A kinetic description of particle fragmentation*

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Abstract

This paper is concerned with the formulation and analysis of particle fragmentation by a mathematical approach of the kinetic theory. We consider a fairly general model which may require a description of the internal configuration of each particle, like internal energy. The fragmentation process is supposed to occur due to the configuration of the corresponding particle; An easy modification would allow to consider the interaction with some external medium (typically a fluid) but we do not deal here with fragmentation processes induced by particles collision. The proposed model is therefore linear and may be analyzed with the use of correct entropies.

Key words: Fragmentation kernels, kinetic equations, Boltzmann equation, coagulation kernels, sprays, polymers.

1 Introduction and setting of the problem

The aim of this paper is to propose a framework to understand and analyze particle fragmentation in a kinetic context. The idea to describe fragmentation in this setting is based on including the mass and the internal energy of

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the particles as additional variables in the distribution function which makes the modeling of the fragmentation kernel clearer in the equation. We analyze this model (the properties of the fragmentation kernel) in the framework of a $L\log L$ theory.

The evolution of complex fluids consisting of a gas interacting with droplets or solid particles is an interesting problem with a wide range of applications in astrophysics, biology, chemistry or meteorology from which we mention here the production of diesel motors, sprays and polymers, for example. The particles (or droplets) inmersed into the fluid could collide and this may produce fragmentation or coagulation between them, which modifies the density or the velocity of the surrounding fluid and thus affects again the evolution of the particles. This problem is usually studied through a two-phase (Eulerian–Eulerian) description of fluids or through a fluid-kinetic description. In this paper we follow the second approach and focus our effort in the analysis and understanding of the kinetic description of the fluid-kinetic coupling. In this sense, we consider here the kinetic (particle) part of the model as isolated of the fluid, so that this can be also viewed as a weak coupling limit model. In a forthcoming paper we will focus our attention on introducing and analyzing some models of interaction with the surrounding fluid.

Before proceeding to the description and analysis of our model, let us briefly summarize the different approaches to this problem studied in the literature and which are relevant in our context. An important direction is the study of the time evolution of the average concentration of particles of a given size in some spatially homogeneous physical system described by the Smoluchowski-type equations, see [2, 10, 28] for a stochastic point of view. On the other hand, a deterministic point of view for the Smoluchowski diffusive models with coagulation-fragmentation kernels has been recently analyzed in [16] for solutions in L^1. The connection betwen the deterministic discrete and the continuous coagulation-fragmentation models (connected to the scale of observation and the degree of saturation of the mixtures) has been investigated in [8, 17, 18]. An introduction to the mathematical investigation of coagulation and fragmentation was first given in [9] (see also [4] or [21]).

In the fluid-kinetic model the density function associated with the particles satisfies a Boltzmann-type equation

$$\frac{\partial f}{\partial t} + (v \cdot \nabla_x) f + \nabla_v \cdot (F f) = Q(f), \tag{1.1}$$

where F is the force per unit mass, which is essentially due to the interaction with the fluid, and $Q(f)$ is the kernel in which collisions, coagulation or fragmentation phenomena must be included. The distribution function f depends on the time $t \in \mathbb{R}_+$, the position $x \in \mathbb{R}^N$, the velocity $v \in \mathbb{R}^N$ and on other variables to be specified such as the diameter of the particles. There are different ways in the literature to determine $Q(f)$ in terms of the features of the problem: kinetic or fluid approximations.

In [12], a general expression of the kernel $Q(f)$ is given. Also, we refer the reader to [3] for a description of the T.A.B. (Taylor Analogy Breakup) model deduced from Taylor's arguments [26]. In this T.A.B. model the fragmentation is due to the increase of the amplitude of the oscillations on the surface of the particles. Once a particle reaches a critical size it breaks up. The model is founded on the analogy between the oscillations on the surface of the particles and the linear instability of harmonic oscillators. However, the T.A.B. model fits experiments only for low fragmentation frequencies, which is in fact related to the turbulent character of the surrounding fluid.

Another interesting approach to determine $Q(f)$ is founded on statistical models based on energy principles. This approach describes the transient evolution of the (particle) bubble-size probability density functions resulting from the break-up of the bubble moving in a turbulent fluid (see the excellent review [15]). In this model it is assumed that the bubble size depends on its diameter, but also on the value of the dissipation rate of the surrounding fluid. Integrating (1.1) in velocity, one obtains the following equation in terms of the number of particles $n = n(t, x, D)$ with size D, located at the position x at time t (see [29]):

$$\frac{\partial n}{\partial t} + \nabla_x \cdot (\bar{v} n) = Q_b(f), \tag{1.2}$$

where $\bar{v}$ is the mean velocity of all particles of size D at a location x at time t. The fragmentation kernel is usually modeled by

$$\begin{aligned} Q_b(t, x, D) = \int_D^\infty \alpha(t, x, D')F(t, x, D, D')g(t, x, D')n(t, x, D')\, dD' \\ - g(t, x, D)n(t, x, D), \end{aligned} \tag{1.3}$$

where $g(D)$ is the fragmentation frequency of particles of size D, $\alpha(D')$ is the mean number of particles resulting from the fragmentation of a mother particle of size D' and $F(t, x, D, D')$ is the size distribution of daughter particles formed from the breakage of a mother particle of size D'. The first term of the kernel accounts for the rate of formation of particles of size D from the fragmentation of particles of diameter larger than D. The second term accounts for the rate of fragmentation of particles of diameter D.

