Mean field limit for interacting particles

P.E. Jabin
Equipe Tosca, Inria, 2004 route des Lucioles, BP 93, 06902 Sophia Antipolis
Laboratoire Dieudonné, Univ. de Nice, Parc Valrose, 06108 Nice cedex 02

Antoine Gerschenfeld
École Normale Supérieure, Paris

0.1 Introduction

The validity of kinetic models as a limit of systems of many interacting particles is still an important open issue. The number of particles to take into account is so large in most applications (plasma physics, galaxies formation...) that the use of continuous models is absolutely required.

The same issues directly arise for the use of particle methods. Those methods rely on the assumption that a large (but not too large) number of "meta-particles" correctly represents the dynamics of a much larger number of real particles. This assumption would be directly implied by the convergence of the system to the unique solution to some equation.

The scaling under consideration here leads to so-called mean field limits. Those limits were classically established under strong regularity assumptions for the interaction, which are not satisfied in many physical situations of interest. We aim at describing those classical approaches but also to present the new ideas recently developed for the singular cases.

We consider N identical particles with positions/velocities (X_i, V_i) in the phase space, interacting through the 2-body interaction kernel $K(x)$, which leads to the evolution equations

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} X_i = V_i, \\ \frac{\mathrm{d}}{\mathrm{d}t} V_i = \frac{1}{N} \sum_j K(X_i - X_j). \end{cases} \tag{0.1}$$

The $1/N$ factor in the second equation is a scaling term so that positions, velocities and accelerations are now of order 1.

The kernel K may take many different forms depending on the physical setting. The guiding example and the one with the most important physical applications $K(x)$ is Coulomb interaction, which reads in dimension d

$$K(x) = -\nabla\phi(x)\ ,\ \phi(x) = \frac{\alpha}{|x|^{d-2}} + \text{ (regular terms)},$$

where $\alpha > 0$ (resp. $\alpha < 0$) corresponds to the repulsive (resp. attractive) case.

In what follows, the dynamics will be considered on the torus $X_i \in \Pi^d$, $d \geq 2$, mainly to simplify the exposition. Note that even then the velocities are still in $\mathbb{R}^d$.

0.2 Well-posedness of the microscopic dynamics

The Cauchy-Lipschitz theorem applies to (0.1) if $K(x)$ is Lipschitz, in which case there exists an unique solution for any initial condition. In the repulsive Coulomb case, it is still possible to apply it by remarking that the energy conservation,

$$E(t) = \frac{1}{N} \sum_i |V_i|^2 + \frac{\alpha}{N^2} \sum_{i \neq j} \frac{1}{|X_i - X_j|} = E(0)\,,$$

implies that the $|X_i - X_j|$ admit a time-independent lower bound in $\frac{1}{N^2}$: one may therefore consider K as Lipschitz on its attainable domain for a given set of initial conditions. One should note, however, that the form of this estimate makes it improper to use in the $N \to \infty$ limit.

It is possible to assume less regularity on K by restricting the set of acceptable initial conditions. In particular, results by (DiPerna and Lions, 1989), (Ambrosio, 2004) and (Hauray, 2005) apply to almost-every initial condition.

0.3 Existence of the macroscopic limit

Given a sequence of initial conditions $Z^{N0} = (X_1^{N0}, .., X_N^{N0}, V_1^{N0}, .., V_N^{N0})$ with corresponding solutions $Z^N(t)$, one expects the empirical density on phase space,

$$f_N(t, x, v) = \frac{1}{N} \sum_i \delta(x - X_i^N(t)) \otimes \delta(v - V_i^N(t)) \,,$$

to converge, in some sense, as $N \to \infty$, to a limit f satisfying an evolution equation, the "limiting dynamics", with initial conditions $f^0 = \lim_N f_N(0, \cdot)$.

