Existence to solutions of a kinetic aerosol model *

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Abstract

Consider small particles of varying size being transported by a fluid. If we allow these particels to coalesce their evolution may be described by the coagulation model

$$f_t + \frac{p}{m}.\nabla_x f = Q(f) \quad .$$

Here f denotes the particle density $f(t, x, m, p)$ of particles with mass $m \in \mathbb{R}^+$, momentum $p \in \mathbb{R}^3$, at time $t > 0$ and position $x \in \mathbb{R}^3$. For a general class of collion operators Q we prove existence of solutions. Under some natural restriction on the initial data we have existence without blowup of the solution.

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1 Introduction

Consider small particles of varying sizes being transported by a fluid. One example is aerosol sprays where tiny bubbles are floating in air; another example is interstellar dust particles which are swept along by hydrogen molecules (modeled as a fluid) in an accretion disc. The interaction between these particles in general may allow for fragmentation or coagulation of particles (like in a Smolouchovski equation) or more complicated interaction; some of this is discussed in the following literature: the model descibed in Baranger [3] includes breakup and oscillations, Hylkema and Villedieu [16] leave out oscillations in their model, Prince and Blanch [23] or Tsouris and Tavlarides [25] descibe the case of bubbles or droplets in flow. For interstellar dynamics or chemically reactive particles, the models may be even more complex (we refer to Illner [17] or Williams [28]).

In this paper we restrict ourselves to the following interaction of particles: assume that these particles may coalesce to form larger particles thus depleting the population of smaller particles by way of increasing the amout of larger size particles. (see Drake [13] for a review of the models, Penrose and Buhagiar [21] where a particle model of coalescence and fragmentation is derived, the so called Becker-Döring equations and Wagner [27] for numerical aspects).

This can be modelled in three regimes: the microscopic, mesoscopic and macroscopic regime. On a microscopic level we consider a finite number of individual particles which move stochastically. The collision of particles leads to the formation of new larger particles (recall we consider only coagulation here) with a certain given rate. In case the number of particles gets very large this approach it is quite costly numerically to handle this model.

The macroscopic regime is probably the most studied up to now. Here the interacting particles are modeled at the contiuum level. Existence results for deterministic models may be found in Amann [2], Collet and Poupaud [9], Dubovskii [14] and finally Laurençot and Mischler [18], [19], [20] for a full answer in the space dependent case. Probabilistic ideas have also been put

to use with success in Aldous [1], Deaconu and Fournier [10] for instance (see also Bertoin [4] and [5] for fragmentation processes). Macroscopic models are easy to write down and the physical identification of the kernels and parameters one can usually readily handel (though it is never trivial); nevertheless they do not take into account the individual velocities of the particles, assuming usually either space independence or a diffusion process for the particles. If a non trivial interaction with a fluid is to be taken into account or if the particles' density is very low and very inhomogeneous, it is necessary to look at the mesoscopic regime.

On a mesoscopic level, the number of microscopically modelled particles has gone to infinity while maintaining constant total mass. We now talk about the particle density $f(t, x, m, p)$ of particles with mass $m \in \mathbb{R}^+$, momentum $p \in \mathbb{R}^3$, at time $t > 0$ and position $x \in \mathbb{R}^3$. The evolution of f is given by a Boltzmann-type equation

$$f_t + \frac{p}{m}.\nabla_x f = Q(f) \tag{1.1}$$

where the collision operator Q is to be specified later. Comparing this with the macroscopic description the latter involves physically measurable variables like mass density, velocity and temperature. The evolution of such macroscopic variables leads to a system of n balance laws ($n \in \mathbb{N}$) which may formally be derived from the mesoscopic description by taking moments and a closure procedure relating the n+1st moment to the first n moments.

We want to address the following question: given a certain class of coagulation processes (to be specified later) and given initially many very small particles, under which circumstances may this flow evolve for some time without leading to a concentration of particles, i.e. without leading to a Dirac delta mass distribution of particles. We proceed to give such an existence result for the *mesoscopic* model under weak assumptions on the initial distribultion of particles.

Existence of solutions to Boltzmann type equations can be very difficult to obtain, possibly requiring the use of renormalized solutions as in the original paper of DiPerna and Lions [11]; We refer to Cercignani's book [8] and Villani [26] for a

review of the issues arising for Boltzmann like models. Fortunately the existence theory for Eq. (1.1) turns out to be much simpler.