An equivalent model is usually used for the bilinear (with respect to the distribution function) coalescence kernel (see [15, 13]), that is omitted here because we focus our attention in this paper on the fragmentation problem. Also, we have omitted in the above description the effects of evaporation or dissolution. Various models have been used to determine $g(t, x, D)$ and $F(t, x, D, D')$, see [15].

Finally, we refer to [7] where a Fokker-Planck-type kernel is considered to model $Q(f)$.

In order to complete the fluid-kinetic description, equation (1.1) should eventually be coupled with the fluid balance laws for the mass, momentum and energy with source terms derived from the interaction between the gas and the particles, see for example [12]. A part of the energy goes to produce fragmentation in the particles.

In this paper, we propose a model to understand the fragmentation process through the evolution of the internal energy of the particles. As we have commented before, we focus our attention here only on the fragmentation process and on the modeling and analysis of the resulting kinetic equation. Our model assumes that the distribution function depends, in addition to the time t, the position x and the velocity v, on the mass and the internal energy. The dependence on the mass is related to the size of the particle while the dependence on the internal energy collects the different effects that could produce fragmentation. In general, the relation between the internal energy, the structure of the particles and the coupling with the turbulent bath provides a crucial information about the mechanism of fragmentation, although to model this is not immediate. This dependence on the internal energy is something more than the dependence on the diameter which is in some sense included in the mass variable. In the previous examples, the internal energy was a function of the surface tension energy (disturbance and oscillations of the drop which depend directly on the velocity of the surrounding fluid) in the T.A.B. model or a function of the turbulent factor which affects the fragmentation frequency in the statistical model described above. Then, in the previous models the fragmentation principles come from the turbulent surrounding fluid. Note that is is often needed to add internal variables in kinetic equations (in applications to biology for instance, see [6]).

The paper is structured as follows: in Section 2 we analyze the principles and conservation laws constituting our model from the discrete interaction. In fact, we establish the necessary conditions on the fragmentation kernel to have mass, energy and momentum conservation without the influence of a surrounding fluid. We will also analyze the conditions in the case of non conservation of momentum, which is the natural case when we deal with the fluid coupling. Section 3 is devoted to prove the well-posedness of our model as well as the main results of this paper based on a $L\log L$ theory. In fact, we first prove an existence and stability result in the context of measure solutions. To avoid concentrations of particles of L^1 solutions we use a combination of nonlinear entropy estimates together with a control on the decay of the solutions for large mass, energy or momentum, which at the same time ensures the absence of particles with infinite velocities. As we will see in this section, adding a variable energy has the consequence of making easier and explicit the analysis and modeling of the fragmentation process. Finally, we explain how one may apply our general framework to study fragmentation models to an example introduce by Hylkema and Villedieu in [11, 12] (see also [25]).

2 Description of the model

We present in this section a general model which takes only into account the fragmentation itself and not a possible coupling with a fluid, for example. We indicate in the last section how such a coupling can easily be obtained from this model.

Consider a particle with mass m' and momentum p'. We will also need another parameter which describes the shape and the possible deformation of the particle. The deformation parameter could typically be a scalar or a vector, the exact choice of which would imply to parametrize the possible forms of the particle, which obviously depends a lot on the exact situation and goes beyond the scope of this paper. Here we simply take for this parameter the internal energy e' of the particle (in general, the internal energy would be a function of the deformation and possibly the mass).

The particle undergoes a binary fragmentation process giving birth to a particle with mass m, momentum p and deformation e and another daughter particle with parameters m^*, p^* and e^*. This process has to conserve total mass and total energy, or at least not increase the total energy if one does not want to rule out the possibility of energy losses. The simplest model would also conserve total momentum, but that is not strictly necessary.

Consequently, the mass m has to be strictly lower than m', with the following relations for m^*, and p^*

$$m^* = m' - m, \quad p^* = P(m', m, p', p). \tag{2.1}$$

If one wants to impose the preservation of total momentum, then $P(p, p') = p' - p$ but in general the following conditions are enough

$$\begin{aligned} &P \in C^\infty(\mathbb{R}^{2N}), \quad P(m', m' - m, p', P(m', m, p', p)) = p, \\ &\frac{P(p', p)}{m' - m} = \frac{p'}{m'} \quad \text{when} \quad \frac{p'}{m'} = \frac{p}{m}. \end{aligned} \tag{2.2}$$

The second condition makes sure that the process is really symmetric and it has for consequence that the jacobian of the transformation $(m, p) \mapsto (m' - m, P(m', m, p', p))$ is exactly 1. The last condition only means that when the velocity of one daughter particle is equal to the velocity of the mother particle then so is the case for the other daughter particle, which is very natural.

