Université de Nice Sophia-Antipolis
Master MathMods - Finite Elements - 2008/2009
V. Dolean

Exercises - Chapter 0 (correction)

Exercise 1.
Find the "stiffness" matrix $\mathbf{K}$ for linear basis functions. If the right hand side f is piecewise linear i.e.

$$f(x) = \sum_{j=1}^{n} f_j \phi_j(x)$$

determine the matrix $\mathbf{M}$ called "mass" matrix such that : $\mathbf{KU} = \mathbf{MF}$.

Answer. The linear basis functions are given by :

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h_i}, \; x \in [x_{i-1}, x_i], \\ \dfrac{x_{i+1} - x}{h_{i+1}}, \; x \in [x_i, x_{i+1}], \\ 0, x \notin [x_{i-1}, x_{i+1}]. \end{cases}$$

According to the expression of the "stiffness" matrix we can write :

$$\begin{aligned} K_{ii} &= \int_0^1 (\phi_i')^2(x)dx = \int_{x_{i-1}}^{x_i} (\phi_i')^2(x)dx + \int_{x_i}^{x_i+1} (\phi_i')^2(x)dx = \frac{1}{h_i} + \frac{1}{h_{i+1}}. \\ K_{i,i+1} &= K_{i+1,i} = \int_0^1 \phi_i'(x)\phi_{i+1}'(x)dx = \int_{x_i}^{x_i+1} \phi_i'(x)\phi_{i+1}'(x)dx = -\frac{1}{h_{i+1}}. \end{aligned}$$

all the other elements being null since in all the other cases the basis functions ϕ_i and ϕ_j cannot be simultaneously non-zero. The right hand side can be written as :

$$b_i = \int_0^1 f(x)\phi_i(x)dx = \sum_{j=1}^{n} f_j \int_0^1 \phi_i(x)\phi_j(x)dx = \sum_{j=1}^{n} M_{ij} f_j, \; M_{ij} = \int_0^1 \phi_i(x)\phi_j(x)dx.$$

Thus, the "mass" matrix is formed by the elements M_{ij} which can be computed as follows (by performing a variable change $x = x_{i-1} + th$ in the integral on $[x_{i-1}, x_i]$ and $x = x_i + th$ in the integral on $[x_i, x_{i+1}]$) :

$$\begin{aligned} M_{ii} &= \int_0^1 \phi_i^2(x)dx = \int_{x_{i-1}}^{x_i} \phi_i^2(x)dx + \int_{x_i}^{x_i+1} \phi_i^2(x)dx \\ &= h_i \int_0^1 t^2 dt + h_{i+1} \int_0^1 (1-t)^2 dt = \frac{h_i + h_{i+1}}{3} \\ M_{i,i+1} &= M_{i+1,i} = \int_0^1 \phi_i(x)\phi_{i+1}(x)dx = \int_{x_i}^{x_i+1} \phi_i(x)\phi_{i+1}(x)dx = h_{i+1} \int_0^1 (1-t)t dt = \frac{h_{i+1}}{6}. \end{aligned}$$

all the other elements being null since in all the other cases the basis functions ϕ_i and ϕ_j cannot be simultaneously non-zero.

Exercise 2.
Give the weak formulation for the two-point boundary value problem :

$$\begin{cases} -u'' + u = f, x \in (0, 1), \\ u(0) = u(1) = 0. \end{cases}$$

Answer. Mutiplying the equation inside the domain by a test function v and by integrating by parts we get :

$$\int_0^1 uv + u'v' = \int_0^1 fv,\ a(u,v) := \int_0^1 uv + u'v'$$

The weak formulation can be written as :

$$\text{find } u \in V = \{v \in L^2(0,1) : v(0) = v(1) = 0\}, \text{ such that } a(u,v) = (f,v),\ \forall v \in V.$$

Exercise 3.
Explain what is wrong in both variational and classical setting for the problem :

$$\begin{cases} -u'' = f, x \in (0,1), \\ u'(0) = u'(1) = 0 \end{cases}$$

that is explain in both contexts why this problem is not well-posed.