The coagulation equation (1.1) has been introduced and studied in the case of spatial homogeneity: $f = f(t, m, p)$ by Roquejoffre and Villedieu [24]. In

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the spatially inhomogeneous case $f = f(t, x, m, p)$ Escobedo, Laurencot and Mischler [15] have studied existence for a more restrictive collision operator than the one we consider, which in their case leads to a preservation of L_p norms. We shall consider a more general class of collision operators and show, that under some natural restriction on the initial data (but stronger than in [15]), we have existence without blowup for any given finite time. The method of our proof is very different from [15] and the two results are really complementary.

Finally we would like to mention that a difficult open question for Eq. (1.1) is whether its limit for an infinite rate of collisions corresponds to pressureless gas dynamics (as it physically should), even only in dimension one (see Bouchut, James [6] and Brenier, Grenier [7], Poupaud, Rascle [22] for the theory of pressureless gas dynamics in one space dimension).

2 The main theorem and the L^∞ bound

We study the following equation on $f(t, x, m, p)$

$$\partial_t f + \frac{p}{m} \cdot \nabla_x f = Q(f) = Q^+(f) - Q^-(f), \quad x, \, p \in \mathbb{R}^d, \, m \in \mathbb{R}^+ \quad (2.1)$$

$$Q^+(f) = \frac{1}{2} \int_0^m \int_{p'} B(m^*, m', p^*, p') f(t, x, m', p') f(t, x, m^*, p^*) \, dp' \, dm'$$

$$Q^-(f) = f(t, x, m, p) \int_0^\infty \int_{p'} B(m, m', p, p') f(t, x, m', p') \, dp' \, dm', \quad (2.2)$$

where $m^* = m - m'$ and $p^* = p - p'$. Here $Q^+(f)$ is the contribution by coagulation of smaller particles, $Q^-(f)$ is the depletion by coagulating with other particles. These are irreversible processes. We have conservation of mass and momentum but no preservation of kinetic energy. We assume the following bound on B

$$B(m^*, m', p^*, p') \leq C \frac{m'^\alpha + m^{*\alpha}}{(\inf(m', m^*))^\beta} \times \left(\frac{|p'|}{m'} + \frac{|p^*|}{m^*} \right), \text{ with } \alpha < 1, \, \beta < 1 + \alpha. \quad (2.3)$$

We say that a function $f \in C^1([0, \, T] \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d)$ is a strong solution to (2.1) if

$$(1 + m^{-1-\beta} + m^k + (|p|/m)^k) f \in L^\infty([0, \, T] \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d), \quad \forall k > 0, \quad (2.4)$$

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if moreover

$$\exists m_0 > 0 \quad \text{s.t.} \quad f(t, x, m, p) = 0, \quad \forall m \leq m_0, \qquad (2.5)$$

and of course if f satisfies (2.1) at every point (note that $Q(f)$ is well defined due to (2.4) and (2.5)). In that case we shall also assume that

$$\exists m_0 > 0 \quad \text{s.t.} \quad f^0(x, m, p) = 0 \text{ if } m \leq m_0. \qquad (2.6)$$

However in general this last assumption is not very reasonnable and we do not know whether strong solutions exist on any time interval. Therefore we are interested in weak solutions.

The two relevant physical quantities for this problem are the total number of particles, the mass and the kinetic energy which we assume to be finite

$$\int_{\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d} (1 + m + |p|^2/m) \, f^0(x, m, p) \, dx \, dm \, dp < \infty. \qquad (2.7)$$

A basic property of equation (2.1) is to preserve the total mass and to decrease the energy and the number of particles, *i.e.* we expect that for any solution f

$$\int_{\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d} (1+m+|p|^2/m) \, f(t, x, m, p) \leq \int_{\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d} (1+m+|p|^2/m) \, f^0(x, m, p).$$
$$(2.8)$$

Given an arbitray time interval $[0, \ T]$, we also define the following quantities

$$\|g\|_{k,m} = \sup_{x,m',p} (1 + |p|^k) \times (1 + |p|/m' + |x|)^{1+0} \times \frac{\mathbb{I}_{m' \leq m}}{m'^{\beta+1}} \times g(x, m', p),$$
$$\|f\|_{k,m}^\infty = \sup_{0 \leq t \leq T} \|f(t, ., ., .)\|_{k,m},$$
$$(2.9)$$

where $1+0$ denotes any exponent larger than 1. Those norms are L^∞ bounds with weights. Without additional assumptions on the rate of coagulation B, it is not known whether the "ordinary" L^∞ norm of the solution can be bounded. Therefore we add weights in (2.9) so as to be able to a priori bound the corresponding norms.