If K is continuous or if $X_i^N(t) \neq X_j^N(t)$ for all t and $i \neq j$, then, posing $K(0) = 0$, one can write the N-body evolution in the form of a Vlasov equation :

$$\begin{cases} \partial_t f_N + v \cdot \nabla_x f_N + (K \star_x \rho_N) \cdot \nabla_v f_N = 0 \\ \rho_N(t, x) = \int \mathrm{d}v f_N(t, x, v) \ . \end{cases} \tag{0.2}$$

Then $f_N \to f$ in weak-$\star$ topology (for the space of Radon measures $M^1(\Pi^d \times \mathbb{R}^d)$), and f solves (0.2) for the initial conditions $\lim_N f_N(0, \cdot)$.

Equation (0.2) cannot be obtained from (0.1) with such an immediate method for any kind of singular interaction $K \notin C_0$. However, even for a Coulomb potential, Eq. (0.2) is well posed provided some assumptions on the initial conditions are made, such as $f(0, \cdot) \in L^1 \cap L^\infty$ and with compact support in velocities; See (Horst, 1981; Lions and Perthame, 1991; Pfaffelmoser, 1992; Schaeffer, 1991). However, the non-linear term $(K \star_x \rho_N) \cdot \nabla_v f_N$ makes the $f_N \to f$ limit highly nontrivial for non-continuous K.

0.4 Physical space models

The above question is easier to solve in the case of hydrodynamics-related models, which evolve according to a first-order equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} X_i = \frac{1}{N} \sum_{j \neq i} \mu_i \mu_j K(X_i - X_j) \,. \tag{0.3}$$

Using $\rho_N(t, x) = \sum_i \mu_i \delta(x - X_i(t))$, it can be rewritten as

$$\partial_t \rho + \nabla_x((K \star \rho)\rho) = 0 \ . \tag{0.4}$$

For instance, in dimension 2, the above yields the incompressible Euler equation for $\mu_i = \pm 1, K(x) = x_\perp / |x^2|$.

As a rule of thumb, the $N \to \infty$ limit is easier to take in this case than in (0.1). A crucial ingredient to the study is a bound on $d_{min}(t) = \inf_{i \neq j} |X_i(t) - X_j(t)|$. This offers direct control over the right-hand term in (0.3),which becomes regular if $d_{min}^N \sim N^{-1/d}$ for singular force terms K (up to a coulombian singularity).

More precisely, assume that, up to time t and for $x \sim x_i$, there exists a locally bounded F such that

$$\left\| \frac{1}{N} \sum_{j \neq i} K(x - X_j(t)) \right\|_{W^{1,\infty}} \leq F\left(\frac{d_{min}}{N^{1/d}}\right) .$$

Let (k, l) be the particles such that $d_{min}(t) = |X_k - X_l|$. If one also assumes that $\mu_k = \mu_l$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} d_{min} = \frac{\mathrm{d}}{\mathrm{d}t}|X_k - X_l| &\geq -\frac{1}{N} \left| \sum_{j \neq k,l} \mu_j (K(X_k - X_j) - K(X_l - X_j)) \right| + o(1) \\ &\geq - \left\| \frac{1}{N} \sum_{j \neq i} K(x - X_j(t)) \right\|_{W^{1,\infty}} d_{min} \geq d_{min} F\left(\frac{d_{min}}{N^{1/d}}\right) .\end{aligned}$$

Hence one can apply Gronwall's lemma to d_{min}, propagating $d_{min} \sim N^{-1/d}$ and ensuring regularity in (0.3). For more on this kind of limits and particularly point-vortex approximations to $2d$ Euler, see for example (Goodman, Hou and Lowengrub, 1990; Schochet, 1996).

Note, however, that such an approach is unapplicable in phase space : in that case, the physical distance bound d_{min} still controls the regularity of the force terms, but the evolution equations only control the phase space distance

$$d^v_{min} = \inf_{i \neq j} (|X_i - X_j| + |V_i - V_j|) \geq d_{min} .$$

Thus one cannot obtain a closed estimate on d_{min}.