Answer. For the classical setting we can see that is u is solution of the problem $u + C$ (where C is a constant) is also solution. Therefore the problem is not well-posed. The weak formulation can be written as :

$$\text{find } u \in V = L^2(0,1), \text{ such that } \int_0^1 u'v' = (f,v),\ \forall v \in V.$$

If we put $v = C$ (where C is a constant) we get that $\int_0^1 f = 0$, that means if f doesn't respect this condition the problem has no solution (this is called compatibility condition). If the condition is fulfilled, the solution is defined only up to a constant.

Exercise 4.
Show that piecewise quadratics have nodal basis consisting of values at nodes x_i together with the midpoints $\frac{1}{2}(x_i + x_{i+1})$. Calculate the stiffness matrix for these elements.

Answer. We denote by ϕ_{2i} the basis functions associated to x_i and by ϕ_{2i+1} those associated to the midpoint $\frac{1}{2}(x_i + x_{i+1})$. They are given by :

$$\phi_{2i}(x) = \begin{cases} \dfrac{2x - x_{i-1} - x_i}{h_i} \cdot \dfrac{x - x_{i-1}}{h_i}\ x \in [x_{i-1}, x_i], \\ \dfrac{2x - x_i - x_{i+1}}{h_{i+1}} \cdot \dfrac{x - x_{i+1}}{h_{i+1}},\ x \in [x_i, x_{i+1}], \\ 0, x \notin [x_{i-1}, x_{i+1}]. \end{cases},\ \phi_{2i+1}(x) = \begin{cases} 4\dfrac{x - x_i}{h_{i+1}} \cdot \dfrac{x_{i+1} - x}{h_{i+1}},\ x \in [x_i, x_{i+1}], \\ 0, x \notin [x_i, x_{i+1}]. \end{cases}$$

The stiffness matrix is again symmetric, with at most 5 non-zero elements on each line which can be computed as follows (by performing a variable change $x = x_{i-1} + th$ in the integral on $[x_{i-1}, x_i]$ and $x = x_i + th$ in the integral on $[x_i, x_{i+1}]$) :

$$\begin{aligned}
K_{2i,2i} &= \int_0^1 (\phi'_{2i})^2(x)dx = \int_{x_{i-1}}^{x_i} (\phi'_{2i})^2(x)dx + \int_{x_i}^{x_i+1} (\phi'_{2i})^2(x)dx, \\
&= \frac{1}{h_i}\int_0^1 (4t-1)^2 dt + \frac{1}{h_{i+1}}\int_0^1 (4t-3)^2 dt = \frac{7}{3}\left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right), \\
K_{2i,2(i+1)} &= K_{2(i+1),2i} = \int_{x_i}^{x_i+1} \phi'_{2i}(x)\phi'_{2(i+1)}(x)dx = \frac{1}{h_{i+1}}\int_0^1 (4t-3)(4t-1)dt = \frac{1}{3h_{i+1}}, \\
K_{2i+1,2i+1} &= \int_{x_i}^{x_i+1} (\phi'_{2i+1})^2(x)dx = \frac{1}{h_{i+1}}\int_0^1 16(2t-1)^2 dt = \frac{16}{3h_{i+1}}, \\
K_{2i,2i+1} &= K_{2i+1,2i} = \int_{x_i}^{x_i+1} \phi'_{2i}(x)\phi'_{2i+1}(x)dx = \frac{1}{h_{i+1}}\int_0^1 (4t-1)(4-8t)dt = -\frac{8}{3h_{i+1}}.
\end{aligned}$$

Exercise 5.
Let $h = \max_{1\leq i\leq n}(x_i - x_{i-1})$. Then,

$$\|u - u_I\| \leq Ch\|u''\|, \forall u \in V,$$

where C is independent of h and u.
Hint : Use first the *homogeneity argument*, then show that :

$$\int_0^1 w(x)^2 dx \leq \tilde{c} \int_0^1 w'(x)^2 dx \tag{1}$$

by utilizing the fact that $w(0) = 0$. How small can you make $\tilde{c}$ if you use both $w(0) = 0$ and $w(1) = 0$?