We call a weak solution to (2.1) on $[0, \ T]$, any function f satisfying (2.8), such that f is a distributional solution to (2.1) and for which the a priori estimates that we are able to prove are true, namely $\|f\|_{k,m}^\infty < \infty$ for any k, m ($Q(f)$ being well defined as a consequence). We are able to obtain an existence result for this notion of weak solutions

Theorem 2.1. *Assume the following mild restriction on the intial data:*

$$\int (1 + m + |p|^2/m) f^0(x, m, p) \, dx \, dm \, dp < \infty$$

$$||f_0||_{k,m} < \infty, \quad \forall k, m \quad .\tag{2.10}$$

$$||f^0||_{k+(4+0)N, 2^{-N}} \to 0 \quad with \quad N \to \infty$$

Then for any $T > 0$, there exists $f(t, x, m, p)$, defined on $[0, T] \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d$, such that it satisfies $||f||_{k,m}^\infty < \infty \, \forall k, m$ and (2.8), and such that f is a solution to (2.1) in a distributional sense.

The main ingredient for this proof is a L^∞ estimate.

Lemma 2.2. *For $T > 0$ and any f^0 satisfying (2.10) and (2.6), then any strong solution f satisfies $||f||_{k,m}^\infty \le C(k, m)$ for a function $C(k, m)$ which does not depend on m_0.*

Before proving this lemma, notice that any strong solution automatically satisfies (2.8).

2.1 Beginning of the proof of Lemma 2.2

We wish to prove L^∞ bounds on f with weights so, to this end, we multiply equation (2.1) by the corresponding quantities,

$$\left(\partial_t + \frac{p}{m} \cdot \nabla_x\right) \cdot \left(\left(1 + \frac{|p|}{m} + |x|\right)^{1+0} \frac{1 + |p|^k}{m^{\beta+1}} f\right) = S_1 + S_2$$

$$= \left(1 + \frac{|p|}{m} + |x|\right)^{0+0} \frac{1 + |p|^k}{m^{\beta+1}} \frac{x \cdot p}{m|x|} f + \left(1 + \frac{|p|}{m} + |x|\right)^{1+0} \frac{1 + |p|^k}{m^{\beta+1}} Q(f).$$

And using the characteristics,

$$||f(t)||_{k,M}^\infty \le ||f^0||_{k,M} + \sup_{x,p,m \le M} \int_0^t (S_1 + S_2)(t - s, x - sp/m, m, p) \, ds$$

$$= ||f^0||_{k,M} + F^1 + F^2.$$

$$\tag{2.11}$$

2.1.1 The contribution from F^1

It is the easiest part. One simply writes

$$
\begin{aligned}
F^1 &\leq \int_0^t \sup_{x,p,m \leq M} \left(1 + \frac{|p|}{m} + |x| \right)^{0+0} \frac{1+|p|^k}{m^{\beta+1}} \frac{|p|}{m} f(t-s, x-sp/m, m, p)\, ds \\
&\leq \int_0^t \|f(s)\|_{k,M}\, ds.
\end{aligned}
$$

$$(2.12)$$

2.1.2 The contribution from F^2

We use the bound (2.3) and the structure of Q in (2.1) to obtain

$$
\begin{aligned}
F^2 \leq M^\alpha &\sup_{x,p,m \leq M} \int_0^t \int_0^{m/2} \int_{p'} \left(1 + \frac{|p|}{m} + |x - sp/m| \right)^{1+0} \frac{1+|p'+p^*|^k}{m^{\beta+1} m'^\beta} \\
&\times \left| \frac{p'}{m'} + \frac{p}{m} \right| f(t-s, x-sp/m, m', p')\, f(t-s, x-sp/m, m^*, p^*)\, dp'\, dm'\, ds.
\end{aligned}
$$

Then, we divide the domain of integration in p' as

$$
\begin{aligned}
F^2 \leq M^\alpha &\sup_{x,p,m \leq M} \int_0^t \int_0^{m/2} \int_{|p'| \geq m'|p|/m} \cdots \\
&+ M^\alpha \sup_{x,p,m \leq M} \int_0^t \int_0^{m/2} \int_{|p'| \leq m'|p|/m} \cdots = F^{12} + F^{22}.
\end{aligned}
$$