0.5 Macroscopic limit in the regular case

0.5.1 Existence and weak solutions

If K is regular enough, it becomes possible to pass to the limit in the non-linear term $(K \star \rho_N)\nabla_v f_N$, thus ensuring the existence of a solution to the limiting dynamics (0.2).

Theorem 0.1 *If K is continuous and if the initial conditions are uniformly bounded in velocity ($|V_i^N(0)| \leq R$ for some R), then there exists a subsequence $f_{\sigma(N)}$ of f_N such that :*

1. $f_{\sigma(N)} \overset{w-\star}{\longrightarrow} f$ *in* $L^\infty(\mathbb{R}_+, M^1(\Pi^d \times \mathbb{R}^d))$ *;*
2. $\rho_{\sigma(N)} \overset{w-\star}{\longrightarrow} \rho = \int f \mathrm{d}v$ *in* $L^\infty(\mathbb{R}_+, M^1(\Pi^d))$ *;*
3. f *is a solution to* (0.2) *in the sense of distribution.*

This theorem proves the existence of measure-valued solutions to (0.2), but its assumptions over K are too weak to ensure their uniqueness (through the convergence of the full sequence f_N, for instance).

Proof First, notice that, as (0.2) conserves probability or total mass,

$$\int \mathrm{d}x\mathrm{d}v f(t,x,v) = 1 \qquad \forall t.$$

Hence $f_N \in L^\infty(\mathbb{R}_+, M^1(\Pi^d \times \mathbb{R}^d))$. This space, as the dual of $L^1(\mathbb{R}_+, \mathcal{C}^0(\Pi^d \times \mathbb{R}^d))$, is weak-$\star$ compact : therefore $f_{\sigma(N)} \overset{w-\star}{\longrightarrow} f$. By construction, it follows that one can take the $N \to \infty$ limit in both linear terms of (0.2).

Then, using

$$|V_i(t)| \leq |V_i(0)| + \frac{1}{N}\sum_{j\neq i}\int_0^t |K(X_i - X_j)| ds \leq R + t\|K\|_\infty\,,$$

one obtains (by compacity of compact-supported continuous functions) that $\rho_{\sigma(N)} = \int f_{\sigma(N)}\mathrm{d}v \overset{w-\star}{\longrightarrow} \rho = \int f\mathrm{d}v$ in $L^\infty(\mathbb{R}_+, M^1(\Pi^d))$.

Given that $\rho, \rho_N \in M^1(\Pi^d)$, $K \star \rho_N(t,\cdot)$ is equicontinuous in x for all fixed t (with the same continuity modulus as K). Moreover, integrating (0.2) over v yields

$$\partial_t \rho_N + \nabla_x \cdot \left(\int v f_N \mathrm{d}v\right) = 0\,,$$

which yields $\partial_t \rho_N \in L^\infty(\mathbb{R}_+, W^{-s,1}(\mathbb{R}^d))$ $\forall s > 1$, so that finally $(K \star \rho_N)$ is equicontinuous in both (x,t) over all $\Pi^d \times [0,T]$.

One may therefore apply Ascoli's theorem : $(K \star \rho_{\sigma(N)}) \to (K \star \rho)$, uniformly over all $\Pi^d \times [0,T]$. Hence $(K \star \rho_{\sigma(N)}) f_{\sigma(N)} \to (K \star \rho) f$ in the sense of distributions, and f is a solution of (0.2) in the same sense.

□

0.5.2 Stability and well-posedness

In order to obtain a stability estimate for the weak solutions derived above, it is necessary to strengthen K's regularity (Braun and Hepp, 1977; Dobrushin, 1979; Spohn, 1991).

