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Basics and some applications of the mortar element method

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We describe the main characteristics of the mortar element method, together with the usual arguments for its numerical analysis in an abstract framework. We illustrate this presentation by describing mortar spectral element and mortar finite element methods. We give three examples of recent applications, concerning the treatment of non homogeneous media, eddy currents in moving conductors, and finite element mesh adaptivity.

1 Introduction

The mortar element method [38, 39] is a domain decomposition technique that allows to take benefit of the presence of the subdomains in order to choose the discretization method the best adapted to the local behavior of the solution of the partial differential equation which must be approximated. This choice can be done a priori or a posteriori regardless of the discretization that is chosen on the adjacent subdomains. Indeed one of the key advantages of the mortar element method when compared with other domain decomposition techniques is that it allows to handle various types of nonconformities with great flexibility:

• Functional nonconformity, i.e., nonconformity in the Hodge sense: The mortar element method relies on variational type discretizations, such as finite elements, spectral methods and wavelets. As a consequence, the discrete problem is constructed via the Galerkin process applied to the variational formulation of the partial differential equations. However, even if the local discrete spaces, i.e., the discrete spaces on each subdomain, are included in the local variational spaces, this is usually no longer the case for the global space since the matching conditions that are enforced on the interfaces between the subdomains are too weak to ensure the conformity. This is specially valid when different discretization methods are used on the subdomains, for instance for the coupling of finite elements and spectral methods, or in the case of sliding meshes.

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• Geometrical nonconformity: An efficient way for handling two- or three-dimensional complex geometries is that the less possible restrictions are enforced on the decomposition into subdomains. The mortar element method allows for handling decompositions of any type, with the only unavoidable condition that, when spectral discretization is used in one subdomain, the intersection of the boundary of this subdomain with the boundary of the computational domain is made of full edges or faces of the subdomain. Even if the numerical analysis of the method is more complex for nonconforming decompositions, this flexibility is a key argument for using the mortar element method.
• Overlapping nonconformity [52, 4]: Here the domain decomposition is chosen with overlap and again different discretizations are chosen over each subdomain. The matching condition in order to transfer the information from one subdomain to the others is done at an interface that lies inside the other subdomains and may even lie inside the discretization pattern of these subdomains. This is particularly attractive to handle complex or even unsteady geometries. However this extension is recent and will only be presented in an example in what follows.

The philosophy of the mortar element method relies on the definition and properties of the functional spaces well-suited for the analysis of the partial differential equation more than on the exact form of the operators. So, we have chosen to present the mortar element methods for the Laplace equation, where the functional space is the Sobolev space of order 1, since the basic arguments for its analysis already exist for this academic problem. We refer to [11, 12] for the treatment of fourth-order problems, the functional space being there the Sobolev space of order 2, and to [8, 58, 80] for problems set in the domains of the curl and divergence operators such as Maxwell's system and Darcy's equations. In Section 2, we describe the discretization of the Laplace equation with Dirichlet boundary conditions in an abstract framework and exhibit the sufficient conditions for the discrete problem to be well-posed. We illustrate this by applications to the finite element discretization, the spectral element discretization and the coupling of both methods. We refer to [42] and [43] for a very interesting work concerning the mortar element method for wavelets, also to [22] and [29] for the use of the mortar element method in the framework of the hp version of the finite element method that we do not consider here and to [3] for mortar finite volumes. Section 3 is devoted to the derivation of a priori error estimates. A key argument is that the choice of integral type matching conditions on the interfaces leads to an optimal evaluation of the consistency error issued from the nonconformity of the discretization. The recent results concerning finite elements and spectral methods are recalled. In Section 4, we explain how the mortar discrete problem can be implemented and solved. In the last section, we present three basic applications of the mortar element method: the spectral element discretization of elliptic equations with discontinuous coefficients, the use of sliding meshes for handling eddy currents in a rotating conductor and also the application of the mortar element method to finite element mesh adaptivity in order to increase the efficiency of the adaptivity. As a conclusion, we give some hints and references for other recent applications.

2 Description of the mortar element method

Let Ω be a bounded connected domain in $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz–continuous boundary $\partial\Omega$. We consider the Laplace equation in Ω with homogeneous Dirichlet boundary conditions on $\partial\Omega$

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

and we recall that, for any data f in $H^{-1}(\Omega)$, it admits the equivalent variational formulation

Find u in $H_0^1(\Omega)$ such that

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega (\mathbf{grad}\, u)(\boldsymbol{x}) \cdot (\mathbf{grad}\, v)(\boldsymbol{x})\, d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x})\, d\boldsymbol{x}. \tag{2}$$

For the construction of a family of finite-dimensional discrete spaces which approximate $H_0^1(\Omega)$, the mortar element method mainly relies on a decomposition of the domain Ω into subdomains without overlap

$$\overline{\Omega} = \cup_{k=1}^K \overline{\Omega}_k \qquad \text{and} \qquad \Omega_k \cap \Omega_{k'} = \emptyset, \quad 1 \le k < k' \le K, \tag{3}$$

where each Ω_k is a connected domain in $\mathbb{R}^d$ with a Lipschitz–continuous boundary. In general the Ω_k are polygons or polyhedra, and the decomposition is said to be geometrically conforming if the intersection of two different subdomains Ω_k is either empty or a vertex or a whole edge or a whole face of both of them. However neither this restriction nor other ones are a priori enforced on the decomposition.

The skeleton $\mathcal{S}$ of the decomposition, equal to $\cup_{k=1}^K \partial\Omega_k \setminus \partial\Omega$, admits a partition without overlap into mortars

$$\overline{\mathcal{S}} = \bigcup_{m=1}^{M^-} \overline{\gamma}_m^- \qquad \text{and} \qquad \gamma_m^- \cap \gamma_{m'}^- = \emptyset, \quad 1 \le m < m' \le M^-, \tag{4}$$

where each γ_m^- is a whole edge ($d = 2$) or face ($d = 3$) of one of the Ω_k, which is then denoted by Ω_m^-. Note that the choice of this decomposition is not unique, however it is decided a priori for all the discretizations we work with. Once it is fixed, we have another partition of the skeleton into non-mortars

$$\overline{\mathcal{S}} = \bigcup_{m=1}^{M^+} \overline{\gamma}_m^+ \qquad \text{and} \qquad \gamma_m^+ \cap \gamma_{m'}^+ = \emptyset, \quad 1 \le m < m' \le M^+, \tag{5}$$

where each γ_m^+ is a whole edge or face of one of the Ω_k, here denoted by Ω_m^+, and either γ_m^+ does not coincide with any $\gamma_{m'}^-$ or, if γ_m^+ is equal to a $\gamma_{m'}^-$, Ω_m^+ is different from $\Omega_{m'}^-$. Note that, even for simple decompositions, M^- is most often different from M^+ (for instance, in the case of the three rectangular subdomains $\Omega_1 =]-1,0[\times]0,1[$, $\Omega_2 =]0,1[\times]0,1[$ and $\Omega_3 =]-1,1[\times]-1,0[$, the pair (M^-, M^+) is either equal to $(2,3)$ or $(3,2)$, according to the choice of the mortar(s) on the line $y = 0$).

Let $\delta = (\delta_1, \ldots, \delta_K)$ be a K–tuple of discretization parameters, one per subdomain. From now on, c stands for a generic constant always independent of the δ_k. For each k, $1 \le k \le K$, we are given a finite-dimensional space $X_{\delta k}$ of functions defined on Ω_k and, only for brevity of the presentation, we assume that $X_{\delta k}$ is contained in $H^1(\Omega_k)$ (even if it is no longer the case when nonconforming finite elements are used on some Ω_k, see [75, 79, 83, 89] for instance). When an edge of Ω_k is a non-mortar γ_m^+, the space of traces of functions of $X_{\delta k}$ on γ_m^+ is denoted by $W_{\delta m}^+$. We also introduce a subspace $\widetilde{W}_{\delta m}^+$ of $W_{\delta m}^+$, usually of the same dimension as $W_{\delta m}^+ \cap H_{00}^{\frac{1}{2}}(\gamma_m^+)$. Finally, we consider the mortar operator Φ which associates with any function v whose restriction to each Ω_k belongs to $H^1(\Omega_k)$ the function in $L^2(\mathcal{S})$ defined by

$$\Phi(v)_{|\gamma_m^-} = \mathrm{Tr}_m^-\big(v_{|\Omega_m^-}\big), \quad 1 \le m \le M^-, \tag{6}$$

where $\mathrm{Tr}_m^\pm$ denotes the trace operator from $H^1(\Omega_m^\pm)$ onto $H^{\frac{1}{2}}(\gamma_m^\pm)$.

We are now in a position to define the mortar element discrete space. It is the space $\mathbb{X}_\delta$ of functions v_δ such that

- their restrictions to each Ω_k, $1 \le k \le K$, belong to $X_{\delta k}$,
- they vanish on $\partial\Omega$,
- the following matching conditions hold on each non-mortar γ_m^+, $1 \le m \le M^+$,

$$\forall \psi_\delta \in \widetilde{W}_{\delta m}^+, \quad \int_{\gamma_m^+} \big(\mathrm{Tr}_m^+(v_{\delta\,|\Omega_m^+}) - \Phi(v_\delta)\big)(\boldsymbol{\tau})\psi_\delta(\boldsymbol{\tau})\,d\boldsymbol{\tau} = 0. \tag{7}$$

Remark: The fact that the space $\widetilde{W}_{\delta m}^+$ is a subspace of $W_{\delta m}^+$ is one of the main characteristics of the mortar element method. Other types of spaces are also encountered, for instance in the case of hybrid methods, see [67], or in the so called "Bavarian" mortar element method, see [86]. Moreover, $\widetilde{W}_{\delta m}^+$ is usually chosen as a subspace of $W_{\delta m}^+$ with positive codimension, in order not to enforce more matching conditions than degrees of freedom (think of an edge γ_m^+ such that its two endpoints belong to $\partial\Omega$).

