Homotopy classes of maps and cohomology

Let π be a commutative compact Lie group $\rightsquigarrow$ $B\pi$ its classifying space

X a space

E a cohomology theory arising from an Ω-spectrum $(E_n)_{n\in\mathbb{Z}}$ so that $E^nX \cong [X, E_n]$

Question: Is the map $[B\pi, X] \longrightarrow \mathrm{Hom}(E^*X, E^*B\pi)$ one to one ?

Some answers:

1) $X = E_n$, W any space then $[W, E_n] \longrightarrow \mathrm{Hom}(E^*E_n, E^*W)$ is one to one

cf [Bousfield-Kan Topology 1972]

Ex Let X be 1-connected with $H\mathbb{Z}^*X$ of finite type in each degree and $H\mathbb{Q}^*X$ free commutative graded $\mathbb{Q}$-algebra then $[W, X_{\mathbb{Q}}] \hookrightarrow \mathrm{Hom}(H\mathbb{Q}^*X, H\mathbb{Q}^*W)$

2) $X = BG$ with G a connected compact Lie group, T = torus

[Adams-Mahmud 1976]:

$$\begin{tikzcd} \mathrm{Rep}(T,G) \arrow[r, "\alpha"] & {[BT, BG]} \arrow[r, "h"] & \mathrm{Hom}(H\mathbb{Q}^*BG, H\mathbb{Q}^*BT) \\ \mathrm{Hom}(T, T_G)/_{W_G} \arrow[u, "\wr"] \arrow[urr, hook, bend right] & & \end{tikzcd}$$

$$\mathrm{Im}\, h = \mathrm{Im}\, h\circ\alpha$$

3) [Miller 1986] X finite CW-complex, p prime number

$$[B\mathbb{Z}/p, X] \hookrightarrow \mathrm{Hom}(H\mathbb{Z}/p^*X, H\mathbb{Z}/p^*B\mathbb{Z}/p) = *$$

$\rightsquigarrow$ [Dwyer-Zabrodsky] For π = finite p-group and G = compact Lie group

$$\mathrm{Rep}(\pi, G) \xrightarrow{\sim} [B\pi, BG]$$

Structure on E^*X

- First of all E^*X is a graded set

- There exists a space $K_E(S)$, for S a graded set, and a map $S \to E^*K_E(S)$ inducing a bijection $\mathrm{Hom}_{grSet}(S, E^*X) \simeq \mathrm{Hom}_{\mathrm{Ho}}[X, K_E(S)]$

$S \mapsto G(S) := E^*K_E(S)$ is a monad on $grSet$ $\rightsquigarrow$ $\mathcal{K}_E :=$ category of G-algebras

Every G-algebra M appears as the coequalizer of a diagram $G^2(M) \rightrightarrows G(M)$

For X a space $E^*(X \to K_E(E^*X))$ makes E^*X a G-algebra

For all space W, $[W, X] \longrightarrow \mathrm{Hom}_{\mathcal{K}_E}(E^*X, E^*W)$ is a bijection if $X \simeq K_E(S)$ for some graded set S

- Every space X has a cosimplicial resolution

$$X \to R(X) \rightrightarrows R^2(X) \to \cdots$$

with $R(X) := K_E(E^*X)$

Cohomology of mapping spaces

$\mathrm{map}(W,X)$ characterized by $\mathrm{Hom}_{\mathrm{htop}}(W\times Z, X) \simeq \mathrm{Hom}_{\mathrm{htop}}(Z, \mathrm{map}(W,X))$

so $\pi_0\, \mathrm{map}(W,X) \simeq [W,X]$

Define for $S \in \mathrm{GrSet}$ $\quad T_{W,E}\, G(S) := E^* \mathrm{map}(W, K_E(S))$ $\quad$ This is functorial in $G(S) \in \mathcal{K}_E$

Then for $M \in \mathcal{K}_E$ define $T_{W,E} M := \mathrm{coeq.}\left(T_{W,E}\, G^2(M) \rightrightarrows T_{W,E}\, G(M) \right)$

From $X \to RX \rightrightarrows R^2 X$ we get a map $T_{W,E}\, E^* X \to E^* \mathrm{map}(W,X)$

Prop For $E = H\mathbb{Z}/p$ or MU or ..

a) $\mathrm{Hom}_{\mathcal{K}_E}(E^*X, E^*) \simeq \pi_0 X$

b) $\mathrm{Hom}_{\mathcal{K}_E}(T_{W,E} M, E^*) \simeq \mathrm{Hom}_{\mathcal{K}_E}(M, E^* W)$

Thm (Lannes 1992 then Dror-Smith, Morel) $\quad$ X with some finiteness hypotheses

$$T_{B(\mathbb{Z}/p)^d,\, H\mathbb{Z}/p}\, H\mathbb{Z}/p^* X \longrightarrow H\mathbb{Z}/p^*\, \mathrm{map}(B(\mathbb{Z}/p)^d, X) \quad \text{is iso}$$

More answers

4) $\pi = (\mathbb{Z}/p)^d$, $E = H\mathbb{Z}/p$, X nilpotent space with E^*X degree-wise finite

[Lannes 1986]: $$[[B\pi, X] \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}_E}(E^*X, E^*B\pi)$$
$$\shortparallel$$
$$\lim_n [Sk_n B\pi, X]$$