The internal energy e^* has to be nonnegative and, in case of energy conservation, it is given by

$$\frac{|p'|^2}{2m'} + e' = \frac{|p|^2}{2m} + \frac{|P(m', m, p', p)|^2}{2(m - m')} + e + e^*.$$

This induces another condition on P. When $P = p' - p$, fragmentation creates kinetic energy which has to be taken from the internal energy. We need to impose a condition to preserve this feature in case of a general P, namely

$$\begin{aligned} E &= \tfrac{|p|^2}{2m} + \tfrac{|P(m',m,p',p)|^2}{2(m-m')} - \tfrac{|p'|^2}{2m'} \\ &\geq c\, \tfrac{|m'\,p - m\,p'|^2}{m'\,m\,(m'-m)}, \quad \text{if} \;\; m \leq m', \end{aligned} \tag{2.3}$$

for a given constant $c \geq 0$. Note that $c = 1$ if $P = p' - p$.

Of course, the process is fully symmetric (the probability of break-up is exactly the same if one swaps the parameters of the two daughter particles).

The total probability of break-up for one particle should decrease as its mass or energy is lower (small particles cannot afford to divide themselves).

Eventually, a particle with a very large velocity should not be created even if its mass is very low. It is possible that the velocity of a daughter particle is different from the one of the mother particle, especially due to the influence of a fluid, but then it is typically limited by the velocity of the fluid.

In order to reproduce in our continuous model the features of the relations between the particles previously described at the discrete level, let us assume the following general properties for the interaction kernel $\mathcal{B}_2$. This kernel describes the probability that a mother particle withh mass m', momentum p' and energy e' breaks up to create a daughter particle with parameters $m,\ p,\ e$ (the parameters of the other daughter particle being computed according the rules above).

- For symmetry reasons, $\mathcal{B}_2(m', p', e', m, p, e)$ satisfies

$$\mathcal{B}_2(m', p', e', m, p, e) \;=\; \mathcal{B}_2(m', p', e', m' - m, P(p', p), e' - e - E). \tag{2.4}$$

- The kernel is also assumed to satisfy the following bounds for some superlinear function Φ and a given number $\alpha > 1$

$$\begin{aligned}\int_0^{m'} \int_{\mathbb{R}^N} \int_o^{e'-E} \left(1 + m + \left(\frac{|p|}{m}\right)^{\alpha}\right) \mathcal{B}_2(m', p', e', m, p, e)\, d(e, p, m) \\ \leq\; C\Phi\left(m',\ \frac{|p'|^2}{2m'} + e'\right),\end{aligned} \tag{2.5}$$

$$\mathcal{B}_2(m', p', e', m, p, e) \leq\; C\Phi\left(m',\ \frac{|p'|^2}{2m'} + e'\right). \tag{2.6}$$

These bounds ensure that the probability of break-up is limited by a "reasonable" increasing function of the total mass and of the total energy. They also make sure that a daughter particle with an infinite velocity p/m cannot be created. We recall that a superlinear function $\Phi(a, b)$ satisfies

$$\Phi(a + a', b + b') \geq \Phi(a, b) + \Phi(a', b').$$

Note that it would be possible to consider a bound with $|p/m - p'/m'|$ instead of only $|p/m|$ without much change in what follows. This would make more sense with respect to Galilean invariance. However, since these models are then typically coupled with a fluid, the order of magnitude of the velocity of the particles is fixed by the fluid and the form chosen here is not irrelevant.

Let $f(t, x, m, p, e) \geq 0$ the distribution function associated with the particles which depends on the time $t \in \mathbb{R}_+$, the position $x \in \mathbb{R}^N$, the moment

$p \in \mathbb{R}^N$, the internal energy $e \in \mathbb{R}_+$ and the mass $m \in \mathbb{R}_+$ and satisfies

$$
\begin{aligned}
\frac{\partial f}{\partial t} + \frac{p}{m} \cdot \nabla_x f \;&=\; -\frac{1}{2} f(t,x,m,p,e)\mathcal{B}_1(m,p,e) \\
&+ \int_m^\infty \int_{\mathbb{R}^N} \int_{e+E}^\infty f(t,x,m',p',e')\mathcal{B}_2(m',p',e',m,p,e)\, d(m',p',e'),
\end{aligned}
\tag{2.7}
$$

where

$$
\mathcal{B}_1(m',p',e') = \int_0^{m'} \int_{\mathbb{R}^N} \int_0^{e'-E} \mathcal{B}_2(m',p',e',m,p,e)\, d(m,p,e). \tag{2.8}
$$

Eq. (2.7) is known as the Bachelier–Chapman-Kolmogoroff–Schmoluchovski equation. When the coupling with the fluid is considered two new terms concerning the force per unit mass for the evolution of the velocity and the law for the evolution of the internal energy must appear in the left–hand side of (2.7).