Example 1: Mortar spectral elements

We assume for simplicity that the Ω_k are rectangles or rectangular parallelepipeds (more complex geometries are handled for instance in [74]) such that the intersection of the boundary of each Ω_k with $\partial\Omega$ is either empty or a corner or a whole edge or a whole face of Ω_k. For each subdomain Ω_k, we fix an integer N_k, take the discretization parameter δ_k equal to N_k^{-1} and choose $X_{\delta k}$ as the space of restrictions to Ω_k of polynomials with d variables and degree $\le N_k$ with respect to each variable. The space $W_{\delta m}^+$ is then the set $\mathbb{P}_{N_m^+}(\gamma_m^+)$ of restrictions to γ_m^+ of polynomials with $d-1$ variables and degree $\le N_m^+$ with respect to each tangential variable (N_m^+ stands for the N_k such that Ω_m^+ is equal to Ω_k). The usual choice for $\widetilde{W}_{\delta m}^+$ is then the space $\mathbb{P}_{N_m^+-2}(\gamma_m^+)$.

Example 2: Mortar finite elements

We assume that the Ω_k are polygons or polyhedra and we introduce a regular family of triangulations $(\mathcal{T}_{kh})_h$ of each Ω_k by triangles or tetrahedra, in the usual sense that

- the union of all elements K of $\mathcal{T}_{kh}$ is equal to $\overline{\Omega}_k$,
- the intersection of two elements K and K' of $\mathcal{T}_{kh}$, if not empty, is a corner or a whole edge or a whole face of both K and K',
- the ratio of the diameter of an element K to the diameter of its inscribed circle or sphere is

bounded by a constant σ independent of K and h.
The parameter δ_k now denotes the maximal diameter of the elements of $\mathcal{T}_{kh}$. For a fixed integer $\ell \geq 1$, the space $X_{\delta k}$ is then the basic finite element space

$$X_{\delta k} = \big\{v_h \in H^1(\Omega_k);\ \forall K \in \mathcal{T}_{kh},\ v_{h\,|K} \in \mathcal{P}_\ell(K)\big\}, \tag{8}$$

where as usual $\mathcal{P}_\ell(K)$ denotes the space of restrictions to K of polynomials with d variables and total degree $\leq \ell$ (different degrees ℓ_k of polynomials can be used according to the Ω_k, we take all the ℓ_k equal to the same value for simplicity). With obvious notations, the space $W^+_{\delta m}$ is then given by

$$W^+_{\delta m} = \big\{\varphi_h \in H^1(\gamma^+_m);\ \forall K \in \mathcal{T}^+_{mh},\ \varphi_{h\,|K\cap\gamma^+_m} \in \mathcal{P}_\ell(K \cap \gamma^+_m)\big\}. \tag{9}$$

In dimension $d = 2$, the space $\widetilde{W}^+_{\delta m}$ is usually defined by

$$\widetilde{W}^+_{\delta m} = \big\{\varphi_h \in W^+_{\delta m};\ \forall K \in \widetilde{\mathcal{T}}^+_{mh},\ \varphi_{h\,|K\cap\gamma^+_m} \in \mathcal{P}_{\ell-1}(K \cap \gamma^+_m)\big\}, \tag{10}$$

where $\widetilde{\mathcal{T}}^+_{mh}$ denotes the set of the two triangles that contain the endpoints of γ^+_m. The choice is less standard in dimension $d = 3$ and we refer to [27] for an example where, as in (10), the degrees of freedom for the functions in $\widetilde{W}^+_{\delta m}$ are the values at the nodes of the principal lattices of order ℓ of the faces $K \cap \gamma^+_m$ of elements K of $\mathcal{T}^+_{mh}$ that do not belong to $\partial\gamma^+_m$.
Remark: A further condition sometimes appears in the definition of the mortar space $\mathbb{X}_\delta$: The functions in $\mathbb{X}_\delta$ are enforced to be continuous at the vertices in dimension $d = 2$, on the edges in dimension $d = 3$, of the Ω_k (the very first definition of the mortar space involved this condition, see [38]). It can be noted that this more stringent definition does not induce any modification in the proofs below. On the other hand, the γ^+_m and γ^-_m can be only parts of edges or faces of the $\Omega^\pm_m$, with the only restriction that they are the union of whole edges or faces of the elements of the triangulations $\mathcal{T}^\pm_{mh}$.
Example 3: Coupling spectral and finite elements
Assume that spectral discretizations are used on part of the subdomains, say on the Ω_k, $1 \leq k \leq K_0$, and that finite elements are used on the other subdomains Ω_k, $K_0 + 1 \leq k \leq K$. Then, the discrete spaces $X_{\delta k}$ are those of Example 1 for $1 \leq k \leq K_0$ and those of Example 2 for $K_0 + 1 \leq k \leq K$. The non-mortars γ^+_m are chosen

- as spectral ones, in the sense that $\widetilde{W}^+_{\delta m}$ is chosen as described in Example 1, if γ^+_m is an edge or a face of a "spectral" subdomain Ω_k, $1 \leq k \leq K_0$,
- as finite element ones, in the sense that $\widetilde{W}^+_{\delta m}$ is chosen as described in Example 2, if γ^+_m is an edge or a face of a "finite element" subdomain Ω_k, $K_0 + 1 \leq k \leq K$,
- either as spectral or finite element ones if γ^+_m is an edge or a face of both a spectral and a finite element subdomains.

The main interest of this coupling is that the advantages of both types of discretizations can be taken into account such as the high accuracy of spectral methods where the solution is smooth or the treatment of complex boundaries by finite elements.

For any data f in $L^2(\Omega)$, the discrete problem can be written as follows:

Find u_δ in $\mathbb{X}_\delta$ such that

$$\forall v_\delta \in \mathbb{X}_\delta, \quad \sum_{k=1}^{K} \int_{\Omega_k} (\mathbf{grad}\, u_\delta)(\boldsymbol{x}) \cdot (\mathbf{grad}\, v_\delta)(\boldsymbol{x})\, d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v_\delta(\boldsymbol{x})\, d\boldsymbol{x}. \tag{11}$$

Remark: In the case of spectral discretizations, the integrals which appear in the previous problem (11) are replaced by appropriate Gauss–Lobatto type quadrature formulas. Since this only adds technical arguments to the numerical analysis of the problem (in fact, the same arguments as for spectral methods without domain decomposition), we prefer not to introduce this numerical integration in the discrete problem for simplicity. Note also that a quadrature formula can be used to replace the integral that appears in condition (7), but it must be carefully chosen since, e.g., the Gauss–Lobatto formula yields matching conditions of pointwise (interpolation) type which are not at all optimal. We refer to [73] for a comparative study of the two approaches.

Remark: The mortar element method can also be seen as a special case of the three–field domain decomposition method, see [41], [49] and [78, §1.7] for instance.

The well-posedness of problem (11) relies on the ellipticity of the bilinear form

$$a(u,v) = \sum_{k=1}^{K} \int_{\Omega_k} (\mathbf{grad}\, u)(\boldsymbol{x}) \cdot (\mathbf{grad}\, v)(\boldsymbol{x})\, d\boldsymbol{x}, \tag{12}$$

on the discrete space $\mathbb{X}_\delta$. Moreover, in order to check the stability of the discrete solution, we wish to prove that the ellipticity constant is independent of δ. To this aim, for a given set of subspaces $\widetilde{W}_m^+$ of $H^{\frac{1}{2}}(\gamma_m^+)$, we introduce the space $\mathbb{X}$ of functions v such that

- their restrictions to each Ω_k, $1 \le k \le K$, belong to $H^1(\Omega_k)$,
- they vanish on $\partial\Omega$,
- the following matching conditions hold on each non-mortar γ_m^+, $1 \le m \le M^+$,

$$\forall \psi \in \widetilde{W}_m^+, \quad \int_{\gamma_m^+} \big(\mathrm{Tr}_m^+(v_{|\Omega_m^+}) - \Phi(v)\big)(\boldsymbol{\tau})\psi(\boldsymbol{\tau})\, d\boldsymbol{\tau} = 0. \tag{13}$$

Note that the $\widetilde{W}_m^+$ are not a priori discrete spaces, so that the space $\mathbb{X}$ is independent of δ. Its main advantage is that it contains both the space $H_0^1(\Omega)$ and all spaces $\mathbb{X}_\delta$ such that each $\widetilde{W}_{\delta m}^+$ contains $\widetilde{W}_m^+$, $1 \le m \le M^+$. We are led to make the following assumption.

Assumption A1: The set of functions v of $\mathbb{X}$ such that $a(v,v)$ is zero is reduced to $\{0\}$.

We also introduce the decomposition-dependent norm

$$\|v\|_{H^1(\cup\Omega_k)} = \Big(\sum_{k=1}^{K} \|v\|_{H^1(\Omega_k)}^2\Big)^{\frac{1}{2}}. \tag{14}$$

Proposition 2.1 *If Assumption $A1$ holds, there exists a constant $\alpha > 0$ such that*

$$\forall v \in \mathbb{X}, \quad a(v,v) \ge \alpha\, \|v\|_{H^1(\cup\Omega_k)}^2. \tag{15}$$

Proof: It can be noted that

$$\|v\|_{H^1(\cup\Omega_k)} = \Big(a(v,v) + \|v\|_{L^2(\Omega)}^2\Big)^{\frac{1}{2}}.$$

Thank to Assumption A1, the kernel of the first term in the right-hand side is $\{0\}$. Moreover, since the embedding of each $H^1(\Omega_k)$ into $L^2(\Omega_k)$ is compact, so is the embedding of $\mathbb{X}$ into

$L^2(\Omega)$. Thus, the desired property is an easy consequence of the Peetre–Tartar lemma, see [62, Chap. I, Thm 2.1].

In order to check that Assumption A1 holds, let us consider a function v in $\mathbb{X}$ such that $a(v,v)$ is equal to zero. Thus v is a constant c_k on each Ω_k. On the domains Ω_k such that $\partial\Omega_k \cap \partial\Omega$ has a positive measure, the constants c_k are zero, thanks to the boundary conditions in $\mathbb{X}$. So Assumption A1 is satisfied if there are enough matching conditions in (13) for all the c_k to be equal. For geometrically conforming decompositions, i.e., if each γ_m^+ is also a $\gamma_{m'}^-$, it suffices that $\widetilde{W}_m^+$ contains the constants. Otherwise:

- In the spectral element case of Example 1, the following results are proved in [38, §A.1]: In dimension $d = 2$, it suffices that $\widetilde{W}_m^+$ contains $\mathbb{P}_{n_m^+}(\gamma_m^+)$, where n_m^+ denotes the number of corners of the Ω_k which are inside γ_m^+. Unfortunately, the result is not so precise in dimension $d = 3$, see [25] and [34, Chap. IV] : There exist nonnegative integers n_m^+ only depending on the decomposition such that Assumption A1 is satisfied. This result is a little disappointing, even if it can be improved for all the decompositions that we have in mind.
- In the finite element case of Example 2, the sufficient conditions are described in [47] in a more general framework, both in dimensions $d = 2$ and $d = 3$ (see [38, §A.2] and [27] for the first results). It suffices for instance that $\widetilde{W}_m^+$ contains a function with a compact support and non zero integral in each intersection $\gamma_m^+ \cap \partial\Omega_k$ which has a positive measure in γ_m^+.
- In the coupling case of Example 3, it suffices that $\widetilde{W}_m^+$ contains the spaces described in the lines above, depending on whether γ_m^+ is of spectral type or of finite element type.

