Université de Nice Sophia-Antipolis
Master MathMods - Finite Elements - 2008/2009
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Basic Concepts - Exercises (Chapter 0)

Exercise 1.
Find the "stiffness" matrix $\mathbf{K}$ for piecewise linear basis functions. If the right hand side f is piecewise linear i.e.

$$f(x) = \sum_{i=1}^{n} f_i \phi_i(x)$$

determine the matrix $\mathbf{M}$ called "mass" matrix such that : $\mathbf{KU} = \mathbf{MF}$.

Exercise 2.
Give the weak formulation for the two-point boundary value problem :

$$\begin{cases} -u'' + u = f, x \in (0,1), \\ u(0) = u(1) = 0 \end{cases}$$

Exercise 3.
Explain what is wrong in both variational and classical setting for the problem :

$$\begin{cases} -u'' = f, x \in (0,1), \\ u'(0) = u'(1) = 0 \end{cases}$$

that is explain in both contexts why this problem is not well-posed.

Exercise 4.
Show that piecewise quadratics have nodal basis consisting of values at nodes x_i together with the midpoints $\frac{1}{2}(x_i + x_{i+1})$. Calculate the stiffness matrix for these elements.

Exercise 5.
Let $h = \max_{1 \le i \le n}(x_i - x_{i-1})$. Then,

$$\|u - u_I\| \le Ch\|u''\|, \ \forall u \in V,$$

where C is independent of h and u.
Hint : Use first the *homogeneity argument*, then show that :

$$\int_0^1 w(x)^2 dx \le \tilde{c} \int_0^1 w'(x)^2 dx$$

by utilizing the fact that $w(0) = 0$. How small can you make $\tilde{c}$ if you use both $w(0) = 0$ and $w(1) = 0$?

Exercise 6.
We denote $a(u,v) = \int_0^1 u'(x)v'(x)dx$ and $V = \{v \in L^2(0,1); a(v,v) < \infty, v(0) = 0\}$. Prove the following *coercivity* results :

$$\|v\|^2 + \|v'\|^2 \le Ca(v,v), \ \forall v \in V$$

Give a value for C.

Exercise 7.
Consider the difference method represented by :

$$-\frac{2}{h_i + h_{i+1}}\left(\frac{U_{i+1} - U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{h_i}\right) = f(x_i).$$

Prove that $\tilde{u}_S = \sum_i U_i \phi_i$ satisfies the following :

$$a(\tilde{u}_S, v) = Q(fv),\ \forall v \in S,\ a(u, v) = \int_0^1 u'(x)v'(x)dx$$

where S consists of piecewise linears and Q denotes the quadrature approximation based on the trapezoidal rule :

$$Q(w) = \sum_{i=0}^{n} \frac{h_i + h_{i+1}}{2} w(x_i).$$

We further define $h_0 = h_{n+1} = 0$ for simplicity of notation.

Exercise 8.
Let Q be give by the previous exercise. Prove that :

$$\left| Q(w) - \int_0^1 w(x)dx \right| \leq Ch^2 \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |w''(x)|dx$$

`Hint` : Observe that the trapezoidal rule is exact for piecewise linears and then use exercise 5.

Exercise 9.
Let u_S the solution of $a(u_S, v) = (f, v), \forall v \in S$, where S consists of piecewise linears and let $\tilde{u}_S$ be as in exercise 7. Prove that :

$$|a(u_S - \tilde{u}_S, v)| \leq Ch^2(\|f'\| + \|f''\|)(\|v\| + \|v'\|)$$

`Hint` : Apply exercise 8 and Schwarz' inequality.

Exercise 10.
Let u_S and $\tilde{u}_S$ be like in the exercise 9. Prove that :

$$\|u_S - \tilde{u}_S\|_E \leq Ch^2(\|f'\| + \|f''\|)$$

`Hint` : Apply exercise 9, pick $v = u_S - \tilde{u}_S$ and apply exercise 6.