[Morel 1996]: For any space X, $[B\pi, X^{\hat{p}}] \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}_H}(H^*X, H^*B\pi)$

where $X^{\hat{p}}$ stands for the p-profinite completion (Artin-Mazur, Sullivan) of X

5) [Notbohm 1991]: G = compact Lie group, T = torus

$$\mathrm{Rep}(T, G) \xrightarrow{\sim} [BT, BG] \hookrightarrow \mathrm{Hom}(H\mathbb{Q}^*BG, H\mathbb{Q}^*BT)$$

[Notbohm-Smith 1991]: G connected compact Lie group, T torus

$$[BT, BG] \xrightarrow{\sim} \mathrm{Hom}_{\lambda\text{-ring}}(K^0BG, K^0BT)$$

6) [Lannes-Dehon 1999]: X = 1-connected space with $H\mathbb{Z}_*X$ free finite type abelian group in each degree then $[[BT, X] \hookrightarrow \mathrm{Hom}(H\mathbb{Q}^*X, H\mathbb{Q}^*BT)$

If moreover $H\mathbb{Q}^*X$ is free as a commutative graded $\mathbb{Q}$-algebra then

$$[[BT, X] \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}_{MU_*}}(MU_*BT, MU_*X) \simeq \mathrm{Hom}_{\lambda\text{-rings}}(K^0X, K^0BT)$$

7) [Dehon 2004] π = commutative compact Lie group, X space with $H\mathbb{Z}_p{}^*X$ torsion free in each degree then

$$T_{B\pi, \hat{MU}}\, \hat{MU}^*X \xrightarrow{\sim} \hat{MU}_c^*\, \mathrm{map}(B\pi, X^{\hat{p}})$$

so $$[B\pi, X^{\hat{p}}] \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}_{\hat{MU}}}(\hat{MU}^*X, MU^*B\pi)$$

but $\quad [B\mathbb{Z}/p, K(\mathbb{Z}/p,2)] \longrightarrow \mathrm{Hom}_{\mathcal{K}_{\hat{MU}}}(MU^* K(\mathbb{Z}/p,2), MU^* B\mathbb{Z}/p)$ is trivial

Link with the Künneth formula

E multiplicative cohomology theory $\qquad E = \hat{MU}, H\mathbb{Z}/p$

"prop" Let W be a space with $E^*W \otimes - : \mathcal{K}_E \to \mathcal{K}_E$ exact then for all $M, N \in \mathcal{K}_E$

$$\mathrm{Hom}_{\mathcal{K}_E}(T_W M, N) \simeq \mathrm{Hom}_{\mathcal{K}_E}(M, \cancel{M}E^* W \otimes N)$$

Proof: - One should then have a Künneth formula : $E^*(W \times Z) \simeq E^*W \otimes E^*Z$ for all space Z

- $N \mapsto \mathrm{Hom}_{\mathcal{K}_E}(G(s), E^*W \otimes N)$ should be representable by a free G-algebra

then compute $\mathrm{map}(W, K_E(s))$

Ex - $E = H\mathbb{Z}/p$, W with $H\mathbb{Z}/p^* W$ degree-wise finite

- $E = \hat{MU}$, W with $H\mathbb{Z}_p^* W$ free and finite type in each degree

- $E = \hat{MU}$, $W = B\Pi$ with $\Pi =$ commutative compact Lie group

Interplay between cohomology theories

One example : $\hat{MU}$ and $H\mathbb{Z}/p$

The standard orientation $MU \to H\mathbb{Z}/p$ gives a functor

$$\mathcal{K}_{MU} \to \mathcal{K}_{H\mathbb{Z}/p} \quad , \quad M \mapsto M/_{\mathcal{G}^1}$$

characterised by $G(S)/_{\mathcal{G}^1} \simeq H\mathbb{Z}/p^* K_{MU}(S)$

For $M \in \mathcal{K}_{MU}$ one get a morphism $\left(T_{B\mathbb{Z}/p, MU} M\right)/_{\mathcal{G}^1} \longrightarrow T_{B\mathbb{Z}/p, H\mathbb{Z}/p}\left(M/_{\mathcal{G}^1}\right)$

Prop a) This morphism is an isomorphism

b) The map $[BV, X^\wedge_p] \longrightarrow \mathrm{Hom}\,(\hat{MU}^* X, \hat{MU}^* BV)$ is one to one for all $V = (\mathbb{Z}/p)^d$ iff $\mathrm{Im}\,(\hat{MU}^* X \to H\mathbb{Z}/p^* X)$ is F. isomorphic to $H\mathbb{Z}/p^* X$

Ex [Tamanoi]: $\mathrm{Im}\,(\hat{MU}^* K(\mathbb{Z}/p, 2) \to H\mathbb{Z}/p^* K(\mathbb{Z}/p, 2)) \simeq \mathbb{F}_p[Q_s \beta \iota_2, s > 0]$

while $H\mathbb{Z}/p^* K(\mathbb{Z}/p, 2) \simeq \mathbb{F}_p[Q_s \iota_2, Q_s \beta \iota_2]$ if $p \neq 2$

$\simeq \mathbb{F}_2[Q_s \iota_2, s \geq 0]$ if $p = 2$

Ex [Hunton - Schuster] $\mathrm{Im}\,(MU^* BG \to H^* BG)$ is F- isomorphic to $H^* BG$ if G is a finite group