Remark: The constant α in estimate (15) depends only on the decomposition. In some cases, this decomposition is deeply linked to the discretization, see [36] for instance, so that different arguments are needed to prove the uniform ellipticity. On the other hand, note that the ellipticity property is much more difficult to establish for fourth-order problems since the kernel of the bilinear form depends not only on the decomposition but also on the discrete spaces (think of harmonic polynomials for instance!). We refer to [11, 12] for the first results concerning these problems in the spectral case and also to [48] for a more general statement in the finite element case.

Theorem 2.2 *If Assumption A1 holds and if each $\widetilde{W}_{\delta m}^+$, $1 \le m \le M^+$, contains $\widetilde{W}_m^+$, for any data f in $L^2(\Omega)$, problem (11) has a unique solution u_δ in $\mathbb{X}_\delta$. Moreover this solution satisfies*

$$\|u_\delta\|_{H^1(\cup\Omega_k)} \le \alpha^{-1}\,\|f\|_{L^2(\Omega)}. \tag{16}$$

Remark: In the case of saddle-point problems such as Stokes and Darcy's system (see Section 5.1), the ellipticity property (15) is not sufficient to prove the well-posedness of the discrete problem. In the mortar spectral element case, a further inf-sup condition is proved in [20] for the Stokes problem and in [8] for the Darcy problem. For the Stokes and Navier–Stokes equations, we also refer to [55] for a pioneering work concerning the coupling of finite elements with spectral discretizations in the mortar framework and to [1, 77] for more recent results in the finite element case. Note to conclude that the Boland and Nicolaides argument [44] provides an efficient tool for these results, since it allows for deriving the global inf-sup condition from the local ones on each subdomain.

3 Numerical analysis

We now wish to prove an a priori error estimate between the solution u of problem (1) and the discrete solution u_δ. When multiplying the first line in (1) by a function u_δ in $\mathbb{X}_\delta$ and integrating by parts on each Ω_k, we observe that

$$a(u, w_\delta) = \int_\Omega f(\boldsymbol{x}) w_\delta(\boldsymbol{x})\, d\boldsymbol{x} - \int_{\mathcal{S}} (\partial_n u)(\boldsymbol{\tau})[w_\delta](\boldsymbol{\tau})\, d\boldsymbol{\tau}, \tag{17}$$

where, if $\boldsymbol{n}$ denotes the unit normal vector going from one Ω_k to another $\Omega_{k'}$, $[w_\delta]$ denotes the jump $w_{\delta\,|\Omega_{k'}} - w_{\delta\,|\Omega_k}$. Next, if Assumption $A1$ holds and if each $\widetilde{W}^+_{\delta m}$, $1 \le m \le M^+$, contains $\widetilde{W}^+_m$, we deduce from Proposition 2.1 that

$$\|u - u_\delta\|^2_{H^1(\cup\Omega_k)} \le \alpha^{-1}\, a(u - u_\delta, u - u_\delta).$$

Adding (17) and subtracting problem (11) then gives, for any v_δ in $\mathbb{X}_\delta$,

$$\|u - u_\delta\|^2_{H^1(\cup\Omega_k)} \le \alpha^{-1} \Big(a(u - u_\delta, u - v_\delta) + \int_{\mathcal{S}} (\partial_n u)(\boldsymbol{\tau})[u_\delta - v_\delta](\boldsymbol{\tau})\, d\boldsymbol{\tau} \Big).$$

Using the continuity of the form $a(\cdot,\cdot)$ leads to the following version of the Second Strang's lemma, see [31],

$$\|u - u_\delta\|_{H^1(\cup\Omega_k)} \le c\, \Big(\inf_{v_\delta \in \mathbb{X}_\delta} \|u - v_\delta\|_{H^1(\cup\Omega_k)} + \sup_{w_\delta \in \mathbb{X}_\delta} \frac{\int_{\mathcal{S}} (\partial_n u)(\boldsymbol{\tau})[w_\delta](\boldsymbol{\tau})\, d\boldsymbol{\tau}}{\|w_\delta\|_{H^1(\cup\Omega_k)}} \Big). \tag{18}$$

Note that the first term on the right-hand side represents the approximation error while the second term is the consistency error.

In order to evaluate the consistency error, we introduce the orthogonal projection operator $\pi^+_{\delta m}$ from $L^2(\gamma^+_m)$ onto $\widetilde{W}^+_{\delta m}$. We recall the following properties of this operator in both cases of Examples 1 and 2: For any nonnegative real numbers s and t, with the further restriction $s \le \ell$ and $t \le \ell$ in the finite element case, and for any function φ in $H^s(\Gamma^+_m)$,

$$\|\varphi - \pi^+_{\delta m}\varphi\|_{H^{-t}(\gamma^+_m)} \le c\, (\delta^+_m)^{t+s}\, \|\varphi\|_{H^s(\gamma^+_m)}. \tag{19}$$

Proposition 3.1 *Let the solution u of problem* (2) *be such that each $u_{|\Omega_k}$ belongs to $H^{s_k+1}(\Omega_k)$,*
(i) with $s_k \ge \frac{1}{2}$ for all k, $1 \le k \le K$, in the case of Example 1 and for the k such that $1 \le k \le K_0$, in the case of Example 3,
(ii) with $\frac{1}{2} \le s_k \le \ell$ for all k, $1 \le k \le K$, in the case of Example 2 and for the k such that $K_0 + 1 \le k \le K$, in the case of Example 3.
The following bound holds for the consistency error issued from problem (11)

$$\sup_{w_\delta \in \mathbb{X}_\delta} \frac{\int_{\mathcal{S}} (\partial_n u)(\boldsymbol{\tau})[w_\delta](\boldsymbol{\tau})\, d\boldsymbol{\tau}}{\|w_\delta\|_{H^1(\cup\Omega_k)}} \le c \sum_{k=1}^K \delta_k^{s_k}\, |\log \delta_k|^{p_k}\, \|u\|_{H^{s_k+1}(\Omega_k)}, \tag{20}$$

where p_k is equal to 1 if one of the edges of Ω_k is a γ^+_m and intersects at least two subdomains $\overline{\Omega}_{k'}$, $k' \neq k$, to zero otherwise.

Proof: On each γ_m^+, $1 \le m \le M^+$, thanks to condition (7), we have

$$\int_{\gamma_m^+} (\partial_n u)(\boldsymbol{\tau})[w_\delta](\boldsymbol{\tau})\, d\boldsymbol{\tau} = \int_{\gamma_m^+} (\partial_n u - \pi_{\delta m}^+ \partial_n u)(\boldsymbol{\tau})[w_\delta](\boldsymbol{\tau})\, d\boldsymbol{\tau}.$$

Note that $[w_\delta]$ belongs to $H^{\frac{1}{2}}(\gamma_m^+)$ when γ_m^+ intersects at most another $\overline{\Omega}_{k'}$. Otherwise it only belongs to $H^{\frac{1}{2}-\varepsilon}(\gamma_m^+)$ for any $\varepsilon > 0$. So, we have

$$\int_{\gamma_m^+} (\partial_n u)(\boldsymbol{\tau})[w_\delta](\boldsymbol{\tau})\, d\boldsymbol{\tau}$$
$$\le \|\partial_n u - \pi_{\delta m}^+ \partial_n u\|_{H^{\varepsilon-\frac{1}{2}}(\gamma_m^+)} \big(\|\mathrm{Tr}_m^+(w_\delta)\|_{H^{\frac{1}{2}-\varepsilon}(\gamma_m^+)} + \|\Phi(w_\delta)\|_{H^{\frac{1}{2}-\varepsilon}(\gamma_m^+)}\big).$$

We recall from [43, Rem. 2.10] that, for $\varepsilon > 0$ and for any part γ of γ_m^+, the extension by zero is continuous from $H^{\frac{1}{2}-\varepsilon}(\gamma)$ into $H^{\frac{1}{2}-\varepsilon}(\gamma_m^+)$ and its norm is bounded by $c\,\varepsilon^{-1}$. When combining this with the trace theorem and using a Cauchy–Schwartz inequality, we obtain

$$\int_{\gamma_m^+} (\partial_n u)(\boldsymbol{\tau})[w_\delta](\boldsymbol{\tau})\, d\boldsymbol{\tau} \le (1 + c\,\varepsilon^{-1})\, \|\partial_n u - \pi_{\delta m}^+ \partial_n u\|_{H^{\varepsilon-\frac{1}{2}}(\gamma_m^+)} \big(\sum_{k \in \mathcal{E}_m^+} \|w_\delta\|^2_{H^1(\Omega_k)}\big)^{\frac{1}{2}},$$

where $\mathcal{E}_m^+$ denotes the set of indices k, $1 \le k \le K$, such that $\gamma_m^+ \cap \partial\Omega_k$ has a positive measure in γ_m^+. Thus, the desired bound follows by applying (19), choosing ε equal to $|\log \delta_m^+|^{-1}$, when it is not zero, and summing over m.