Dealing with the first term, we write

$$
\begin{aligned}
F^{12} \leq 2M^\alpha &\sup_{x,p,m \leq M} \int_0^t \int_0^{m/2} \int_{p'} \left(1 + \frac{|p'|}{m'} + |x - sp/m| \right)^{1+0} \frac{1+|p'+p^*|^k}{m^{\beta+1} m'^\beta} \\
&\times \frac{|p'|}{m'} f(t-s, x-sp/m, m', p')\, f(t-s, x-sp/m, m^*, p^*)\, dp'\, dm'\, ds,
\end{aligned}
$$

or, since $m^* \geq m/2$

$$
\begin{aligned}
F^{12} \leq 2M^{1+\alpha} &\int_0^t \int_0^{M/2} \|f(s)\|_{k+4+0,m'} \|f(s)\|_{k,M-m'}\, dm'\, ds \\
&\times \int_{p'} \frac{dp'}{1+|p'|^{3+0}}.
\end{aligned}
$$

$$(2.13)$$

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Next, for F^{22}, we remark that coagulation rules imply that $|p|/m$ is less than the maximum of $|p'|/m'$ and $|p^*|/m^*$. As a consequence, in this case $|p|/m \leq |p^*|/m^*$, and

$$F^{22} \leq 4M^\alpha \sup_{x,p,m \leq M} \int_0^t \int_0^{m/2} \int_{p'} \left(1 + \frac{|p^*|}{m^*} + |x - sp/m|\right)^{1+0} \frac{1 + |p' + p^*|^k}{m^{\beta+1} m'^\beta}$$
$$\times \left(\frac{|p'|}{m'} + \frac{|p|}{m}\right) f(t-s, x-sp/m, m', p') f(t-s, x-sp/m, m^*, p^*) dp' dm' ds.$$

Introducing as previously the corresponding multipliers, we end up with

$$F^{22} \leq 2M^\alpha \sup_{x,p,m \leq M} \int_0^{m/2} \|f\|_{k+4+0,m'}^\infty \|f\|_{k,m-m'}^\infty$$
$$\times \int_0^t \int_{p'} \frac{1 + |p|/m}{(1 + |p'|/m' + |x - sp/m|)^{1+0}} \times \frac{dp' \, ds}{1 + |p'|^{3+0}} \, dm'.$$

The term on the second line is clearly bounded by

$$C \sup_{p',m'} \int_0^t \frac{1 + |p|/m}{(1 + |x - sp/m|)^{1+0}} \, ds.$$

Now we use the lemma

Lemma 2.3. *There exists a constant C (depending on T) such that $\forall \, t \leq T$, $\forall \, x, v \in \mathbb{R}^d$*

$$\int_0^t \frac{ds}{(1 + |x - sv|)^{1+0}} \leq \frac{C}{1 + |v|}.$$

This implies that

$$F^{22} \leq \frac{CM^\alpha}{t} \int_0^{M/2} \|f\|_{k+4+0,m}^\infty \|f\|_{k,M-m}^\infty \, dm. \tag{2.14}$$

Proof of Lemma 2.3. It is a simple computation. First consider the case where $t \times |v| \leq |x|/2$, then for any $0 \leq s \leq t$

$$\frac{1}{(1 + |x - sv|)^{1+0}} \leq \frac{4}{(1 + |x|)^{1+0}}.$$

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And so

$$\int_0^t \frac{ds}{(1+|x-sv|)^{1+0}} \leq \frac{4t}{(1+|x|)^{1+0}} \leq \frac{4t}{1+|x|} \leq \frac{4t}{1+2t\,|v|}.$$

This last quantity is dominated by $2/(1+|v|)$ if $t \leq 1/2$ and by $4t/(1+|v|)$ if $t > 1/2$; Consequently

$$\int_0^t \frac{ds}{(1+|x-sv|)^{1+0}} \leq \frac{2+4t}{1+|v|}.$$

Otherwise, of course

$$\int_0^t \frac{ds}{(1+|x-sv|)^{1+0}} \leq \int_0^{3t} \frac{ds}{(1+|x-sv|)^{1+0}}.$$