Let us analyze the conservation laws satisfied by this kind of model. Denote $Q = \{q = (m,p,e) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+\}$ and Q' the same set when the variables $q' = (m',p',e')$ are involved.

Concerning mass conservation, by applying the Fubini theorem we can write (at least formally)

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^3}\int_Q m\, f(t,x,q)\, dx\, dq = &-\frac{1}{2}\int_{\mathbb{R}^3}\int_Q m\, f(t,x,q)\mathcal{B}_1(q)\, dx\, dq \\
&+ \int_{\mathbb{R}^3}\int_{Q'}\int_0^{m'}\int_{\mathbb{R}^3}\int_0^{e'-E} m\, f(t,x,q')\mathcal{B}_2(q',q)\, dx\, dq\, dq'.
\end{aligned}
\tag{2.9}
$$

Applying the Fubini theorem again, taking into account (2.8) and rearranging the terms in (2.9) we find

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}\int_Q m\, f(t,x,q)\, dx\, dq \\
&\quad = \frac{1}{2}\int_{\mathbb{R}^3}\int_{Q'}\int_0^{m'}\int_{\mathbb{R}^3}\int_0^{e'-E} (m+m^*)\, f(t,x,q')\mathcal{B}_2(q',q)\, dq\, dq'\, dx \\
&\qquad - \int_{\mathbb{R}^3}\int_{Q'}\int_0^{m'}\int_{\mathbb{R}^3}\int_0^{e'-E} m\, f(t,x,q')\mathcal{B}_2(q',q)\, dq\, dq'\, dx,
\end{aligned}
\tag{2.10}
$$

where $m^* = m' - m$, and for the other variables $q^* = (m^*, p^*, e^*)$, with $p^* = P(p',p)$ and $e^* = e' - e - E$, the $*$ index representing the other particle created by the break-up. Since the relation (2.4) implies that $\mathcal{B}_2(q', q) = \mathcal{B}_2(q', q^*)$ and the change of variable from q to q^* preserves the volume by (2.2), equation (2.10) means that the mass is preserved along the time evolution:

$$\frac{d}{dt} \int_{\mathbb{R}^3} \int_Q m\, f(t, x, q)\, dx\, dq \;=\; 0. \tag{2.11}$$

In a similar way we can deduce that the momentum is also preserved, provided $P(p', p) = p' - p$:

$$\frac{d}{dt} \int_{\mathbb{R}^3} \int_Q p\, f(t, x, m, p, e)\, dx\, d(m, p, e) \;=\; 0, \tag{2.12}$$

as well as the energy (kinetic plus internal energy) is:

$$\frac{d}{dt} \int_{\mathbb{R}^3} \int_Q (\frac{p^2}{2m} + e)\, f(t, x, m, p, e)\, dx\, d(m, p, e) \;=\; 0. \tag{2.13}$$

3 Existence results

3.1 Statement of the results

The aim of this section is to prove the existence of solutions (in a suitable sense) to Equation (2.7) together with the structure introduced in the previous section. We use weak solutions (precisely defined below) and we get two results, one for solutions which are measures and another one where we prove that mass concentration cannot occur if there was none initially.

Equation (2.7) for the distribution function is linear, hence there is no real difficulty with the regularity of the solution and we may even work with measures. However, if the function $\Phi(a, b)$ in the bounds for $\mathcal{B}_2$ is increasing faster than $a + b$, it is necessary to have more than only the boundedness of mass, momentum and energy. Therefore, we demand that the initial data f^0 satisfy

$$\int_{\mathbb{R}^3} \int_Q \tilde{\Phi}\left(m,\, \frac{p^2}{2m} + e\right)\, f^0(x, q)\, dq\, dx \;<\; \infty, \tag{3.1}$$

for some superlinear function $\tilde{\Phi}(a, b)$ with $\tilde{\Phi}(a, b)/\Phi(a, b) \to +\infty$ as $a + b \to +\infty$. We point out that if f^0 is a measure which is not absolutely continuous with respect to the Lebesgue measure, the previous integral has of course to be taken as the integral against the measure itself.

Another very natural assumption which we need is that initially there are no infinite velocities, or more precisely that for some exponent $\alpha' > 1$

$$\int_{\mathbb{R}^3} \int_Q \left(\frac{|p|}{m} \right)^{\alpha'} f^0(x,q)\, dq\, dx \ < \ \infty. \tag{3.2}$$

The exponent α' is not necessarily related to α. However if $\alpha' < \alpha$ then the kernel $\mathcal{B}_2$ also satisfies the estimate (2.5) with α' instead of α. Then, if $\alpha' > \alpha$, f^0 satisfies the estimate (3.2) for α. Therefore, we are free to assume that $\alpha = \alpha'$.

