertical arrows are coverings, and j, J are connected.

- τ_{assoc} is the $\mathfrak{F}_{sym}$-operad for associative monoids
- τ_{planar} is the $\mathfrak{F}_{cyc}$-operad for planar structures
- $i^*(\tau_{planar}) = \tau_{assoc}$ (τ_{planar} is the “cyclic” version of τ_{assoc})
- $\tau_{ribbon} = j_!(\tau_{planar})$

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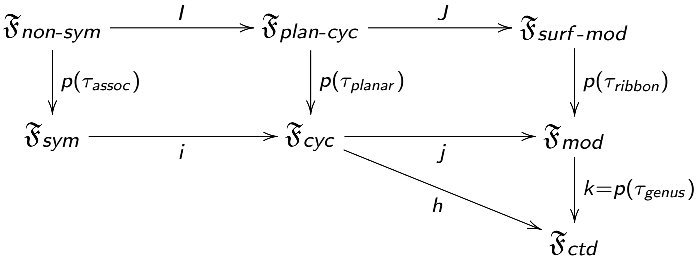

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Proposition (Doubek, B-K)

The set $j_!(\tau_{planar})(\gamma, n)$ is in bijection with either

- equ. cl. of one-vertex ribbon graphs with γ loops and n flags
- $\{(g, s; p_1, \dots, p_\nu) \mid n = p_1 + \dots + p_\nu \text{ and } 1 - 2g = \nu + s - \gamma\}$
- topological types of bordered oriented surfaces of genus g with s punctures and ν boundaries having p_i marked points each

Corollary (Markl, B-K)

The morphisms of the Feynman category $\mathfrak{F}_{surf\text{-}mod}$ can be considered as genus-labeled "polycyclic" graphs and $J(\imath) = \imath$.

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A Feynman category $\mathfrak{F}$ is *cubical* if there is a degree function $\deg : Mor(\mathfrak{F}) \to \mathbb{N}_0$ such that

- $\deg(\phi \circ \psi) = \deg(\phi) + \deg(\psi)$
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Remark

In the non-unital case without constants, the Feynman categories $\mathfrak{F}_{sym}, \mathfrak{F}_{cyc}, \mathfrak{F}_{mod}, \mathfrak{F}_{non\text{-}sym}, \mathfrak{F}_{plan\text{-}cyc}, \mathfrak{F}_{surf\text{-}mod}$ are cubical. The degree of ϕ is the number of edges of the representing graph Γ_ϕ.

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In the non-unital case without constants, the Feynman categories $\mathfrak{F}_{sym}, \mathfrak{F}_{cyc}, \mathfrak{F}_{mod}, \mathfrak{F}_{non\text{-}sym}, \mathfrak{F}_{plan\text{-}cyc}, \mathfrak{F}_{surf\text{-}mod}$ are cubical. The degree of ϕ is the number of edges of the representing graph Γ_ϕ.

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Definition ($W_{\mathfrak{F}}$-construction)

Let P be an operad over a cubical Feynman category $\mathfrak{F}$. Put

$$(W_{\mathfrak{F}}P)(B) = \left(\coprod_{\phi \in \mathfrak{F}(A,B)} P(A) \times_{\mathrm{Aut}_{\mathfrak{F}}(\phi)} [0,1]^{\deg(\phi)} \right) / \sim$$

where identifications are on faces of $[0,1]^{\deg(\phi)}$ according to coarser factorisations of ϕ. $\mathrm{Aut}_{\mathfrak{F}}(\phi)$ acts on both sides.
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For any cubical Feynman category $\mathfrak{F}$, the category of topological $\mathfrak{F}$-operads admits a *transferred model structure*. If P has an underying cofibrant $\mathcal{V}$-collection then $W_{\mathfrak{F}}P$ is a *cofibrant $\mathfrak{F}$-operad*.

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Example (cubically subdivided convex polytopes)

- $W_{sym}(\tau_{assoc})$(rooted corolla)=associahedron
- $W_{cyc}(\tau_{planar})$(corolla)=cyclohedron

Proposition (B-K)

Let $\phi : \mathfrak{F} \to \mathfrak{F}'$ be a functor of cubical Feynman categories.

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  no for transferred *projective* model structure, but yes for transferred *equivariant* model structure, cf. Vogt.
- Since $p_!(\mathbb{1}_{plan\text{-}cyc}) = \tau_{planar}$, $j_!(W\tau_{planar})$ decomposes according to $p_!J_!(W_{plan\text{-}cyc}\mathbb{1})$. What about derived modular envelopes of other cyclic operads ?
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