Answer. In the following we will use the *homogeneity argument* as in the lecture. According to the definition of the two norms, it is sufficient to prove the estimate piecewise, i.e :

$$\int_{x_{j-1}}^{x_j} (u - u_I)'(x)^2 dx \leq c(x_j - x_{j-1})^2 \int_{x_{j-1}}^{x_j} u''(x) dx$$

with $C = \sqrt{c}$. This inequality can be re-written in terms of error by denoting : $e = u - u_I$ (note that u_I is piecewise linear and therefore its second derivative cancels) and then by performing a variable change $x = x_{j-1} + t(x_j - x_{j-1})$ (an affine mapping from the interval $[x_{j-1}, x_j]$ to $[0, 1]$) as follows :

$$\int_0^1 \tilde{e}(t)^2 dt \leq c \int_0^1 \tilde{e}''(t) dt,\ \tilde{e}(t) = e(x_{j-1} + t(x_j - x_{j-1})).$$

Now, using some results of the lecture, it is enough to prove (1) for $w = \tilde{e}$. Note that $w(0) = w(1) = 0$ since the interpolation error will be zero at all nodes. Therefore :

$$w(x) = \int_0^x w'(t) dt$$

by using Schwarz' inequality we get :

$$\begin{aligned}
\int_0^1 w(x)^2 dx &= \int_0^1 \left(\int_0^x 1 \cdot w'(t) dt\right)^2 dx \leq \int_0^1 \left(\int_0^x dt\right) \cdot \left(\int_0^x w'(t)^2 dt\right) dx \\
&\leq \int_0^1 x \cdot \left(\int_0^x w'(t)^2 dt\right) dx \leq \int_0^1 x \cdot \left(\int_0^1 w'(t)^2 dt\right) dx \\
&= \left(\int_0^1 w'(t)^2 dt\right) \cdot \int_0^1 x dx = \frac{1}{2} \int_0^1 w'(t)^2 dt,
\end{aligned}$$

the constant is thus $\tilde{c} = \frac{1}{2}$.
If $w(0) = w(1) = 0$, we can consider this function as periodic with period $T = 1$ and write its Fourier series as follows :

$$w(x) = \sum_k a_k \sin(k\pi x) = \sum_{k\neq 0} a_k \sin(k\pi x) \Rightarrow w'(x) = \sum_{k\neq 0} k\pi a_k \cos(k\pi x)$$

Using Parseval's equality we get :

$$\int_0^1 w(x)^2 dx = \frac{1}{2} \sum_{k\neq 0} a_k^2,\ \int_0^1 w'(x)^2 dx = \frac{1}{2} \sum_{k\neq 0} (k\pi)^2 a_k^2,$$

which proves the optimal inequality (the best $\tilde{c} = \frac{1}{\pi^2}$) :

$$\int_0^1 w(x)^2 dx \leq \frac{1}{\pi^2} \int_0^1 w'(x)^2 dx.$$

Exercise 6.
We denote $a(u,v) = \displaystyle\int_0^1 u'(x)v'(x)dx$ and $V = \{v \in L^2(0,1); a(v,v) < \infty, v(0) = 0\}$. Prove the following *coercivity* results :

$$\|v\|^2 + \|v'\|^2 \le Ca(v,v),\ \forall v \in V$$

Give a value for C.