This integral is the largest when v is parallel to x, i.e. $x = \alpha v$ with $\alpha \leq 3t$, because $|x-sv|$ may then vanish. We compute it in this (worst) situation and moreover we may assume that $|v| \geq 1$, since if $|v| \leq 1$ then the integral is bounded by t which less than $2t/(1+|v|)$. Now assuming that $|v| \geq 1$

$$\int_0^t \frac{ds}{(1+|x-sv|)^{1+0}} \leq 2 \times \int_0^t \frac{ds}{(1+s|v|)^{1+0}} \leq \frac{2}{|v|} \times \int_0^\infty \frac{ds}{(1+s)^{1+0}} \leq \frac{2C}{|v|}.$$

So eventually the lemma is proved. $\square$

2.2   Conclusion of the proof of Lemma 2.2

Combining Estimates (2.12), (2.13) and (2.14) together, we have (here the constant C behaves like $T(1+m)$)

$$\|f(t)\|_{k,m} \leq \int_0^t \|f(s)\|_{k,m}\, ds + \|f^0\|_{k,m}$$
$$+ C\, m^\alpha \int_0^{m/2} \|f\|_{k+4+0,m'}^\infty \|f\|_{k,m-m'}^\infty\, dm'. \tag{2.15}$$

Now as initially we have

$$\|f^0\|_{k,m} = G(k,m) < \infty, \quad \forall k. \tag{2.16}$$

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The estimate (2.15) becomes

$$
\begin{aligned}
\|f(t)\|_{k,m} \leq & G(k,m) + \int_0^t \|f(s)\|_{k,m}\, ds \\
& + C\, m^\alpha \int_0^{m/2} \|f\|_{k+4+0,m'}^\infty \|f\|_{k,m-m'}^\infty\, dm'.
\end{aligned}
\tag{2.17}
$$

By a Gronwall lemma, this reduces to

$$
\|f(t)\|_{k,m} \leq C\, e^T\, G(k,m) + C\, e^T\, m^\alpha \int_0^{m/2} \|f\|_{k+4+0,m'}^\infty \|f\|_{k,m-m'}^\infty\, dm'.
$$

Taking the supremum in t, we get

$$
\|f\|_{k,m}^\infty \leq C G(k,m) + C\, m^\alpha \|f\|_{k+4+0,m/2}^\infty \int_0^{m/2} \|f\|_{k,m-m'}^\infty\, dm'.
\tag{2.18}
$$

Applying another Gronwall lemma but with m as the "time variable", we find

$$
\|f\|_{k,m}^\infty \leq e^{Cm^\alpha \|f\|_{k+4+0,m/2}^\infty} G(k,m).
\tag{2.19}
$$

Given our approximation procedure, there exists m_0 such that $\|f\|_{k,m_0}^\infty = 0$ for all k and we wish to prove estimates on $\|f\|_{k,m}^\infty$ uniform in m_0. We consider the sequences

$$
u_n = \|f\|_{k+(4+0)(N-n),m_0 2^n}^\infty, \quad v_n = G(k+(4+0)(N-n), m_0 2^n),
$$

they satisfy the relation

$$
u_{n+1} \leq e^{Cm_0^\alpha 2^{n\alpha} u_n} v_{n+1}.
$$

Of course we may choose the integer N as we wish. First of all, let us notice that for any integer M, $G(k+(4+0)n, M\, 2^{-n})$ converges toward zero thanks to the hypothesis of Theorem 2.1. Furthermore as $G(k+(4+0)n, M\, 2^{-n}) \leq G(k+(4+0)(n-\log M), 2^{-n-\log M})$ for $M < 1$, the same hypothesis implies that $\sup_n G(k+(4+0)n, M\, 2^{-n})$ exists and converges towards zero with M. Hence we may define M such that $e^C \times \sup_n G(k+(4+0)n, M\, 2^{-n}) < 1$. Now we take for N the largest integer such that $M > m_0 \times 2^N$. We recall that $v_{N-n} = G(k+(4+0)n, m_0\, 2^{N-n})$ is precisely $G(k+(4+0)n, M\, 2^{-n})$ and therefore and $e^C v_n \leq 1$ for all $n \leq N$.