Theorem 0.2 *If $K \in W^{1,\infty}$ and $f^1, f^2 \in L^\infty(\mathbb{R}_+, M^1(\Pi^d \times \mathbb{R}^d))$ are two solutions to (0.2) with compact support in velocity, then*

$$\|f^1(t) - f^2(t)\|_{W^{-1,1}(\Pi^d\times\mathbb{R}^d)} \leq C\|f_0^1 - f_0^2\|_{W^{-1,1}(\Pi^d\times\mathbb{R}^d)} \exp[C\|\nabla K\|_{L^\infty} t] \qquad (0.5)$$

This stability estimate yields the convergence of the f_N to the limiting dynamics, as well as the uniqueness and well-posedness of its solutions : in practice, it acts as a limit to the concentration of the corresponding measures. One should also note that, although the exponential growth of this estimate is certainly not optimal, the C constant only depends on the total masses of f^1 and f^2, as shown below.

Proof For $\gamma \in \{1,2\}$, the characteristics of each solution, defined as

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}X^\gamma(t,x,v) = V^\gamma(t,x,v) \\ \frac{\mathrm{d}}{\mathrm{d}t}V^\gamma(t,x,v) = (K\star\rho^\gamma)(t,X^\gamma) \end{cases} \text{for } \begin{cases} X(0,x,v) = x \\ V(0,x,v) = v \end{cases},$$

are well-defined by Cauchy-Lipschitz (since $K \in \mathcal{C}^1$, $\rho^\gamma \star K$ is Lipschitz), and verify the corresponding estimate

$$|\nabla X^\gamma| + |\nabla V^\gamma| \leq Ce^{C\|\nabla K\|_\infty t}\ . \tag{0.6}$$

Moreover, letting $\mathcal{L} = \{\phi \in \mathcal{C}^1(\Pi^d \times \mathbb{R}^d) : \|\phi\|_\infty \leq 1, \|\nabla\phi\|_\infty \leq 1\}$, one has

$$\begin{aligned}
\|f^1(t) - f^2(t)\|_{W^{-1,1}} &= \sup_{\phi\in\mathcal{L}} \int \mathrm{d}x\mathrm{d}v\phi(x,v)\left(f^1(t,x,v) - f^2(t,x,v)\right) \\
&= \sup_{\phi\in\mathcal{L}} \int \mathrm{d}x\mathrm{d}v \left(\phi(X^1(t),V^1(t))f^1(0,x,v) - \phi(X^2(t),V^2(t))f^2(0,x,v)\right) \\
&\leq \sup_{\phi\in\mathcal{L}} \int \mathrm{d}x\mathrm{d}v\phi(X^1(t),V^1(t))\left|f^1(0,x,v) - f^2(0,x,v)\right| \\
&\quad + \sup_{\phi\in\mathcal{L}} \int \mathrm{d}x\mathrm{d}v\left|\phi(X^1(t),V^1(t)) - \phi(X^2(t),V^2(t))\right| f^2(0,x,v) \\
&\leq \|\nabla(X^1,V^1)\|_\infty \|f_0^1 - f_0^2\|_{W^{-1,1}} + \|(X^1,V^1) - (X^2,V^2)\|_\infty \|f_0^2\|_{L^1}\ .
\end{aligned}$$

Since the first term in the above is bounded by (0.6), it is sufficient to bound $\|(X^1,V^1) - (X^2,V^2)\|_\infty$. Given that $\frac{\mathrm{d}}{\mathrm{d}t}|X_1 - X_2| \leq |V_1 - V_2|$ and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|V_1 - V_2| &\leq |(K\star\rho^1)(t,X^1) - (K\star\rho^2)(t,X^2)| \\
&\leq |(K\star\rho^1)(t,X^1) - (K\star\rho^1)(t,X^2)| + |(K\star(\rho^1-\rho^2))(t,X^2)| \\
&\leq |X^1 - X^2|\|\nabla K\|_\infty \int \rho^1\mathrm{d}x + \|\nabla K\|_\infty\|\rho^1 - \rho^2\|_{W^{-1,1}}\ ,
\end{aligned}$$