Answer. Using the previous exercise it is easy to see that :

$$\|v\| \le c\|v'\|,\ c = \frac{1}{2} \Rightarrow \|v\|^2 + \|v'\|^2 \le (c^2+1)\|v'\|^2 = (c^2+1)a(v,v) \Rightarrow C = \frac{5}{4}$$

Furthermore, if the space V were given by $V = \{v \in L^2(0,1); a(v,v) < \infty, v(0) = v(1) = 0\}$, then C attains its optimal value $C = \dfrac{1+\pi^4}{\pi^4}$.

Exercise 7.
Consider the difference method represented by :

$$-\frac{2}{h_i + h_{i+1}}\left(\frac{U_{i+1} - U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{h_i}\right) = f(x_i). \qquad (2)$$

Prove that $\tilde{u}_S = \sum_i U_i\phi_i$ satisfies the following :

$$a(\tilde{u}_S, v) = Q(fv),\ \forall v \in S,\ a(u,v) = \int_0^1 u'(x)v'(x)dx$$

where S consists of piecewise linears and Q denotes the quadrature approximation based on the trapezoidal rule :

$$Q(w) = \sum_{i=0}^{n} \frac{h_i + h_{i+1}}{2} w(x_i).$$

We further define $h_0 = h_{n+1} = 0$ for simplicity of notation.

Answer. The relation (2) can be re-written as :

$$\mathbf{K}U = \mathbf{F},\ \mathbf{F} = \left(\frac{h_i + h_{i+1}}{2} f(x_i)\right)_{1 \le i \le n-1},\ U = (U_i)_{1 \le i \le n-1}$$

If we write v as a linear combination of basis elements of S : $v = \sum_i V_i\phi_i$ with $V_i = v(x_i)$ and denote $V = (v(x_i))_{1 \le i \le n-1}$ we see that by linearity of a w.r.t. al components we have :

$$\begin{aligned} a(\tilde{u}_S, v) &= a(\sum_i U_i\phi_i, \sum_j V_j\phi_j) = \sum_i\sum_j a(\phi_j,\phi_i)U_iV_j = (\mathbf{K}U, V) \\ &= (\mathbf{F}U, V) = \sum_{i=0}^{n} \frac{h_i + h_{i+1}}{2} f(x_i)v(x_i) = Q(fv). \end{aligned}$$

The difference method is thus equivalent to a piecewise polynomial approximation where the right hand side is approximated with a trapezoidal rule.

Exercise 8.
Let Q be give by the previous exercise. Prove that :

$$\left|Q(w) - \int_0^1 w(x)dx\right| \le Ch^2 \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |w''(x)|dx \qquad (3)$$

Hint : Observe that the trapezoidal rule is exact for piecewise linears and then use exercise 5.

Answer. Let $w_I \in S$ be the piecewise linear interpolant of w. We have that the trapezoidal rule is exact for w_I and since $w_I(x_i) = w(x_i)$ we have :

$$\int_0^1 w_I(x)dx = Q(w_I) = Q(w).$$

If we denote by $e = w - w_I$ the equation (3) becomes :

$$\left|\int_0^1 e(x)dx\right| \leq Ch^2 \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |e''(x)|dx$$

By using the *homogeneity argument* it is enough to prove that :

$$\left|\int_{x_{i-1}}^{x_i} e(x)dx\right| \leq C(x_i - x_{i-1})^2 \int_{x_{i-1}}^{x_i} |e''(x)|dx \Leftrightarrow \left|\int_0^1 \tilde{e}(t)dt\right| \leq C \int_0^1 |\tilde{e}''(t)|dt$$

where $\tilde{e}(t) = e(x_{i-1} + t(x_i - x_{i-1}))$. To simplify the notations we denote $w = \tilde{e}$ and we see that $w(0) = w(1) = 0$ and by Rolle's theorem there exist ξ such that $w(\xi) = 0$. We further obtain that :