Then by induction it is easy to check that $u_n \leq 1$ for $0 \leq n \leq N$. Of course $u_0 = v_0 \leq 1$ and if it is true for u_n then

$$u_{n+1} \leq e^{C(m_0 2^n)^\alpha} v_{n+1} \leq e^C v_{n+1} \leq 1.$$

Consequently $\|f\|_{k,m}^\infty$ is bounded for all k and for all m up to at least $M/2$. Now to conclude the proof, we apply (2.19) for $m = M$, then $m = 2M$ and so on, and we deduce that for any $m > 0$, $\|f\|_{k,m}^\infty$ is bounded uniformly in m_0, as M is bounded from below uniformly in m_0.

3 Existence of solutions to 2.1

3.1 Existence of strong solutions for a small time

As usually we are going to use a fixed point argument. Having chosen a given f^0 (and thus a critical mass m_0 such that (2.6) holds), we define the space Ω in which we work: Ω is the space of all non negative functions $f \in C^1([0, t_0] \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d)$, with $f(0, x, m, p) = f^0(x, m, p)$, satisfying (2.4) and such that (2.5) holds for the same parameter m_0 as for f^0. The distance on this space is

$$
\begin{aligned}
d(f, g) = & \|(1 + m^{-\beta-1}(f - g)\|_{L^\infty} \\
& + \sum_{k=1}^{\infty} 2^{-k} \min\left(1, \ \left\|(m^k + (|p|/m)^k) \, |f - g|(t, x, m, p)\right\|_{L^\infty}\right).
\end{aligned}
$$

We define the operator T on Ω as follows : For $f \in \Omega$, Tf is the solution to the equation

$$
\begin{aligned}
\partial_t Tf + \frac{p}{m} \cdot \nabla_x Tf = & \frac{1}{2} \int_0^m \int_{p'} B(m^*, m', p^*, p') f' \, f^* \, dm' \, dp' \\
& - Tf(t, x, m, p) \times \int_0^\infty \int_{p'} B(m, m', p, p') \, f(t, x, m', p') \, dp' \, dm', \quad (3.1)
\end{aligned}
$$

$$Tf(0, x, m, p) = f^0(x, m, p).$$

This is a linear transport equation including a given source term and absorption. All coefficients are regular and well defined since $f \in \Omega$ and (2.3), therefore the existence of a C^1 solution is ensured. Moreover the source term is always non negative and so the solution Tf is also non negative.

Then the source term, which we denote by $Q^+(f)(t, x, m, p)$, vanishes if $m \leq m_0$ because in this case both m' and m^* are necessarily less than m_0 too and f satisfies (2.5). Consequently Tf also satisfies (2.5) with m_0.

Now thanks to (2.3), we have for example

$$m^{-\beta-1} Q^+(f) \leq m^{-\beta-1} \int_{m/2}^m \int_{p'} \frac{m'^\alpha}{m^{*\beta}} \times \left(\frac{|p'|}{m'} + \frac{|p^*|}{m^*} \right) f' \, f^* \, dm' \, dp'$$

$$\leq 2^{1+\beta} \left\| m^{-\beta} (1 + |p|/m) \, f \right\|_{L^\infty} \times \int_{m/2}^m \int_{p'} \frac{1}{1 + (m')^2} \times \frac{1}{1 + |p'|^4}$$

$$\times (1 + (m')^2) \times (1 + |p|^4) \times (m')^{\alpha-\beta-1} \left(1 + \frac{|p'|}{m'} \right) f'.$$

And as a consequence

$$m^{-\beta-1} Q^+(f) \leq C \left\| (1 + m^{-\beta-1} + (|p|/m)^q) \, f \right\|_{L^\infty}$$
$$\times \left\| (1 + m^{12+2\alpha-4\beta} + m^{-\beta-1} + (|p|/m)^{5r}) \, f \right\|_{L^\infty},$$

by a Hölder estimate and with $1/q + \beta/(1+\beta) = 1$ and $1/r + (1+\beta)/(1+\beta-\alpha) = 1$ or $r = 2$ whichever is larger. With the same kind of computation, we may show that

$$(m^k + (|p|/m)^k) \, Q^+(f) \leq C \left\| (m^{-2\beta} + (|p|/m)^{2k+2}) \, f \right\|_{L^\infty}$$
$$\times \left\| (m^{2\alpha+2k} + (|p|/m)^{2k+2}) \, f \right\|_{L^\infty}.$$

From that it is easy to deduce that T is continuous on Ω and with the same kind of bounds that it is contracting provided t_0 is small enough, which gives a strong solution on $[0, \, t_0]$. We leave the details to the reader but we note that the time t_0 is bounded from below by a quantity which depends only on the $\|f^0\|_{k,m}$, $k, m \in \mathbb{R}_+$.