Note that the $|\log \delta_k|$ are most often negligible (moreover they can be replaced by $|\log \delta_k|^{\frac{3}{4}}$ according to a recent result [54] and by 1 in the finite element case of Example 2) and that they disappear in the case of a geometrically conforming decomposition. Evaluating the approximation error is much more complex, specially in dimension $d = 3$, so we only give the main ideas of the proof. In dimension $d = 2$, it relies on the following argument:
– taking v_δ^1 equal, on each Ω_k, to the interpolate of u at the Gauss-Lobatto nodes in the case of Example 1, to the Lagrange interpolate of u on the principal lattice of order ℓ associated with each element of $\mathcal{T}_{kh}$ in the case of Example 2, with obvious extension to the case of Example 3 (note that each $v^1_{\delta\,|\Omega_k}$ belongs to $X_{\delta k}$),
– lifting on each Ω_m^- the jump of v_δ^1 at each corner of an Ω_k which is inside γ_m^- by multiplying it by a quadratic or piecewise affine function which vanishes on $\partial\Omega_m^- \setminus \gamma_m^-$ and denoting by v_δ^2 the sum of v_δ^1 and of these liftings,
– taking v_δ^3 equal to

$$v_\delta^3 = \mathcal{L}_{\delta m}^+ \circ \tilde{\pi}_{\delta m}^+ \big(\Phi(v_\delta^2) - \mathrm{Tr}_m^+(v^2_{\delta\,|\Omega_m^+})\big),$$

where $\mathcal{L}_{\delta m}^+$ is a continuous lifting operator from $W_{\delta m}^+ \cap H_{00}^{\frac{1}{2}}(\gamma_m^+)$ into the space of functions in $X_{\delta m}^+$ vanishing on $\partial\Omega_m^+ \setminus \gamma_m^+$ (the existence of such an operator and its properties are stated in [40, §III.3] in the case of Example 1 and in [40, §IX.4] in the case of Example 2), while the operator $\tilde{\pi}_{\delta m}^+$ takes its values in $W_{\delta m}^+ \cap H_{00}^{\frac{1}{2}}(\gamma_m^+)$ and is defined, for any smooth enough function v, by

$$\forall \psi_\delta \in \widetilde{W}_{\delta m}^+, \quad \int_{\gamma_m^+} \big(v - \tilde{\pi}_{\delta m}^+ v\big)(\boldsymbol{\tau}) \psi_\delta(\boldsymbol{\tau})\, d\boldsymbol{\tau} = 0.$$

Indeed, thanks to this choice, it is readily checked that the function $v_\delta = v_\delta^2 + v_\delta^3$ now belongs to $\mathbb{X}_\delta$. We refer to [39, §4] for the details of this proof and to [34, Chap. IV] for the improvement in the case of a geometrically conforming decomposition.

Proposition 3.2 *If the assumptions of Proposition* 3.1 *are satisfied, the following bound holds for the approximation error issued from problem* (11) *in the case of dimension* $d = 2$

$$\inf_{v_\delta \in \mathbb{X}_\delta} \|u - v_\delta\|_{H^1(\cup\Omega_k)} \leq c\,(1+\lambda_\delta)^{\frac{1}{2}} \sum_{k=1}^{K} \delta_k^{s_k}\, \|u\|_{H^{s_k+1}(\Omega_k)}, \tag{21}$$

where λ_δ *is equal*
(i) to 0 *in the case of a geometrically conforming decomposition, in the finite element case of Example* 2 *or if all non-mortars* γ_m^+ *are of finite element type in the case of Example* 3*,*
(ii) otherwise, to the maximum of the λ_m^+*,* $1 \leq m \leq M_+$ *in the case of Example* 1 *or such that* γ_m^+ *is of spectral type in the case of Example* 3*, with each* λ_m^+ *equal to the maximum of the* N_k/N_m^+ *for all* Ω_k *such that* $\partial\Omega_k \cap \gamma_m^+$ *has a positive measure in* γ_m^+*.*

This estimate is optimal since λ_δ is most often bounded independently of δ in practical situations. However the analysis is much more complex in the case of dimension $d = 3$. We refer to [18, 25, 27, 28, 46] for several results in this direction, however we only consider the case of a geometrically conforming decomposition for brevity. The next result is proved in [34, Chap. IV] in the spectral case.

Proposition 3.3 *If the assumptions of Proposition* 3.1 *are satisfied, the following bound holds for the approximation error issued from problem* (11) *in the case of dimension* $d = 3$*, when the decomposition is geometrically conforming and in the spectral case of Example* 1

$$\inf_{v_\delta \in \mathbb{X}_\delta} \|u - v_\delta\|_{H^1(\cup\Omega_k)} \leq c \sum_{k=1}^{K} \delta_k^{s_k}\, \|u\|_{H^{s_k+1}(\Omega_k)}. \tag{22}$$

Combining the results of Propositions 3.1 to 3.3 with (18) leads to the final error estimate.

Theorem 3.4 *If the assumptions of Proposition* 3.1 *are satisfied, the following a priori error estimate holds between the solution* u *of problem* (2) *and the solution* u_δ *of problem* (11) *either in dimension* $d = 2$ *or in the case of a geometrically conforming decomposition and in the spectral case of Example* 1 *in dimension* $d = 3$

$$\|u - u_\delta\|_{H^1(\cup\Omega_k)} \leq c\,(1+\lambda_\delta)^{\frac{1}{2}} \sum_{k=1}^{K} \delta_k^{s_k}\, |\log \delta_k|^{p_k}\, \|u\|_{H^{s_k+1}(\Omega_k)}, \tag{23}$$

where the quantities p_k *and* λ_δ *are introduced in Propositions* 3.1 *and* 3.2*, respectively.*

Estimate (23) gives a good idea of the interest of the mortar element method in a large number of situations.

4 Numerical implementation

The linear system associated with problem (11) is usually implemented via one of the two following techniques: the construction of a basis of the constrained spaces or the introduction of a Lagrange multiplier to handle the matching conditions. We successively describe these implementation techniques. Next, we briefly explain how they can be combined with more standard domain decomposition algorithms and give some references for that. Further algorithms and many more details on those which are described here can be found, e.g., in [88] and the references therein.

4.1 Construction of a basis of the constrained spaces

The main idea, due to [72] (see also [80, Chap. 6]), consists in extracting from a basis of all $X_{\delta k}$ a basis of $\mathbb{X}_\delta$. We describe it in the case of two subdomains Ω_1 and Ω_2 for simplicity, and we assume that there exists only one non-mortar γ_1^+, with Ω_1^+ equal to Ω_1.

We introduce the Lagrange basis functions φ_j^k, $1 \le j \le J_\delta^k$, which are

- in the spectral case of Example 1, associated with the nodes of the Gauss–Lobatto formula on Ω_k (so that J_δ^k is equal to $(N_k+1)^d$),
- in the finite element case of Example 2, associated with the nodes of the principal lattices of order ℓ of all elements K of $\mathcal{T}_{kh}$.

Among them, we denote by φ_j^k, $1 \le j \le J_{\delta 0}^k$, those associated with nodes inside Ω_k and by φ_j^k, $J_{\delta 0}^k + 1 \le j \le J_{\delta *}^k$, those associated with the nodes belonging to $\mathcal{S}$ but not to $\partial\Omega$.

The basis $\mathcal{B}$ is built as the union of three disjoint sets $\mathcal{B}^1$, $\mathcal{B}^2$ and $\mathcal{B}^{\mathcal{S}}$, where

- $\mathcal{B}^1$ is the set of the φ_j^1, $1 \le j \le J_{\delta 0}^1$, extended by zero to Ω_2,
- $\mathcal{B}^2$ is the set of the φ_j^2, $1 \le j \le J_{\delta 0}^2$, extended by zero to Ω_1.

The construction of $\mathcal{B}^{\mathcal{S}}$ is a little more complex. With each function φ_j^1, $J_{\delta 0}^1 + 1 \le j \le J_{\delta *}^1$, we associate the function $\varphi_j^{\mathcal{S}}$ equal to φ_j^1 on Ω_1 and to the linear combination $\sum_{i=J_{\delta 0}^2+1}^{J_{\delta *}^2} q_{ij}\,\varphi_i^2$ on Ω_2, where the q_{ij} are such that

$$\forall \psi_\delta \in \widetilde{W}_{\delta 1}^+, \quad \int_{\gamma_1^+} \big(\varphi_j^1 - \sum_{i=J_{\delta 0}^2+1}^{J_{\delta *}} q_{ij}\,\varphi_i^2\big)(\boldsymbol{\tau})\psi_\delta(\boldsymbol{\tau})\,d\boldsymbol{\tau} = 0. \tag{24}$$

It can be checked from the definition of $\widetilde{\mathbb{W}}_{\delta 1}^+$ that system (24) defines the q_{ij} in a unique way. The basis $\mathcal{B}^{\mathcal{S}}$ is then the set of the $\varphi_j^{\mathcal{S}}$, $J_{\delta 0}^1 + 1 \le j \le J_{\delta *}^1$.

Let Q be the rectangular matrix with coefficients equal to the q_{ij}, $J_{\delta 0}^2 + 1 \le i \le J_{\delta *}^2$, $J_{\delta 0}^1 + 1 \le j \le J_{\delta *}^1$. The discrete solution u_δ admits the following expansion in the basis $\mathcal{B}$

$$u_\delta = \sum_{j=1}^{J_{\delta 0}^1} u_j^1\,\varphi_j^1 + \sum_{j=1}^{J_{\delta 0}^2} u_j^2\,\varphi_j^2 + \sum_{j=J_{\delta 0}^1+1}^{J_{\delta *}^1} u_j^{\mathcal{S}}\,\varphi_j^{\mathcal{S}}, \tag{25}$$

so that the vector U of unknowns is made of the u_j^1, u_j^2 and $u_j^{\mathcal{S}}$. Moreover, the function u_δ can be written on each Ω_k,

$$u_{\delta|\Omega_1} = \sum_{j=1}^{J_{\delta 0}^1} u_j^1\, \varphi_j^1 + \sum_{j=J_{\delta 0}^1+1}^{J_{\delta *}^1} u_j^{\mathcal{S}}\, \varphi_j^1,$$

$$u_{\delta|\Omega_2} = \sum_{j=1}^{J_{\delta 0}^2} u_j^2\, \varphi_j^2 + \sum_{i=J_{\delta 0}^2+1}^{J_{\delta *}^2} \sum_{j=J_{\delta 0}^1+1}^{J_{\delta *}^1} u_j^{\mathcal{S}}\, q_{ij}\, \varphi_i^2. \quad (26)$$

Hence the vector made by these new coefficients on each Ω_k are equal to $\widetilde{Q}U$, where $\widetilde{Q}$ is block diagonal, with two blocks equal to the identity and the last one equal to the rectangular matrix Q. As a consequence, problem (11) results into the square linear system

$$\widetilde{Q}^T \widetilde{A}\, \widetilde{Q}\, U = \widetilde{Q}^T \widetilde{F}, \quad (27)$$

where
(i) the matrix $\widetilde{A}$ is symmetric and block-diagonal, one block per subdomain Ω_k and each block corresponding to a discrete Laplace equation on an Ω_k with Dirichlet boundary conditions on $\partial\Omega_k \cap \partial\Omega$ and Neumann boundary conditions on $\partial\Omega_k \cap \mathcal{S}$,
(ii) $\widetilde{Q}^T$ denotes the transposed matrix of $\widetilde{Q}$, so that the matrix $\widetilde{Q}^T \widetilde{A}\widetilde{Q}$ is symmetric.
Only the dimension of the vector U is equal to the dimension of $\mathbb{X}_\delta$. Indeed, the vector $\widetilde{F}$ and the square matrix $\widetilde{A}$ are larger, which seems necessary in order to enforce that $\widetilde{A}$ is block diagonal.