Now we may define the notion of solution we use. Assume that a measure $df(t,.,.)$ satisfies for some $T > 0$

$$\int_{\mathbb{R}^3 \times Q} \left(1 + \frac{|p|^\alpha}{m^\alpha} + \tilde{\Phi}\left(m, \frac{p^2}{2m} + e \right) \right) df(t,x,q) \ \in \ L^\infty([0,\ T]). \tag{3.3}$$

Then f (or df) is a weak solution to the equation (2.7) with initial data f^0 (or df^0) satisfying (3.1) and (3.2) if and only if for any C^1 function ψ of the variables t, x, m, p and e with compact support in $[-T,\ T] \times \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+$, the following identity is fulfilled:

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3 \times Q} \Big(-\partial_t \psi(t,x,q) - \frac{p}{m} \cdot \nabla_x \psi(t,x,q) \Big)\ df(t,x,q) = \\ &\qquad \int_{\mathbb{R}^3 \times Q} \psi(0,x,q)\, df^0(x,q) - \frac{1}{2} \int_0^T \int_{\mathbb{R}^3 \times Q} \mathcal{B}_1(q)\, \psi(t,x,q)\, df(t,x,q) \\ &\qquad + \int_0^T \int_{\mathbb{R}^3 \times Q} \psi(t,x,q) \times \int_m^\infty \int_{\mathbb{R}^N} \int_{e+E}^\infty \mathcal{B}_2(q',q)\, df(t,x,q')\, dq\, dt. \end{aligned} \tag{3.4}$$

This definition makes sense since the bounds (3.3) and (2.6) together ensure that $\mathcal{B}_2(q',q)\, df(t,x,q')$ is indeed integrable in x, q'; then, the definition (2.8) and the bounds (2.5) and (3.3) show that $\mathcal{B}_1\, df$ is also integrable.

Our first result therefore reads

Theorem 3.1 *Assume that $f^0 \in M^1(\mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+)$ is a nonnegative measure satisfying the estimates (3.1) and (3.2). Then, there exists a non negative measure f in $L^\infty([0,\ T],\ M^1(\mathbb{R}^3 \times Q))$ for all $T > 0$, satisfying (3.3), which solves equation (2.7) in the sense of* (3.4).

Remark. The solution is obtained as a limit of classical solutions to (2.7) which exist at least for a short time interval.

In this setting, even if the initial data is initially in L^1, it does not necessarily remain so for all positive times. A natural question is therefore whether it is possible to avoid concentrations of particles.

A first attempt in this direction would be to impose nonlinear estimates on f^0 like an entropy $\int f^0 \log f^0 < \infty$ and show that they are preserved in time. So first, it is important to note that the entropy is in fact $\int m\, f^0 \log f^0$. Indeed, what counts here is more the mass of the particles than their number. Technically (in the computations), it is also clear that the right quantity is $\int m\, f \log f$ and not $\int f \log f$.

It may seem strange on a closer look to consider the entropy: the problem is linear, the entropy is neither decreasing nor even conserved and there is no clear and self-imposing reason to use it. Actually, we might have used other quantities like L^p norms of $m^k f$ for instance and the same kind of computations would work for these norms. However, the required assumption to prove estimates for the entropy is more simple and the following computation more natural, although this is more a consequence of the mathematical properties of the logarithm than of any real physical considerations.

Therefore, we assume that

$$\int_{\mathbb{R}^3 \times Q} m\, f^0(x,q) \log f^0(x,q)\, d(q,x) < \infty. \tag{3.5}$$

We also assume (3.1) but with a more demanding condition on $\tilde{\Phi}$, namely

$$\frac{\tilde{\Phi}(a,b)}{\log (a+b)\, \Phi(a,b) \log \Phi(a,b)} \longrightarrow +\infty, \quad \text{as } a+b \to \infty. \tag{3.6}$$

We now have the following boundedness property for the entropy

Theorem 3.2 *Consider a solution f to (2.7), obtained as a limit of classical solutions, satisfying the hypotheses of Theorem 3.1. If in addition f^0 satisfies (3.5) and (3.1) with (3.6), then f belongs to $L^\infty([0,\ T],\ L^1(\mathbb{R}^3 \times Q))$ for all $T > 0$ and*

$$\int_{\mathbb{R}^3 \times Q} m\, f(t,x,q) \log f(t,x,q)\, d(q,x) \ \in\ L^\infty([0,\ T]). \tag{3.7}$$

Remark. It is possible to have a corresponding result for $\|m^k\, f\|_{L^p}$ instead of the entropy by suitably modifying the condition (3.6). In our context we cannot ensure the uniqueness of solutions. We also note that some a priori estimates in our results can be considered as extensions of the Smoluchovski fragmentation problem without spatial dependence.