3.2 Weak stability

We prove the following

Proposition 3.1. *Let $f_n \in L^\infty([0, \, t_0], \, L^1(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d))$ be a sequence of solutions to (2.1) in the distributional sense with uniformly bounded mass and energy, and $\|f_n\|_{k,m}^\infty$ uniformly bounded in n, for any $m > 0$, $k > 0$. Then up to extracting a subsequence, f_n converges towards $f \in L^\infty([0, \, t_0], \, L^1(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d))$ for the weak$-*$ topology of measures, a solution to (2.1) in the distributional sense with bounded mass and energy, $f(t = 0, .) = f^0 = \lim f_n(t = 0, .)$.*

Notice that if a function $g(t, x, m, p)$ satisfies $\|g\|_{k,m}^\infty < \infty$ for all k, m, then $Q(g)$ is bounded in L^∞ in time with values in L^1_{loc} in x, m, p; And so is $(p/m)\, g$. Consequently if g is also a solution to (2.1) in the distributional sense, then $\partial_t g$ belongs to L^∞ in time with values in a (local) Sobolev space with negative exponent. And therefore, traces in time of g are well defined.

Proof. First of all, we note that because of the uniform bounds on

$$\int_{\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d} (1 + m + |p|^2/m)\, f_n(t, x, m, p)\, dx\, dm\, dp,$$

we may extract a subsequence f_n, which we denote by the same indices, which converges to $f \in L^\infty([0,\ t_0],\ M^1(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d))$. Moreover $\|f\|_{k,m}^\infty$ is finite for all $m > 0$ and thus $f \in L^\infty([0,\ t_0],\ L^1(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d))$ and its mass and energy are bounded.

Then $(|p|/m)^{1+0} f_n \in L^\infty_{loc}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d)$ uniformly in n and consequently

$$\frac{p}{m} f_n \longrightarrow \frac{p}{m} f \text{ in } \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d).$$

Now as $\|f_n\|_{k,m}$ is uniformly bounded for any m, $Q^+(f_n)$ is uniformly bounded in $L^\infty_{loc}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d)$. For $Q^-(f_n)$ we write, using (2.3)

$$\int_0^m \int_{p'} B(m, m', p, p')\, f'_n dp'\, dm' \leq C m^\alpha (1 + |p|/m) \int_0^m \int_{p'} \frac{1 + |p'|/m'}{m'^\beta} f'_n dp' dm'$$

$$\leq C m^{1+\alpha}(1 + |p|/m) \times \|f_n\|_{4,m}^\infty,$$

and

$$\int_m^M \int_{p'} B(m, m', p, p')\, f'_n dp'\, dm' \leq \frac{C\, M^\alpha}{m^\beta}(1 + |p|/m) \int_m^M \int_{p'} (1 + |p'|/m')\, f'_n$$

$$\leq \frac{C\, M^\alpha}{m^\beta}(1 + |p|/m) \times \|f_n\|_{4,M}^\infty.$$

Therefore denoting

$$Q^-_M(f_n) = f_n(t, x, m, p) \times \int_0^M \int_{p'} B(m, m', p, p')\, f'_n dp'\, dm',$$

We also know that $Q^-_M(f_n)$ is uniformly bounded in L^∞_{loc} for $m \leq M$. Finally, again for $m \leq M$ and because $\alpha < 1$

$$\int_M^\infty \int_{p'} B(m, m', p, p')\, f'_n dp'\, dm' \leq \frac{C}{m^\beta}(1 + |p|/m) \int_M^\infty \int_{p'} m'^\alpha (1 + |p'|/m')\, f'_n$$

$$\leq \frac{C\, M^{\alpha-1}}{m^\beta}(1 + |p|/m) \int_0^\infty \int_{p'} |p'|\, f'_n\, dp'\, dm',$$

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and thus denoting $\Omega_R = [0,\ t_0] \times B(0,R) \times [0,R] \times B(0,R)$,

$$\|R_M(f_n)\|_{L^1(\Omega_R)} = \left\| f_n(t,x,m,p) \int_M \int_{p'}^\infty B(m,m',p,p')\, f'_n \right\|_{L^1(\Omega_R)}$$

$$\leq C\, M^{\alpha-1}\, t_0\, R^{2d+1}\, \|f_n\|_{4,R}^\infty \times \int_{\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d} |p|\, f_n(0,x,m,p).$$