The integrals in the matching conditions involve discrete functions defined on different non-matching grids and, as a consequence, the computation goes through the intersection of two different meshes on $\mathcal{S}$. In dimension $d = 2$, this is rather easy [50] since $\mathcal{S}$ is a one-dimensional curve. In dimension $d = 3$, finding this intersection becomes a hard task. The use of quadrature formulas to evaluate these integrals highly increases the efficiency of the implementation. However, using a standard Galerkin approximation and replacing the exact integration by a quadrature formula based only on the mortar or non-mortar side does not yield optimal results [73]. The best approximation error requires a quadrature formula based on the mortar side and the consistency error requires one on the non-mortar side. As a consequence, we are led to work with different test and trial spaces, resulting into a Petrov-Galerkin approach. A different but symmetric approach relies on the introduction of a third discretization on the skeleton, totally independent of those on the mortar and non-mortar sides. A quadrature formula is then defined on the mesh triangles of this third mesh and projected on those of the mortar and non-mortar sides (see [81] for a concrete application of this approach in three dimensions). We refer to [80] for a detailed analysis of the discretization relying on the previous considerations.

4.2 Introduction of a Lagrange multiplier

We introduce the space $\widetilde{\mathbb{W}}_\delta = \prod_{m=1}^{M^+} \widetilde{W}_{\delta m}^+$, and the space $\widetilde{\mathbb{X}}_\delta$ of functions v_δ such that
- their restrictions to each Ω_k, $1 \le k \le K$, belong to $X_{\delta k}$,

- they vanish on $\partial\Omega$.

We also define the bilinear form

$$c(v_\delta, \boldsymbol{\psi}_\delta) = \sum_{m=1}^{M^+} \int_{\gamma_m^+} \big(\mathrm{Tr}_m^+(v_{\delta\,|\Omega_m^+}) - \Phi(v_\delta)\big)(\boldsymbol{\tau})\psi_m(\boldsymbol{\tau})\,d\boldsymbol{\tau}, \quad \text{with} \quad \boldsymbol{\psi}_\delta = (\psi_1, \dots, \psi_{M^+}). \tag{28}$$

So $\mathbb{X}_\delta$ is the space of all functions in $\widetilde{\mathbb{X}}_\delta$ such that $c(v_\delta, \boldsymbol{\psi}_\delta)$ is zero for all $\boldsymbol{\psi}_\delta$ in $\widetilde{\mathbb{W}}_\delta$. We now consider the discrete problem

Find $(u_\delta, \boldsymbol{\lambda}_\delta)$ *in* $\widetilde{\mathbb{X}}_\delta \times \widetilde{\mathbb{W}}_\delta$ *such that*

$$\begin{aligned} &\forall v_\delta \in \widetilde{\mathbb{X}}_\delta, \quad \sum_{k=1}^{K} \int_{\Omega_k} (\mathbf{grad}\, u_\delta)(\boldsymbol{x}) \cdot (\mathbf{grad}\, v_\delta)(\boldsymbol{x})\,d\boldsymbol{x} + c(v_\delta, \boldsymbol{\lambda}_\delta) \\ &\qquad = \int_\Omega f(\boldsymbol{x}) v_\delta(\boldsymbol{x})\,d\boldsymbol{x}, \\ &\forall \boldsymbol{\psi}_\delta \in \widetilde{\mathbb{W}}_\delta, \quad c(u_\delta, \boldsymbol{\psi}_\delta) = 0. \end{aligned} \tag{29}$$

The next statement is classical, and we refer to [18] for its detailed proof.

Proposition 4.1 *If Assumption* $A1$ *holds and if each* $\widetilde{W}_{\delta m}^+$, $1 \le m \le M^+$, *contains* $\widetilde{W}_m^+$, *for any data* f *in* $L^2(\Omega)$, *problem* (29) *has a unique solution* $(u_\delta, \boldsymbol{\lambda}_\delta)$ *in* $\widetilde{\mathbb{X}}_\delta \times \widetilde{\mathbb{W}}_\delta$. *Moreover, problems* (11) *and* (29) *are equivalent, in the sense that*
(i) for each solution $(u_\delta, \boldsymbol{\lambda}_\delta)$ *of problem* (29), *the function* u_δ *is a solution of problem* (11),
(ii) for each solution u_δ *of problem* (11), *there exists a* $\boldsymbol{\lambda}_\delta$ *in* $\widetilde{\mathbb{W}}_\delta$ *such that* $(u_\delta, \boldsymbol{\lambda}_\delta)$ *is a solution of problem* (29).

According to the arguments in [18], it can also be checked that $\boldsymbol{\lambda}_\delta$ is an approximation of $\partial_n u$, where u is the solution of problem (2). The main interest of problem (29) is that it results into a square linear system of type

$$\begin{pmatrix} \widetilde{A} & C \\ C^T & 0 \end{pmatrix} \begin{pmatrix} \widetilde{U} \\ \Lambda \end{pmatrix} = \begin{pmatrix} \widetilde{F} \\ 0 \end{pmatrix}, \tag{30}$$

where the matrix $\widetilde{A}$ and the vector $\widetilde{F}$ are the same as in system (27). System (30) can be solved via an Uzawa algorithm for instance. We refer to [78, Chap. 3] for more details on the way of solving it.

4.3 Solution algorithms

The simplest approach for solving the mortar element discretization is to consider the first representation of the space with "constrained" basis. Indeed under the formulation (27) a global conjugate gradient method can be designed that consists in computing from a current approximation U^p at iteration p, the residual $\mathcal{R}^p = \widetilde{Q}^T\widetilde{F} - \widetilde{Q}^T\widetilde{A}\widetilde{Q}U^p$. This is particularly well suited for parallel implementation since the matrix A is block diagonal, see [1, 26, 68]. From the degrees of freedom U^p we derive the nodal values over each subdomain. The complexity is very low since

• the matrix $\widetilde{Q}$ is nontrivial only on a set of dimension equal to the number of degrees of freedom on the skeleton,
• the expensive part, i.e., the multiplication by the matrix $\widetilde{A}$ is done independently over each subdomain,
• the residuals are finally added through the multiplication by $\widetilde{Q}^T$.
The implementation is thus very natural and "embarrassingly parallel", the convergence rate is the same as the one of the conjugate gradient method for solving the mono-domain discretization. In order to accelerate the convergence, preconditioners can be proposed based on incomplete factorization, but a better way is to combine the standard techniques that have proved their efficiency on conforming domain decompositions, such as Schwarz, substructuring or FETI type approaches [5, 53, 56, 57, 82]. The coupling of the mortar element method with multigrid algorithms is analyzed in [87] and [85].

A more precise description of the previous techniques can be found in [78, 88]. We also refer to the recent book [84] for a review of all these methods and even more.

5 Some applications

We have chosen to present three basic applications, the first one in the spectral context and the last two ones in the finite element context. Hints and references for other applications are given at the end of the section. We present a few numerical experiments in the two-dimensional case. For brevity, we prefer to omit the description of three-dimensional tests and refer to [59] and [73] among others for recent simulations of different physical phenomena.

5.1 Treatment of non homogeneous porous media

By non homogeneous media, we mean that one or several coefficients involved in the partial differential equation are piecewise constant. So it seems natural to use a domain decomposition which takes into account these discontinuities, in the sense that the coefficients are constant on each subdomain, but which can involve a larger number of subdomains for numerical reasons. We have chosen to present this idea for the Darcy equations that model the flow of a viscous incompressible fluid in a porous medium

$$\begin{cases} \boldsymbol{u} + \alpha\, \mathbf{grad}\, p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{31}$$

The parameter α is a piecewise constant positive function which represents the porosity of the inhomogeneous medium divided by the viscosity of the fluid. It must be observed that this system can equivalently be written as a second-order elliptic equation with Neumann boundary conditions, and we refer to [33] for the analysis of the mortar spectral element discretization of this equation, to [15] for the mathematical and numerical comparison of the two discretizations issued from the two formulations of the problem in a specific geometry.

It is readily checked that problem (31) admits the following equivalent variational formulation

Find $(\boldsymbol{u}, p)$ *in* $L^2(\Omega)^d \times \big(H^1(\Omega) \cap L^2_0(\Omega)\big)$ *such that*

$$\forall \boldsymbol{v} \in L^2(\Omega)^d, \quad \int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x} + \int_\Omega \boldsymbol{v}(\boldsymbol{x}) \cdot \alpha(\boldsymbol{x})\, (\mathbf{grad}\, p)(\boldsymbol{x})\, d\boldsymbol{x} = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x},$$
$$\forall q \in H^1(\Omega), \quad \int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot (\mathbf{grad}\, q)(\boldsymbol{x})\, d\boldsymbol{x} = 0, \tag{32}$$

where $L^2_0(\Omega)$ stands for the space of functions in $L^2(\Omega)$ with a null integral on Ω. Note that this problem is not of usual saddle-point type since three bilinear forms are involved in its formulation. However its well-posedness is proved in [2] according to the arguments given in [32]: For any data $\boldsymbol{f}$ in $L^2(\Omega)^d$, problem (32) admits a unique solution $(\boldsymbol{u}, p)$.

We now consider a decomposition of the domain Ω as introduced in (3), with the assumptions that each subdomain Ω_k satisfies the further condition stated in Example 1 and that the function α is constant, equal to α_k, on each domain Ω_k. For a parameter δ, we introduce two discrete spaces:

- The space $\mathbb{Z}_\delta$ of functions in $L^2(\Omega)^d$ such that their restrictions to each Ω_k, $1 \le k \le K$, belong to $X^d_{\delta k}$, where $X_{\delta k}$ is the space described in Example 1,
- The space $\overline{\mathbb{X}}_\delta$ of functions in $L^2_0(\Omega)$ such that their restrictions to each Ω_k, $1 \le k \le K$, belong to the space $X_{\delta k}$ described in Example 1 and which satisfy condition (7) again for the space $\widetilde{W}^+_{\delta m}$ described in Example 1.