3.2 Proof of Theorem 3.1

Obtaining classical solutions to equation (2.7) in a short time interval (a time interval depending on the size of the initial data) is extremely simple and we simply admit it. Therefore, Theorem 3.1 is as usual a direct consequence of a stability result which, in this case, reads

Proposition 3.1 *Consider a sequence $f_n \in C^1([0,\ T]\times\mathbb{R}^3\times Q)$, uniformly bounded in $L^\infty([0,\ T], M^1(\mathbb{R}^3\times Q))$, satisfying the estimate (3.3), of solutions to equation (2.7) in the weak sense (3.4). Assume that the corresponding initial data f_n^0 satisfy the estimates (3.1) and (3.2) uniformly in n. Then, any weak* limit f of f_n in $L^\infty([0,\ T], M^1(\mathbb{R}^3\times Q))$ satisfies (3.3) and is a solution to (2.7).*

Proof. Up to the extraction of a subsequence, we may assume that the sequence f_n converges toward f weakly in $L^\infty([0,\ T], M^1(\mathbb{R}^3\times Q))$. The main point in the proof is to show that all f_n satisfy the estimate (3.3) uniformly in n.

Let us first compute, for any positive function ϕ,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3\times Q}\phi\left(m,\frac{p^2}{2m}+e\right) f_n(t,x,q)\,d(q,x) = \\
&\qquad -\frac{1}{2}\int_{\mathbb{R}^3\times Q}\phi\left(m,\frac{p^2}{2m}+e\right) f_n(t,x,q)\mathcal{B}_1(q)\,d(q,x)\\
&\qquad +\int_{\mathbb{R}^3\times Q'}\int_0^{m'}\int_{\mathbb{R}^3}\int_0^{e'-E}\phi\left(m,\frac{p^2}{2m}+e\right) f_n(t,x,q')\mathcal{B}_2(q',q)\,dq\,d(q',x).
\end{aligned}
$$

Using formula (2.8) for $\mathcal{B}_1$ and introducing the $*$ variables as in the conservation of mass, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3\times Q}\phi\left(m,\frac{p^2}{2m}+e\right) f_n(t,x,q)\,d(q,x) = \\
&=\frac{1}{2}\int_{\mathbb{R}^3\times Q'}\int_0^{m'}\int_{\mathbb{R}^3}\int_0^{e'-E}\left(\phi\left(m,\frac{p^2}{2m}+e\right)+\phi\left(m^*,\frac{|p^*|^2}{2m^*}\right)-\phi\left(m',\frac{|p'|^2}{2m'}\right)\right)\\
&\qquad\qquad \times f_n(t,x,q')\mathcal{B}_2(q',q)\,dq\,d(q',x).
\end{aligned}
$$

Hence, if ϕ is superlinear, then

$$
\frac{d}{dt}\int_{\mathbb{R}^3\times Q}\phi\left(m,\frac{p^2}{2m}+e\right) f_n(t,x,q)\,d(q,x)\le 0. \tag{3.8}
$$

Consequently, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3\times Q} \tilde{\Phi}\left(m, \frac{p^2}{2m}+e\right) f_n(t,x,q)\, d(q,x) \\
&\qquad\qquad \leq \int_{\mathbb{R}^3\times Q} \tilde{\Phi}\left(m, \frac{p^2}{2m}+e\right) f_n^0(x,q)\, d(q,x) \leq C.
\end{aligned}
\tag{3.9}
$$

We may now prove the two other bounds. Applying the Fubini theorem and then the estimate (2.5), we find

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3\times Q} \left(1+\frac{|p|^\alpha}{m^\alpha}\right) f_n(t,x,q)\, d(q,x) \\
&\qquad \leq \int_{\mathbb{R}^3\times Q'} \int_0^{m'} \int_{\mathbb{R}^3} \int_0^{e'-E} \left(1+\frac{|p|^\alpha}{m^\alpha}\right) f_n(t,x,q')\mathcal{B}_2(q',q)\, dq\, d(q',x) \\
&\qquad \leq C \int_{\mathbb{R}^3\times Q'} \Phi\left(m', \frac{|p'|^2}{2m}+e'\right) f_n(t,x,q')\, d(q',x).
\end{aligned}
$$

The bound (3.9) implies that the sequence f_n satisfies (3.3) uniformly in n.

Since this estimate is linear in f_n, we know that it is also satisfied by$f$. It is now easy to check that we can pass to the limit in every term of equation (2.7). Notice that, to do so, we need a uniform control for each term, hence the conditions $\alpha > 1$ and $\tilde{\Phi}(a,b)/\Phi(a,b) \to +\infty$ as $a+b \to \infty$. $\square$

3.3 Proof of Theorem 3.2

We may assume that f is C^1. If we prove that the estimate (3.7) holds in terms of only (3.3) and (3.5), the theorem will immediately follow.