one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f^1(t) - f^2(t)\|_{W^{-1,1}} \leq C\|\nabla K\|_\infty\|f^1(t) - f^2(t)\|_{W^{-1,1}},$$

from which (0.5) is derived by Gronwall's lemma. □

0.6 Well-posedness for singular kernels

The above establishes the well-posedness of the limiting dynamics in the case $K \in \mathcal{C}^1$, while hinting that $K \in \mathcal{C}^0$ is likely insufficient to reach a similarly satisfactory conclusion. These kernels remain far from the Coulomb-like interactions one would like to use in practice. It is however possible to obtain further results by exploiting the particular nature of these kernels, namely $K \in \mathcal{C}^1(\mathbb{R}^d \setminus \{0\})$.

0.6.1 The weakly singular case

For kernels less singular in 0 than $1/|x|$ (so this never contains the Coulombian case), it is still possible to derive well-posedness from Gronwall-type estimates (Hauray and Jabin, 2007). Such a distinction between $K = o(1/|x|)$ and the rest makes sense physically as if $K = -\nabla\phi$ it exactly corresponds to the cases of bounded vs unbounded potentials ϕ.

Theorem 0.3 *Given a kernel $K \in \mathcal{C}^1(\mathbb{R}^d \setminus \{0\})$ such that $|K(x)| \underset{x\to 0}{\sim} \frac{1}{|x|^\alpha}$ with $\alpha < 1$ and a sequence Z^{N0} of initial conditions with uniform compact support such that*

$$d_{min}(0) = \min_{i\neq j} \left(|X_i^{N0} - X_j^{N0}| + |V_i^{N0} - V_j^{N0}|\right) \geq cN^{-\frac{1}{2d}} \ ,$$

then there exists $c' > 0$ such that $d_{min}(t) \geq c'N^{-\frac{1}{2d}}$ for any $t > 0$. Then the sequence of N-body solutions f_N converges weakly towards the unique solution $f \in L^1 \cap L^\infty$ of (0.2)*, which is compactly supported.*

The $\alpha < 1$ condition above is probably close to optimal, although it remains far from the Coulomb case. This stems from the need to bound the integrals of the force along the trajectories of close particles, which take the form

$$\int \frac{\mathrm{d}t}{|X + Vt|^\alpha} < \infty \ .$$

Similarly, the condition on $d_{min}(0)$ is quite remote from physical reality, as the probability of having it satisfied vanishes to 0 for sets of particles with random initial positions and velocities. However, this does not preclude its practical use, for instance for numerical purposes. It is a strong assumption on its own, since

$$f_N^0 \xrightarrow{w} f^0 \text{ and } d_{min}(0) \geq \frac{c}{N^{\frac{1}{2d}}} \Rightarrow f_N^0 \overset{weak-*\ M^1}{\longrightarrow} f^0 \in L^1 \cap L^\infty \ .$$

A similar, stronger theorem can be proven for the physical space models (Hauray, 2008) :

Theorem 0.4 *If $|K(x)| \sim 1/|x|^\alpha$ with $\alpha < d-1$, then, for any sequence of initial data X^{N0} such that*

$$d_{min}(0) = \min_{i\neq j} |X_i^{N0} - X_j^{N0}| \geq cN^{-\frac{1}{d}} \ ,$$

the dynamics (0.3) *with $\mu_i = 1$, $\frac{\mathrm{d}}{\mathrm{d}t}X_i = \frac{1}{N}\sum_{j\neq i} K(X_i - X_j)$, verify $d_{min}(t) \geq c'N^{-\frac{1}{d}}$ for all $t \leq T$, and for all $t \geq 0$ if $\nabla \cdot K = 0$. Thus the sequence (ρ_N) converges towards the unique solution ρ to* (0.4)*.*

The condition $\alpha < d-1$ is a noticeable improvement on the Vlasov case, as the $2D$ Euler equation ($\alpha = 1$) is now the limiting case ; however, the assumption on d_{min} remains just as strong.