The discrete problem now reads

Find $(\boldsymbol{u}_\delta, p_\delta)$ *in* $\mathbb{Z}_\delta \times \overline{\mathbb{X}}_\delta$ *such that*

$$\forall \boldsymbol{v}_\delta \in \mathbb{Z}_\delta, \quad \int_\Omega \boldsymbol{u}_\delta(\boldsymbol{x}) \cdot \boldsymbol{v}_\delta(\boldsymbol{x})\, d\boldsymbol{x} + b_\alpha(\boldsymbol{v}_\delta, p_\delta) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}_\delta(\boldsymbol{x})\, d\boldsymbol{x},$$
$$\forall q_\delta \in \overline{\mathbb{X}}_\delta, \quad b(\boldsymbol{u}_\delta, q_\delta) = 0, \tag{33}$$

where the bilinear forms $b(\cdot,\cdot)$ and $b_\alpha(\cdot,\cdot)$ are defined by

$$b(\boldsymbol{u}, q) = \sum_{k=1}^K \int_{\Omega_k} \boldsymbol{u}(\boldsymbol{x}) \cdot (\mathbf{grad}\, q)(\boldsymbol{x})\, d\boldsymbol{x},$$
$$b_\alpha(\boldsymbol{v}, p) = \sum_{k=1}^K \alpha_k \int_{\Omega_k} \boldsymbol{v}(\boldsymbol{x}) \cdot (\mathbf{grad}\, p)(\boldsymbol{x})\, d\boldsymbol{x}. \tag{34}$$

In order to optimize the numerical analysis of this problem, we introduce the quantities (see [2])

$$\|\boldsymbol{u}\|_{\alpha^{-1}} = \Big(\sum_{k=1}^K \alpha_k^{-1}\, \|\boldsymbol{u}\|^2_{L^2(\Omega_k)^d}\Big)^{\frac12}, \qquad \|p\|_{\alpha*} = \Big(\sum_{k=1}^K \alpha_k\, \|\mathbf{grad}\, p\|^2_{L^2(\Omega_k)^d}\Big)^{\frac12}. \tag{35}$$

Indeed, the same arguments as for Proposition 2.1 yield that, if Assumption A1 holds, the seminorm $\|\cdot\|_{\alpha*}$ is a norm on $\overline{\mathbb{X}}_\delta$. Moreover, when taking $\boldsymbol{v}_\delta$ equal to $\alpha^{-1}\,\boldsymbol{u}_\delta + \mathbf{grad}\, p_\delta$ in the first line of problem (33), we derive the estimate

$$\|\boldsymbol{u}_\delta\|_{\alpha^{-1}} + \|p_\delta\|_{\alpha*} \le \sqrt{2}\, \|\boldsymbol{f}\|_{\alpha^{-1}}. \tag{36}$$

Since problem (33) results into a square linear system, this estimate obviously yields its well-posedness.

Proposition 5.1 *If Assumption* $A1$ *holds and if each* $\widetilde{W}^+_{\delta m}$, $1 \le m \le M^+$, *contains* $\widetilde{W}^+_m$, *for any data* $\boldsymbol{f}$ *in* $L^2(\Omega)^d$, *problem* (33) *has a unique solution* u *in* $\mathbb{X}_\delta$. *Moreover this solution satisfies estimate* (36).

To prove an a priori error estimate between the solutions of problem (32) and (33), we observe that the following identity holds for any $(\boldsymbol{w}_\delta, r_\delta)$ in $\mathbb{Z}_\delta \times \overline{\mathbb{X}}_\delta$

$$\begin{aligned}\forall \boldsymbol{v}_\delta \in \mathbb{Z}_\delta, \quad & \int_\Omega (\boldsymbol{u}_\delta - \boldsymbol{w}_\delta)(\boldsymbol{x}) \cdot \boldsymbol{v}_\delta(\boldsymbol{x})\, d\boldsymbol{x} + b_\alpha(\boldsymbol{v}_\delta, p_\delta - r_\delta) \\ & \qquad = \int_\Omega (\boldsymbol{u} - \boldsymbol{w}_\delta)(\boldsymbol{x}) \cdot \boldsymbol{v}_\delta(\boldsymbol{x})\, d\boldsymbol{x} + b_\alpha(\boldsymbol{v}_\delta, p - r_\delta), \\ \forall q_\delta \in \overline{X}_\delta, \quad & b(\boldsymbol{u}_\delta - \boldsymbol{w}_\delta, q_\delta) = b(\boldsymbol{u} - \boldsymbol{w}_\delta, q_\delta) - \int_{\mathcal{S}} (\boldsymbol{u} \cdot \boldsymbol{n})(\boldsymbol{\tau})[q_\delta]\, d\boldsymbol{\tau}.\end{aligned} \tag{37}$$

So the same arguments as for estimate (18), combined with triangle inequalities, lead to

$$\begin{aligned}& \|\boldsymbol{u} - \boldsymbol{u}_\delta\|_{\alpha^{-1}} + \|p - p_\delta\|_{\alpha *} \\ & \quad \le c \Big(\|\boldsymbol{u} - \boldsymbol{w}_\delta\|_{\alpha^{-1}} + \|p - r_\delta\|_{\alpha *} + \sup_{q_\delta \in \overline{\mathbb{X}}_\delta} \frac{\int_{\mathcal{S}} (\boldsymbol{u} \cdot \boldsymbol{n})(\boldsymbol{\tau})[q_\delta](\boldsymbol{\tau})\, d\boldsymbol{\tau}}{\|q_\delta\|_{\alpha *}} \Big).\end{aligned} \tag{38}$$

Bounding the approximation error in $\mathbb{Z}_\delta$ is derived by introducing the orthogonal projection operator from $L^2(\Omega)^d$ provided with the norm $\|\cdot\|_{\alpha^{-1}}$ onto $\mathbb{Z}_\delta$, and evaluating the approximation error in $\overline{\mathbb{X}}_\delta$ and the consistency error relies on the same arguments as in Section 3.

Theorem 5.2 *Let the solution* $(\boldsymbol{u}, p)$ *of problem* (32) *be such that each* $(\boldsymbol{u}_{|\Omega_k}, p_{|\Omega_k})$, $1 \le k \le K$, *belongs to* $H^{s_k}(\Omega_k)^d \times H^{s_k+1}(\Omega_k)$ *for some* $s_k > \frac{1}{2}$. *In the case of dimension* $d = 2$, *the following error estimate holds between this solution and the solution* $(\boldsymbol{u}_\delta, p_\delta)$ *of problem* (33)*:*

$$\begin{aligned}& \|\boldsymbol{u} - \boldsymbol{u}_\delta\|_{\alpha^{-1}} + \|p - p_\delta\|_{\alpha *} \\ & \quad \le c\,(1 + \lambda_\delta)^{\frac{1}{2}} \sum_{k=1}^{K} \delta_k^{s_k} \left(|\log \delta_k|^{p_k}\, \alpha_k^{-\frac{1}{2}}\, \|\boldsymbol{u}\|_{H^{s_k}(\Omega_k)^d} + \alpha_k^{\frac{1}{2}}\, \|p\|_{H^{s_k+1}(\Omega_k)} \right),\end{aligned} \tag{39}$$

where the quantities p_k *and* λ_δ *are introduced in Propositions* 3.1 *and* 3.2, *respectively.*

Estimate (39) is fully optimal, at least for a geometrically conforming decomposition, and the possibly high ratios between the different values of the α_k are correctly taken into account by the weighted norms. Moreover the same results still hold when the integrals in (33) are replaced by Gauss-type quadrature formulas and also for more complex subdomains, according to the arguments in [74].

5.2 Eddy currents with sliding meshes

Eddy currents are generated in a conducting material by temporal variations of the surrounding magnetic field, as it results from Lenz's law (see [45, Chap. 8] for a more precise explanation of this phenomenon). These variations are induced either when the conductor does not move but is embedded in a magnetic field with a time varying source or when the magnetic field is constant but the conductor is moving. The first case can be treated without difficulties by a standard finite element or spectral approach, but not the second, where a part of the domain is moving with respect to the remaining part.

The eddy current system is represented by Maxwell's equations in the low frequency approximation, i.e., the displacement currents are neglected with respect to the induced ones. Let $\boldsymbol{E}$ be the electric field, $\boldsymbol{H}$ the magnetic field, $\boldsymbol{B}$ the magnetic induction, $\boldsymbol{J}_s$ the source current and $\boldsymbol{J}$ the induced or *eddy* current. We assume that the magnetic permeability μ and the electric conductivity σ are piecewise constant scalar functions on Ω. The equations that must be solved in $\Omega \times [0, T]$, with $T > 0$, are Faraday's law and Ampère's theorem, and read

$$\mathbf{curl}\, \boldsymbol{E} = -\partial_t \boldsymbol{B}, \qquad \mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J} + \boldsymbol{J}_s. \tag{40}$$

They are combined with two constitutive relations and the standard Gauss conservation law,

$$\boldsymbol{B} = \mu\, \boldsymbol{H}, \qquad \boldsymbol{J} = \sigma\, \boldsymbol{E}, \qquad \operatorname{div} \boldsymbol{B} = 0. \tag{41}$$

The system is made complete by appropriate boundary conditions on $\partial\Omega \times [0, T]$ and initial conditions at $\Omega \times \{0\}$.

The computation of the space and time distribution of eddy currents is of great importance for performance prediction and device design in the case of electric engines. For all important examples, we can divide the computational domain into at least two subdomains: Ω_1 which contains the moving part (thus, the conductor) and the remaining part $\Omega_2 = \Omega \setminus \overline{\Omega_1}$ which usually contains the magnetic field sources. Here the skeleton $\mathcal{S}$ of the decomposition is reduced to the interior of $\overline{\Omega_1} \cap \overline{\Omega_2}$. This non-overlapping decomposition is chosen in such a way that the interface $\mathcal{S}$, which can be physical or not, is invariant with respect to the motion of Ω_1, i.e., $\mathcal{S}$ is a *sliding interface*, and Ω_1 moves in contact with Ω_2 along $\mathcal{S}$. Note that the idea of sliding meshes was first considered in [7] and [6].