First of all note that, from Theorem 3.1, the estimate (3.3) where $\tilde{\Phi}$ satisfies (3.6) is still true, hence

$$
\int_{\mathbb{R}^3\times Q} \tilde{\Phi}\left(m, \frac{p^2}{2m}+e\right) f(t,x,q)\, d(q,x) \;\in\; L^\infty([0,\ T]). \tag{3.10}
$$

Let us then compute

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3\times Q} m\, f(t,x,q)\log f(t,x,q)\, d(q,x)\\
&\quad = -\frac{1}{2}\int_{\mathbb{R}^3\times Q}\mathcal{B}_1(q) m f(t,x,q)(1+\log f(t,x,q))\, d(q,x)\\
&\quad + \int_{\mathbb{R}^3\times Q'}\int_0^{m'}\int_{\mathbb{R}^3}\int_0^{e'-E}\mathcal{B}_2(q',q) m f(t,x,q')\\
&\qquad \times (1+\log f(t,x,q))\, dq\, d(q',x).
\end{aligned} \tag{3.11}$$

Because of the hypothesis (2.6), there exists an increasing function $\psi(q')$ of the form

$$\psi\left(m', \frac{|p'|^2}{m'} + e'\right),$$

such that

$$\sup_{m,p,e}\int_m^\infty\int_{\mathbb{R}^3}\int_{e+E}^\infty \frac{\mathcal{B}_2(q',q)}{\psi(q')}\, dq' \le C. \tag{3.12}$$

This function ψ may be chosen as a polynomial function times $\Phi(q')$. We emphasize that such a function always exists thanks to condition (2.3), which states that E is positive in all circumstances and which is needed only here.

Now we bound the second term in the right-hand side of (3.11) as follows

$$\begin{aligned}
&\int_{\mathbb{R}^3\times Q'}\int_0^{m'}\int_{\mathbb{R}^3}\int_0^{e'-E}\mathcal{B}_2(q',q) m f(t,x,q')(1+\log f(t,x,q))\, dq\, d(q',x)\\
&= \int_{\mathbb{R}^3\times Q'}\int_0^{m'}\int_{\mathbb{R}^3}\int_0^{e'-E}\frac{\mathcal{B}_2(q',q)}{\psi(q')} m\psi(q') f(t,x,q')(1+\log f(t,x,q))\, dq\, d(q',x).
\end{aligned}$$

We apply the following duality inequality to the two terms $a = \psi(q')f(t,x,q')$ and $b = 1+\log f(t,x,q)$:

$$a\cdot b \le a(1+\log a) + b e^{b-1},$$

and obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3\times Q'}\int_0^{m'}\int_{\mathbb{R}^3}\int_0^{e'-E}\mathcal{B}_2(q',q) m f(t,x,q')(1+\log f(t,x,q))\, dq\, d(q',x)\\
&\le \int_{\mathbb{R}^3\times Q'}\int_0^{m'}\int_{\mathbb{R}^3}\int_0^{e'-E}\frac{\mathcal{B}_2(q',q)}{\psi(q')} m\psi(q') f(t,x,q')\\
&\qquad \times (1+\log f(t,x,q')+\log\psi(q'))\, dq\, d(q',x)\\
&\quad + \int_{\mathbb{R}^3\times Q}\int_m^\infty\int_{\mathbb{R}^3}\int_{e+E}^\infty \frac{\mathcal{B}_2(q',q)}{\psi(q')} m f(t,x,q)(1+\log f(t,x,q))\, dq'\, d(q,x).
\end{aligned}$$

Because of the definition of ψ, we may bound the last term and get

$$\int_{\mathbb{R}^3\times Q'}\int_0^{m'}\int_{\mathbb{R}^3}\int_0^{e'-E}\mathcal{B}_2(q',q)mf(t,x,q')(1+\log f(t,x,q))\,dq\,d(q',x)$$
$$\leq \int_{\mathbb{R}^3\times Q'}\int_0^{m'}\int_{\mathbb{R}^3}\int_0^{e'-E}\mathcal{B}_2(q',q)mf(t,x,q')(1+\log f(t,x,q'))\,dq\,d(q',x)$$
$$+\int_{\mathbb{R}^3\times Q'}\mathcal{B}_2(q',q)mf(t,x,q')\log\psi(q')\,dq\,d(q',x)$$
$$+C\int_{\mathbb{R}^3\times Q}mf(t,x,q)(1+\log f(t,x,q))\,d(q,x).$$