We now wish to use the result of velocity averaging proved in [12]. The only "difficulty" is that DiPerna, Lions and Meyer use the velocity variable $v = p/m$. Consequently we change variable and define

$$g_n(t,x,m,v) = m^d\, f_n(t,x,m,mv).$$

The equation satisfied by g_n is of course

$$\partial_t g_n + v \cdot \nabla_x g_n = S_n^1 + S_n^2,$$

with

$$S_n^1(t,x,m,v) = m^d\, Q^+(f_n)(t,x,m,mv) + m^d\, Q_M^-(f_n)(t,x,m,mv),$$
$$S_n^2(t,x,m,v) = m^d\, R_M(f_n)(t,x,m,mv).$$

As localization in t,x,m,v implies localization in t,x,m,p, the term S_n^1 is uniformly bounded in L_{loc}^∞. Moreover

$$\|S_n^2(t,x,m,v)\|_{L^1(\Omega_R)} = \|R_M(f_n)(t,x,m,p)\|_{L^1([0,\ t_0] \times B(0,R) \times [0,\ R] \times B(0,R^2))}.$$

The variable m is now only a parameter and the result from DiPerna, Lions and Meyer can be applied directly. Consequently we obtain that for any $\phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$, $\int \phi(m,v) g_n\, dm\, dv$ is the sum of a term uniformly bounded in $H_{loc}^{1/2}$ by $C\, M^\alpha$ and a term bounded in L_{loc}^1 by $C\, M^{\alpha-1}$.
Notice that for any $\psi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$, we may define $\phi(m,v) = \psi(m,mv)$ and ϕ also belongs to C_c^∞. Therefore $\int \psi(m,p) f_n dm\, dp$, which is precisely equal to $\int \phi(m,v) g_n\, dm\, dp$, is also the sum of a term uniformly bounded in $H_{loc}^{1/2}$ by $C\, M^\alpha$ and a term bounded in L_{loc}^1 by $C\, M^{\alpha-1}$.
We immediately deduce that $\int \psi(m,p) f_n\, dm\, dp$ converges strongly in L_{loc}^1 toward $\int \psi(m,p) f$. As those averages are moreover uniformly bounded in L_{loc}^∞, the convergence holds in all L_{loc}^p, $p < \infty$.
Decomposing the integrals in $Q(f_n)$ in integrals over a compact support in m and p and a small remainder, as we have just done, it is easy to conclude

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that $Q(f_n)$ converges strongly in L^1_{loc} towards $Q(f)$. Therefore f is also a solution to (2.1).

It only remains to explain why $f(t = 0, .) = \lim f_n(t = 0, .)$. For that note that we have proved that $p/m\, f_n$ and $Q(f_n)$ are uniformly bounded in L^1_{loc} and consequently $\partial_t f_n$ is uniformly bounded in $L^\infty([0, \; t_0] \times W^{-1,1}_{loc})$. This ensures compactness in time and so we indeed have $f(t = 0, .) = f^0 = \lim f_n(t = 0, .)$. $\square$

3.3 Conclusion of the existence proof for equation 2.1

We first take a sequence $f^0_n \in C^1$ which approximates f^0 in L^1. We may take this sequence such that $\|f^0_n\|_{k,m}$ is bounded by $2\|f^0\|_{k,m}$ for all $k > 0, \; m > 0$, the mass and energy of f^0_n is the same as the ones of f^0 and each f^0_n satisfies (2.6) but with different parameters m^n_0 of course since $m^n_0 \to 0$.

We know that there exists a time t_0 such that we have a sequence f_n of strong solutions to (2.1) on $[0, \; t_0]$ with $f_n(t = 0, .) = f^0_n$, f_n has uniformly bounded mass and energy and all $\|f_n\|^\infty_{k,m}$ are uniformly bounded.

We may apply Prop. 3.1 to obtain a solution f to (2.1), in the distributional sense, with $f(t = 0, .) = f^0$, bounded mass and energy and $\|f\|_{k,m} < \infty$, $\forall k, m > 0$.

Prop. 2.2 eventually enables us to start again the procedure at the time t_0 and to obtain a weak solution on any time interval $[0, \; T]$.

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