When Ω_1 is moving with respect to Ω_2, we have to choose the reference system with respect to which we write the eddy current problem. Let $\mathcal{R}_i$ be the reference system linked to Ω_i, $i = 1, 2$. If $\boldsymbol{v}$ is the conductor velocity, the appropriate form of Ohm's law in $\mathcal{R}_2$ reads

$$\boldsymbol{J} = \sigma\, (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \quad \text{in} \quad \Omega_1 \times [0, T] \qquad \text{and} \qquad \boldsymbol{J} = \sigma\, \boldsymbol{E} \quad \text{in} \quad \Omega_2 \times [0, T]. \tag{42}$$

The motion of Ω_1 is directly taken into account in the convective term $\boldsymbol{v} \times \boldsymbol{B}$. This is a typical feature of the Eulerian description, i.e., the use of a *unique* reference system for both subdomains Ω_k.

To get rid of the explicit velocity term, the idea is to use as many different reference systems as the number of subdomains, that is, in our case, to reformulate with respect to $\mathcal{R}_1$ the equations in Ω_1 and with respect to $\mathcal{R}_2$ the equations in Ω_2. This is the Lagrangian description, where the observer is attached to the subdomain under consideration and describes the events from his material point of view. Indeed, the explicit velocity term disappears from Ohm's law, provided that each subdomain is treated in its own "co-moving" frame ($\mathcal{R}_k$ with Ω_k). If two different reference systems are used, one has to couple both of them by suitable transmission conditions at the sliding interface. We underline the fact that for the analysis of eddy current problems in domains with moving parts, the freedom in the choice of the reference frame is valid only when the motion can be considered as quasi-stationary with respect to electrodynamics. This freedom is a consequence of the low frequency limit and would not be possible with the full system of Maxwell's equations. Thanks to this property of the eddy current model, we can adopt the "piecewise Lagrangian approach" (a Lagrangian approach in each part), see [80, Chap. 4] for instance. This allows us to work with independent meshes and discretizations in the Ω_k. Searching the solution of the problem in Ω is equivalent to looking for the solutions of the subproblems stated in the Ω_k which satisfy some transmission conditions at the interface $\mathcal{S}$.

The transmission conditions at the interface express the continuity through $\mathcal{S}$ of the tangential components of $\boldsymbol{H}$ and $\boldsymbol{E}$ and of the normal components of $\boldsymbol{B}$ and $\boldsymbol{J}$, for all t in $[0, T]$, see [21]. Let $r_t : \Omega_1 \to \Omega_1$ be the motion operator acting on Ω_1 and r_{-t} its inverse. We adopt the notation $\Omega_1(t) = r_t\Omega_1(0)$ where $\Omega_1(0)$ is the initial configuration of Ω_1. The material particle occupies a definite position $\boldsymbol{x}$ in the initial configuration $\Omega_1(0)$ and its changing position in the present configuration $\Omega_1(t)$ at time t is given by $r_t\boldsymbol{x}$. If u is a scalar function (e.g., a scalar potential) defined on Ω and u_i denotes its restriction $u_{|\Omega_i}$ to Ω_i, the transmission conditions on $\mathcal{S} \times [0, T]$ read

$$u_1(r_{-t}\boldsymbol{x}, t) = u_2(\boldsymbol{x}, t), \qquad c(r_{-t}\boldsymbol{x})\, \partial_n u_1(r_{-t}\boldsymbol{x}, t) = c(\boldsymbol{x})\, \partial_n u_2(\boldsymbol{x}, t), \tag{43}$$

where c is a scalar function of the space position (e.g., depending on the material parameters). In the case where $\boldsymbol{u}$ is a vector field (e.g., a vector potential), the transmission conditions (43) are written for the either tangential or normal component of $\boldsymbol{u}$ that is physically continuous across $\mathcal{S}$ (see [16, 51] for more details). To apply a finite element or spectral discretization, we rewrite the eddy current problem in variational form. Note that only the left-hand condition in (43) (which is the *essential* one) is explicitly enforced on both the unknowns and test functions. The right-hand condition in (43) (which is the *natural* one) is handled in the variational formulation and can be recovered by integration by parts [50].

The eddy current problem is then discretized in time by a finite difference scheme and in space by finite elements or a spectral type method. For the space discretization, the main problem is to verify the essential transmission condition at each time t in $[0, T]$. In a conforming approach, this condition is exactly satisfied, whereas in a nonconforming one, such as in the mortar framework, this condition is weakly enforced, i.e., it is re-written in terms of suitable Lagrange multipliers and the jump of the traces. This weak formulation allows for more flexibility in the space mesh (the space discretization parameter is completely independent of the time step) but it is a little more costly, as explained in Section 4.1.

Figure 1 presents the mesh near the interface (left part) and the curves of isovalues of the magnetic vector potential (right part) in a cross section of the considered device. The meshes are non-matching at the sliding interface and the mortar element method is used to take into account the conductor movement.

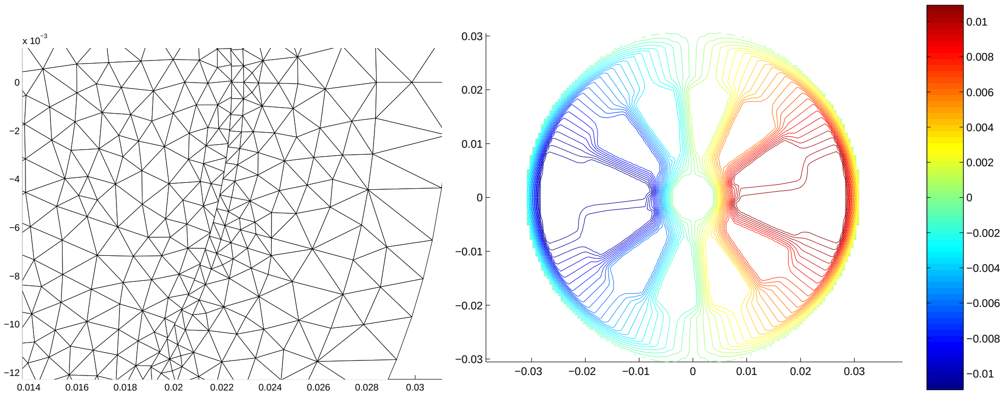

Figure 1

(collaboration with the Laboratoire de G´enie Electrique de Paris, U.M.R. 8507 C.N.R.S.)

5.3 Mesh adaptivity

A very simple way for locally refining a triangular mesh in dimension $d = 2$ consists in cutting the marked triangles into four equal triangles by joining the middle of the edges, and this process can be iterated as many times as needed. However it leads to a partition of the domain which is no longer a triangulation since hanging nodes appear on some edges of the triangles that have been cut. A number of methods have been proposed to solve this difficulty; let us quote among others the Delaunay algorithm. We choose here another approach which consists in handling the nonconforming parts of the decomposition by the mortar element method.

We assume that Ω is a polygon. Let $(\mathcal{T}_h^0)_{h^0}$ be a regular family of triangulations of Ω, in the sense made precise in Section 2, where as usual h^0 denotes the maximal diameter of the triangles in $\mathcal{T}_h^0$. Assuming that the family $(\mathcal{T}_h^{n-1})_{h^{n-1}}$, we construct a new family of triangulations as follows:

• For arbitrary positive integers m, we cut some elements of $\mathcal{T}_h^{n-1}$ into 2^{2m} subtriangles by iteratively joining the midpoints of the edges of these elements;

• We denote by $\mathcal{T}_h^{n,k}$, the set of triangles such that their area is equal to 2^{-2k} times the area of the triangle of $\mathcal{T}_h^0$ in which they are contained;

• We denote by $\mathcal{K}_h^n$ the set of integers k such that $\mathcal{T}_h^{n,k}$ is not empty and by $\Omega^{n,k}$ the open domain such that $\overline{\Omega}^{n,k}$ is the union of the triangles of $\mathcal{T}_h^{n,k}$;

• We call $\mathcal{T}_h^n$ the union of all $\mathcal{T}_h^{n,k}$, $k \in \mathcal{K}_h^n$.

It is readily checked that the $\Omega^{n,k}$ form a partition of the domain Ω as described in (3). So we use the decompositions (4) and (5) of the skeleton with the further assumption that each $\gamma_m^{\pm}$ is a whole edge of one of the elements of $\mathcal{T}_h^n$.

We are now interested in the finite element discretization of problem (1) which relies on the partition $\mathcal{T}_h^n$. So we consider the discrete spaces $X_{\delta k}$ and $\widetilde{W}_{\delta m}^{+}$ defined in (8) and (10), respectively, with the triangulation $\mathcal{T}_{kh}$ replaced by $\mathcal{T}_h^{n,k}$. Let $\mathbb{X}_h^n$ denote the subspace of the corresponding space $\mathbb{X}_\delta$ defined in Section 2, made of functions which are continuous at the endpoints of the γ_m^+ (this further condition make the analysis easier and also simplifies the implementation). We consider the following discrete problem

Find u_h^n in $\mathbb{X}_h^n$ such that

$$\forall v_h \in \mathbb{X}_h^n, \quad a_{hn}(u_h^n, v_h) = \int_\Omega f(\boldsymbol{x}) v_h(\boldsymbol{x})\, d\boldsymbol{x}, \tag{44}$$

where the bilinear form $a_{hn}(\cdot,\cdot)$ is defined by

$$a_{hn}(u, v) = \sum_{k \in \mathcal{K}_h^n} \int_{\Omega^{n,k}} (\mathbf{grad}\, u)(\boldsymbol{x}) \cdot (\mathbf{grad}\, v)(\boldsymbol{x})\, d\boldsymbol{x}. \tag{45}$$

Due to the continuity of the functions of $\mathbb{X}_h^n$ at the endpoints of the γ_m^+, the next result is obvious.

Proposition 5.3 *For any data f in $L^2(\Omega)$, problem* (44) *has a unique solution u in $\mathbb{X}_h^n$.*

The main difficulty for the a priori analysis of this problem is to prove the uniform ellipticity of the form $a_{hn}(\cdot,\cdot)$ since the decomposition we consider highly depends on the discretization. We refer to [37, Prop. 2.1] for the proof of the following result, which requires an assumption on the choice of the γ_m^+

Assumption A2: For $1 \le m \le M^+$,

(i) either γ_m^+ is the union of several edges of triangles contained in $\overline{\Omega}_m^+$,

(ii) or γ_m^+ intersects at most another domain $\Omega^{n,k} \neq \Omega_m^+$.