Inserting this into (3.11), we end up with

$$\frac{d}{dt}\int_{\mathbb{R}^3\times Q} m\,f(t,x,q)\log f(t,x,q)\,d(q,x)$$
$$\leq \frac{1}{2}\int_{\mathbb{R}^3\times Q'}\int_0^{m'}\int_{\mathbb{R}^3}\int_0^{e'-E}\mathcal{B}_2(q',q)(m+m^*-m')f(t,x,q')$$
$$\times\,(1+\log f(t,x,q'))\,dq\,d(q',x)$$
$$+C\int_{\mathbb{R}^3\times Q}mf(t,x,q)(1+\log f(t,x,q))\,d(q,x)$$
$$+\int_{\mathbb{R}^3\times Q'}\mathcal{B}_2(q',q)mf(t,x,q')\log\psi(q')\,dq\,d(q',x).$$

The first term in the right-hand side vanishes by the definition of m^*. The last term is bounded because $\log\psi(q')$ is dominated by $\log\Phi$ plus a constant times $\log(1+m'+|p'|^2/2m'+e')$. Therefore, thanks to the estimate (2.5) the term is dominated by the estimate (3.10) (note that it is here that we need the condition (3.6)).

A simple Gronwall lemma now proves the estimate (3.7), which finishes the proof of Theorem 3.2. □

4 Application: The model by Hylkema and Villedieu

In his thesis [11, 12], J. Hylkema proposed a model for the dynamics of bubbles in a turbulent gas. Our purpose in this section is to describe the model, explain how it fits in our framework and how our existence results may be applied to it.

We deal here only with the fragmentation process and forget about all other couplings between the fluid and the particles.

Hence, the velocity field $u(t,x)$ of the fluid is a given data and the model will only determine how a particle may break given a particular velocity. The way to know when and how a particle of size m' and momentum p' breaks down is through its Weber number, which is

$$We(t,x,m',p') = 2\frac{\rho_g}{\sigma}\left|\frac{p'}{m'} - u(t,x)\right|^2 m'^{1/3}, \tag{4.1}$$

where ρ_g is the mass density of the gas and σ the surface tension. When this number is larger than a critical value W_c, the bubble has a probability $1/\tau$ to break, where

$$\tau(t,x,m',p') = C\frac{\sqrt{\rho_p\sigma}}{\rho_g} \times \frac{m'^{1/6}}{(\log W_e)^{1/4}}, \tag{4.2}$$

with ρ_p the mass density of the particles.

The internal energy of a particle is not taken into account by this model and hence the unknown in the equation is a function f of the time t, space x and momentum p variables. This equation is quite close to (2.7) and reads

$$\begin{aligned} \frac{\partial f}{\partial t} \;+\; &\frac{p}{m}\cdot\nabla_x f = -B(t,x,m,p)f(t,x,m,p) \\ &+ \int_0^\infty \int_{\mathbb{R}^N} B(t,x,m',p')H(t,x,m,p,m',p')f(t,x,m',p')\,dp'dm'. \end{aligned} \tag{4.3}$$

The two kernels B and H are defined as follows

$$\begin{aligned} B(t,x,m,p) &= \frac{1}{\tau(t,x,m,p)}\mathbb{I}_{We(t,x,m,p)\geq W_c}, \\ H(t,x,m,p,m',p') &= C\frac{m'^{1/3}}{\bar{m}^{4/3}}e^{-m/\bar{m}} \times \phi\left(\frac{p}{m} - \bar{v}\right), \\ \bar{m} &= C\frac{m'}{We(t,x,m',p')} \times \frac{1}{\sqrt{We - W_c}}, \\ \bar{v} &= u(t,x) + \frac{p'/m' - u}{1 + C(m'/m)^{2/9}}. \end{aligned} \tag{4.4}$$

Notice that in the original model in [12], the function ϕ was a Dirac mass (which is quite reasonable to smooth a bit).

Except for the absence of an internal energy this model satisfies almost all the assumptions we required in Section 2. In particular, assuming that the velocity u of the fluid is bounded, we have

$$B(t,x,m',p') \le C_k(1+m'^{1/6}) \times \left(\frac{p'^2}{m'}\right)^k, \quad \forall k>0,$$

$$\int_0^\infty \int_{\mathbb{R}^N} \left(1+\frac{|p|}{m}\right)^{2/3} H(t,x,m,p,m',p')\, dm\, dp \le Cm'^{1/3} \times \bar{m}^{-1/3} \times |\bar{v}|$$
$$\le C(1+m'^{1/3}) \times \left(1+\frac{p'^2}{m'}\right),$$

$$H(t,x,m,p,m',p') \le Cm'^{1/3} \times \left(1+\frac{p'^2}{m'}\right)^2.$$

However, this model is too simple to satisfy some basic physical assumptions and in particular the size of a daughter particle could be larger than the size of the mother particle, though not in average.

This is not at all a problem from the point of view of the mathematical analysis, and more precisely the results stated in Section 3 are also true for the equation (4.3) with the definitions (4.4). The proof would be exactly the same, so we do not reproduce it here and refer the reader to Section 3 for details.

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