$$\begin{aligned}
\left|\int_0^1 w(x)dx\right| &= \left|\int_0^1 \int_0^x w'(t)dtdx\right| = \left|\int_0^1 \int_0^x \int_\xi^t w''(\tau)d\tau dtdx\right| \leq \int_0^1 \int_0^x \left|\int_\xi^t w''(\tau)d\tau\right| dtdx \\
&\leq \int_0^1 \int_0^1 \left|\int_\xi^t w''(\tau)d\tau\right| dtdx = \int_0^1 \left|\int_\xi^t w''(\tau)d\tau\right| dt \\
&\leq \int_0^\xi \int_t^\xi |w''(\tau)|d\tau dt + \int_\xi^1 \int_\xi^t |w''(\tau)|d\tau dt \leq \int_0^\xi \int_0^\xi |w''(\tau)|d\tau dt + \int_\xi^1 \int_\xi^1 |w''(\tau)|d\tau dt \\
&\leq \xi \int_0^\xi |w''(\tau)|d\tau + (1-\xi) \int_\xi^1 |w''(\tau)|d\tau \leq \max\{\xi, 1-\xi\} \int_0^1 |w''(x)|dx
\end{aligned}$$

The constant is then given by : $C = \max\{\xi, 1-\xi\}$.

Exercise 9.
Let u_S the solution of $a(u_S, v) = (f, v), \forall v \in S$, where S consists of piecewise linears and let $\tilde{u}_S$ be as in exercise 7. Prove that :

$$|a(u_S - \tilde{u}_S, v)| \leq Ch^2(\|f'\| + \|f''\|)(\|v\| + \|v'\|) \tag{4}$$

Hint : Apply exercise 8 and Schwarz' inequality.

Answer. By applying exercises 7 and 8 we get :

$$\begin{aligned}
|a(\tilde{u}_S - u_S, v)| &= |Q(fv) - (f, v)| = \left|Q(fv) - \int_0^1 (fv)(x)\right| \leq Ch^2 \int_0^1 |(fv)''(x)|dx \\
&= Ch^2 \int_0^1 |f''(x)v(x) + 2f'(x)v'(x)|dx
\end{aligned}$$

By applying Schwarz's inequality we further obtain :

$$\begin{aligned}
\int_0^1 |f''(x)v(x) + 2f'(x)v'(x)|dx &\leq \int_0^1 |f''(x) \cdot v(x)|dx + \int_0^1 |f'(x) \cdot v'(x)|dx + \int_0^1 |1 \cdot f'(x)v'(x)|dx \\
&\leq \|f''\|\|v\| + \|f'\|\|v'\| + \|f'v'\| \leq \|f''\|\|v\| + \|f'\|\|v'\| + C\|(f'v')'\| \\
&= \max\{1, C\}(\|f''\|\|v\| + \|f'\|\|v'\| + \|f''v'\|) \\
&\leq \max\{1, C\}(\|f''\|\|v\| + \|f'\|\|v'\| + \|f''\|\|v'\| + \|f'\|\|v\|) \\
&\leq C(\|f'\| + \|f''\|)(\|v\| + \|v'\|)
\end{aligned}$$

Exercise 10.
Let u_S and $\tilde{u}_S$ be like in the exercise 9. Prove that :

$$\|u_S - \tilde{u}_S\|_E \leq Ch^2(\|f'\| + \|f''\|)$$

`Hint` : Apply exercise 9, pick $v = u_S - \tilde{u}_S$ and apply exercise 6.

Answer. We plug $v = u_S - \tilde{u}_S$ into (4) and we get :

$$\|u_S - \tilde{u}_S\|_E^2 = a(u_S - \tilde{u}_S, u_S - \tilde{u}_S) \leq Ch^2(\|f'\| + \|f''\|)(\|u_S - \tilde{u}_S\| + \|u'_S - \tilde{u}'_S\|)$$

The coercivity of a (the application of the exercise 6) gives :

$$\|u_S - \tilde{u}_S\| \leq C\|u_S - \tilde{u}_S\|_E \text{ and } \|u'_S - \tilde{u}'_S\| \leq C\|u_S - \tilde{u}_S\|_E$$

and the conclusion follows directly.