Proposition 5.4 *If Assumption $A2$ holds, there exists a constant $\alpha > 0$ independent of h and n such that*

$$\forall v_h \in \mathbb{X}_h^n, \quad a_{nh}(v_h, v_h) \ge \alpha\, \|v_h\|^2_{H^1(\cup\Omega^{n,k})}. \tag{46}$$

Thanks to this ellipticity property, proving the a priori error estimate relies on the same arguments as in Section 3. We refer to [36, §2] for the details.

Theorem 5.5 *Let the solution u of problem* (1) *be such that each $u_{|\Omega^{n,k}}$, $k \in \mathcal{K}_h^n$, belongs to $H^{s_k+1}(\Omega^{n,k})$ for some $s_k > \frac{1}{2}$. If Assumption A2 holds, the following error estimate holds between this solution and the solution u_h^n of problem* (44)*:*

$$\|u - u_h^n\|_{H^1(\cup\Omega^{n,k})} \le c\,(1 + \mu_\delta) \sum_{k \in \mathcal{K}_h^n} (h^{n,k})^{s_k}\, \|u\|_{H^{s_k+1}(\Omega^{n,k})}, \tag{47}$$

where

(i) $h^{n,k}$ denotes the maximal diameter of the triangles in $\mathcal{T}_h^{n,k}$,

(ii) the parameter μ_δ is the maximum of the μ_m, $1 \le m \le M^+$, where each μ_m denotes the maximal ratio k_+/k where Ω_m^+ is equal to $\Omega^{n,k}$ and k_+ denote the maximum of the k' such that γ_m^+ intersects $\overline{\Omega}^{n,k'}$.

Estimate (47) is fully optimal, at least when the $\mathcal{K}_h^n$ are contained in a set of integers which is bounded independently of h and n. Moreover, despite the nonconformity of the discretization, optimal a posteriori estimates are proved in [36, §3]. So we think that this method provides a very efficient tool for adaptivity. The previous results have been recently extended to the Stokes problem in [35] and to the heat equation in [30].

To conclude, we describe a numerical experiment performed on the code FreeFem++ [64]. In the square $\Omega =]-1,1[^2$, we take f equal to 100 times the characteristic function of the domain $]0,\frac{1}{2}[^2$. Figure 2 presents the final adapted mesh and the curves of isovalues of the corresponding discrete solution. Note that the final mesh does not reflect the nearly concentric structure of the solution because the initial mesh does not and that we stop the adaptivity iteration when the Hilbertian sum of the indicators is small enough.

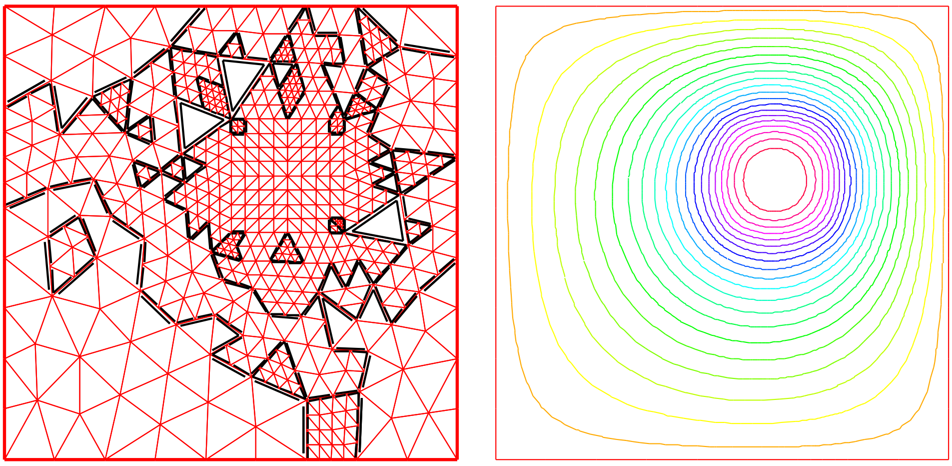

Figure 2
(courtesy of F. Hecht)

Adaptivity by the mortar element method in the framework of spectral elements, first proposed in [9], has been developed in [76] and more recently in [59].

5.4 And others

We have described three applications of the mortar element method, but there exist many other ones and, among them, we wish to quote the following topics.

- The interaction between two solids

The mortar element method has become an efficient tool for handling the contact between two solids. Indeed, in the finite element case for instance, each of these solids is provided with its own triangulation, and the contact between them is modeled by an appropriate condition which can often be treated as a mortar matching conditions. The case of a fixed contact zone has first been analyzed in [23]. The more complex and more realistic situation of unilateral contact where the contact zone is an unknown of the problem has been studied in [66, 24, 71, 19] for elastic solids, in [60] for elastic–viscoplastic solids and in [65, 10] in the case of contact with friction. This also seems to be the first application of the mortar element method to variational inequalities.

- The Navier–Stokes equations with discontinuous boundary data

We consider an elliptic problem with boundary condition equal to 1 on a part of the boundary and to 0 elsewhere. When extended to the Navier–Stokes equations with zero normal values on the boundary of the domain and tangential values presenting the same type of discontinuity, this problem is known as the driven cavity problem. The mortar element method seems well suited to handle this type of difficulty since a partition of the domain Ω into two subdomains Ω_1 and Ω_2 can be introduced such that the boundary data are constant on each $\partial\Omega \cap \partial\Omega_k$; so the discretization takes into account the discontinuity.

The main difficulty here is that neither the solution of the Laplace equation nor the velocity of the Navier–Stokes equations with this type of boundary conditions has a square integrable gradient. This leads to use weighted Sobolev spaces, the distance being essentially the distance to the boundary $\partial\Omega$ up to a fixed positive power and to introduce a weighted version of the mortar element method. The existence of a solution for the weighted formulation of the problem and the convergence of the corresponding mortar spectral element of the discretization are proved in [13, 14] for the Stokes and Navier–Stokes equations, respectively.

- The coupling of fluids and structures

For fluid-structure interaction problems, and more generally for problems related to multiphysics there is another reason for referring to the mortar element methodology. Indeed, the operators used in the modeling of the different phenomena taking place in adjacent domains, here the fluid part and the structure part, can be of different orders, leading to different *functional spaces* on both parts of the interface and thus to different discretization spaces as well. In our specific case, the coupling between the fluid subdomain and the structure subdomain implies that the velocities of the fluid and the structure coincide (essential coupling condition) and the normal stresses are also equal (natural coupling condition). As always the variational statement for this coupled problem involves spaces in which the essential interface condition is the only that appears explicitly and the mortar condition deals only with this one. In the case of the finite element approach, the difficulty is that the nature of the finite elements is different: They are, e.g., H^1–conforming ones of Lagrange type for the fluid and H^2–conforming ones of Hermite type for the structure (see [40, Chap. VIII] for precise definitions). This forbids the exact equality of the discrete velocities, in this sense the mortar coupling provides the natural approach for relaxing the equality. For the analysis of a simplified problem in this context, we refer to [63]. It appears that the mortar element method builds a natural frame and yields optimal error bounds, in contrast with approaches based on interpolation.

- The problems of shells

An extension of the mortar element method to the Discrete Kirchhoff Triangles (called DKT in what follows) approximation for shells problem is presented and analyzed in [70] and numerical results are shown in [69]. The basic idea of the DKT approximation is to neglect the shear strain energy and to introduce on the discrete model some Kirchhoff–Love relations between the rotations β and the displacements. For the shell equation, many functions have to be matched, first the tangential displacements $u_{h\alpha}$ for $\alpha = 1, 2$ and then the transversal displacement u_{h3}. The rotations β_h associated with these displacements have also to be matched. For the two first components of the displacement, the matching is easy since these functions are independent and are involved in a second order equation, their natural space is $H^1(\Omega)$ and the standard mortar element method for piecewise quadratic finite elements is used. The originality of the generalization relies on the following approach: We start from the value u_{h3}^+ to derive the rotations β^+ by using the DKT conditions. The next step is then to recover the rotation β^- on the mortar side and at this level we handle separately the tangential and transverse components. The final step is to reconstruct the displacement on the mortar side by integrating the rotations, in order to maintain as much as possible the Kirchhoff–Love conditions. This way of matching leads to optimal consistency and approximation errors.

- The coupling $\boldsymbol{T} - \Phi$

We consider again the eddy current problem and still use the notation of Section 5.2. We are now interested in conductors that can move anywhere in the global domain. ¿From the point of view of the problem formulation, we assume that $\overline{\Omega}_1$ is contained in Ω_2 and we work with two different unknowns:
- a scalar function Φ in the whole computational domain Ω_2,
- a vector field $\boldsymbol{T}$ in the conducting part Ω_1.

The equations for these unknowns are derived from Maxwell's system (40) by writing

$$\boldsymbol{H} = \boldsymbol{T} - \mathbf{grad}\,\Phi \quad \text{in } \Omega_1, \qquad \boldsymbol{H} = \boldsymbol{T}_s - \mathbf{grad}\,\Phi \quad \text{in } \Omega_2 \setminus \overline{\Omega}_1, \tag{48}$$

where $\boldsymbol{T}_s$ represents a source term. Indeed, we have $\boldsymbol{J} = \mathbf{curl}\,\boldsymbol{T}$ and $\boldsymbol{J}_s = \mathbf{curl}\,\boldsymbol{T}_s$.

The novelty with respect to Section 5.2 is that the domain decomposition cannot be done without overlap, due to the lack of sliding interfaces in the configuration of the problem (as in the case of an electrodynamic levitation system for instance). In the mortar element framework, one has to deal with overlapping decompositions, as already hinted in the introduction. In this particular case, we use two different meshes on Ω_2, that is considered as a non-mortar domain, and on Ω_1, that plays the role of a mortar domain: the mesh in Ω_1 is neither a restriction nor a refinement of the mesh of Ω_2. The transmission conditions concern the continuity of Φ and of the tangential component of $\boldsymbol{T}$ through $\mathcal{S} = \partial\Omega_1$. Again the mortar approach is used to weakly enforce this continuity. The computation of the associated integrals now goes through the intersection of the three-dimensional supports of the trial functions defined in Ω_2 with the two-dimensional supports of the Lagrange multipliers defined on $\mathcal{S}$. This computation can be efficiently performed by using quadrature formulas [73, 61]. For this particular problem, standard Lagrange finite elements are used to approximate the scalar potential Φ everywhere in Ω_2 while the vector potential $\boldsymbol{T}$ is approximated in Ω_1 by edge elements (see [45, Chap. V] for a general description of edge elements).

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