Proof As shown in 0.4, the uniqueness of the limit results from the estimate on d_{min}. Since $K \sim |x|^{-\alpha}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}|X_i - X_k| \geq -\frac{1}{N}(|K(X_i - X_k)| + |K(X_k - X_i)|) - \frac{1}{N}\sum_{j\neq i,k}(|K(X_i - X_j)| + |K(X_k - X_j)|)$$
$$\geq -\frac{|X_i - K_k|}{N}\sum_{j\neq i,k}\left(\frac{C}{|X_i - X_j|^{\alpha+1}} + \frac{C}{|X_k - X_j|^{\alpha+1}}\right) - \frac{2C}{Nd_{min}^{\alpha}} .$$

In order to bound $\frac{1}{N}\sum_{j\neq i}\frac{C}{|X_i - X_j|^{\alpha+1}}$, let $N_k = |\{j \neq i \text{ such that } |X_i - X_j| \in [2^k d_{min}, 2^{k+1} d_{min}]\}|$. By definition of d_{min}, $N_k = 0$ for any $k < 0$; furthermore, as Π has diameter 1, $N_k = 0$ as well for $k > k_0 = -\log_2 d_{min}$. Hence

$$\frac{1}{N}\sum_{j\neq i}\frac{C}{|X_i - X_j|^{\alpha+1}} \leq \frac{C_d}{N}\sum_{k=0}^{k_0}\frac{N_k}{2^{k(\alpha+1)}d_{min}^{\alpha+1}} .$$

By defintion of d_{min}, $N_k \leq C_d 2^{kd}$ as all particles are farther than d_{min} from each other; therefore, as $\alpha + 1 < d$,

$$\frac{1}{N}\sum_{j\neq i}\frac{C}{|X_i - X_j|^{\alpha+1}} \leq \frac{C_d}{N}\sum_{k=0}^{k_0}\frac{2^{k(d-\alpha-1)}}{d_{min}^{\alpha+1}} \leq \frac{C_d}{N}\frac{2^{k_0(d-\alpha-1)}}{d_{min}^{\alpha+1}} \leq \frac{C_d}{Nd_{min}^{d}} .$$

Gathering the estimates yields

$$\frac{\mathrm{d}}{\mathrm{d}t}|X_i - X_k| \geq -|X_i - X_k|\frac{C_d}{Nd_{min}^{d}} - \frac{C}{Nd_{min}^{\alpha}} .$$

Hence, choosing (i, k) such that $d_{min} = |X_i - X_k|$, one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}d_{min} \geq -d_{min}\frac{C_d}{Nd_{min}^{d}} - \frac{C}{Nd_{min}^{\alpha}} = -d_{min}\frac{N^{-1}}{d_{min}^{d}}(C_d + Cd_{min}^{d-\alpha}) ,$$

from which we may bound d_{min} by Gronwall's lemma. The rest of the theorem follows easily from here. □

0.7 An almost-everywhere approach

Attempting to derive a stability estimate, e.g. a bound on $|X_i^N(t, Z^{N0}) - X_i^N(t, Z^{N0} + \delta)|$ for small δ, while avoiding Cauchy-Lipschitz / Gronwall-like methods, one can only succeed for almost-all initial data, for some definition of "almost-all". As the dimension N of the system of differential equations changes, quantitative estimates are needed, unless the more traditional approaches to ODE's. Such an estimate was derived in

(Crippa and De Lellis, 2008) in a finite dimensional framework, and a natural idea followed in a future work by Barré, Hauray and Jabin is to try to extend it like

$$\int_{(\Pi^d\times\mathbb{R}^d)^N} \mathrm{d}\mathbb{P}[Z^{N0}] \log\left(1+\frac{1}{N\|\delta\|_\infty}\sum_{i=1}^N \big(|X_i^N(t,Z^{N0})-X_i^N(t,Z^{N0}+\delta)| \right. \\ \left. +\ |V_i^N(t,Z^{N0})-V_i^N(t,Z^{N0}+\delta)|\big)\right) \le C(1+t)\ , \quad (0.7)$$

where the measure $\mathbb{P}$ on initial configurations determines the meaning of "almost-everywhere", and thus must be chosen carefully.

In order to prove (0.7), one has to differentiate the integral in t, then perform the change of variables $Z^{N0}\to Z^N(t)$ (which has Jacobian 1) . One then needs to estimate integrals of the form

$$\int_{(\Pi^d\times\mathbb{R}^d)^N} \frac{\mathrm{d}\mathbb{P}_t(Z^{N0})}{|X_1^0-X_2^0|^{\alpha+1}}\,,$$

where $\mathbb{P}_t$ is the image of $\mathbb{P}$ by the flow at time t, which should remain finite for $\alpha<d-1$.

Note that one has to be careful doing the estimate as a simple direct computation would require a bound on

$$\int \mathrm{d}\mathbb{P}_t(Z^{N0}) \max_i \left(\frac{1}{N}\sum_{j\neq i}\frac{1}{|X_i^0-X_j^0|^{\alpha+1}}\right) = +\infty\,.$$

The proof involves the following conditions on $\mathbb{P}$:

$$\forall t\,,\int_{\Pi^{d(N-k)}\times\mathbb{R}^d} \mathbb{P}_t(Z^{N0})\mathrm{d}X_{N-k}^0..\mathrm{d}X_N^0\mathrm{d}V_1^0..\mathrm{d}V_N^0 \le C^k\,,$$

which can be checked if $\mathbb{P}$ is flow-invariant, and but would be very hard to investigate otherwise. If K derives from a potential, $K=-\nabla\phi$ with $\phi\ge 0$, one such invariant is the energy

$$H_N=\frac{1}{N}\sum_{i=1}^N|V_i^N|^2+\frac{1}{N^2}\sum_{i\neq j}\phi(X_i^N-X_j^N)\,,$$

and one can choose $\mathbb{P}(Z^{N0})\propto e^{-H_N(Z^{N0})}$ or $\mathbb{P}(Z^{N0})\propto e^{-NH_N(Z^{N0})}$. The first choice leads to an easier proof, but the corresponding sequences (f_N^0) of initial conditions are such that $f_N^0 \xrightarrow{a.s.} 0$, and is therefore very restrictive; the second implies that $f_N(0)\xrightarrow{a.s.}\rho(x)e^{-|v|^2}$, where ρ minimizes

$$\int_{\Pi^{2d}}\frac{1}{2}\phi(x-y)\rho(x)\rho(y)\mathrm{d}x dy+\int_\Pi \rho(x)\log\rho(x)\mathrm{d}x\,.$$

In both cases, (0.7) can be used to perturb the dynamics instead of the initial conditions. For instance, by letting $f_N^{(\delta)}$ be the evolution according to a regularized kernel $K_\delta=\mathbb{1}_{\{|x|>\delta\}}K$ with the same initial conditions, one gets

$$\int_{(\Pi^d\times\mathbb{R}^d)^N} \mathrm{d}\mathbb{P}(Z^{N0}) \log\left(1+\frac{1}{|\delta|}\|f_N(t)-f_N^{(\delta)}(t)\|_{W^{-1,1}}\right) \le C(1+t)\,,$$

which yields the convergence of (f_N) towards a (unique) solution f of (0.2) with $f^0 = \lim_N f_N(0)$. Hence the two choices of $\mathbb{P}$ prove the stability of the problem's two stationary solutions (zero and thermal equilibrium) by Dirac masses.

Further application of this approach would require considering non-invariant measures for $\mathbb{P}$...

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