# Exercises - Chapter 3 (Correction)

**Exercise 1.**
**Let $\mathcal{P}_k$ denote the set of all polynomials of degree less than or equal to $k$ in one variable.**

**Let $\widehat{K} = [0\,,1]$, the following triplets $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ are they finite elements? In the favorable case, give the nodal basis of $\widehat{\mathcal{P}}$.**

**(a)** $\widehat{\mathcal{P}} = \mathcal{P}_1$**,** $\widehat{\mathcal{N}} = \{N_1\,, N_2\}$ **where** $N_1(v) = v(0)$ **and** $N_2(v) = v(1)$**.**

**Answer**
• $\widehat{K} \subset \mathbb{R}$ is a bounded closed set, $\mathring{\widehat{K}} \neq \emptyset$, $\partial\widehat{K}$ is smooth.
• $\dim \mathcal{P}_1 = 2$.
• $\dim \mathcal{N} = 2 = \dim \widehat{\mathcal{P}}'$ the dual of $\mathcal{P}_1$.
Let $\alpha_1$, $\alpha_2$ be reals such that $\alpha_1 N_1(v) + \alpha_2 N_2(v) = 0\ \forall v \in \widehat{\mathcal{P}}$. Then, for $v \in \widehat{\mathcal{P}}$ such that $v(0) \neq 0$ and $v(1) = 0$, one gets $\alpha_1 = 0$. By choosing $v \in \widehat{\mathcal{P}}$ such that $v(0) = 0$ and $v(1) \neq 0$, one gets $\alpha_2 = 0$. Then $\widehat{\mathcal{N}}$ is a basis of $\widehat{\mathcal{P}}'$.
Therefore $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ is a finite element.

• **The nodal basis** $(\widehat{\phi}_1\,,\widehat{\phi}_2)$ can be calculated by using the barycentric coordinates $\widehat{\lambda}_1$, $\widehat{\lambda}_2$ associated to the nodes $\widehat{a}_1 = 0$, $\widehat{a}_2 = 1$, that is

$$\left\{\begin{array}{ccccc} \widehat{\lambda}_1(\widehat{x}) & + & \widehat{\lambda}_2(\widehat{x}) & = & 1\,, \\ \widehat{\lambda}_1(\widehat{x})\,O\widehat{a}_1 & + & \widehat{\lambda}_2(\widehat{x})\,O\widehat{a}_2 & = & O\widehat{M} = \widehat{x}\,, \end{array}\right.$$

where $O$ is the origine, $O\widehat{a}_i$ the vector whose ends are $O$ and $\widehat{a}_i$, $O\widehat{M}$ the vector whose ends are $O$ and $\widehat{M}$, $\widehat{M}$ being the current point.
Since $O\widehat{a}_1 = 0$ and $O\widehat{a}_2 = 1$, on gets $\widehat{\lambda}_2(\widehat{x}) = \widehat{x}$ and $\widehat{\lambda}_1(\widehat{x}) = 1 - \widehat{x}$. Therefore the nodal basis $(\widehat{\phi}_1\,,\widehat{\phi}_2)$ is given by

$$\left\{\begin{array}{ccccc} \widehat{\phi}_1(\widehat{x}) & = & \widehat{\lambda}_1(\widehat{x}) & = & 1-\widehat{x} \\ \widehat{\phi}_2(\widehat{x}) & = & \widehat{\lambda}_2(\widehat{x}) & = & \widehat{x} \end{array}\right. \iff \left\{\begin{array}{ccc} \widehat{\phi}_1 & = & \widehat{\lambda}_1\,, \\ \widehat{\phi}_2 & = & \widehat{\lambda}_2\,. \end{array}\right.$$

**(b)** $\widehat{\mathcal{P}} = \mathcal{P}_2$**,** $\widehat{\mathcal{N}} = \{N_1\,, N_2\,, N_3\}$ **where** $N_1(v) = v(0)$**,** $N_2(v) = v(1)$ **and** $N_3(v) = v(1/2)$**.**

**Answer**
• $\widehat{K} \subset \mathbb{R}$ is a bounded closed set, $\mathring{\widehat{K}} \neq \emptyset$, $\partial\widehat{K}$ is smooth.
• $\dim \mathcal{P}_2 = 3$.
• $\dim \mathcal{N} = 3 = \dim \widehat{\mathcal{P}}'$ the dual of $\mathcal{P}_2$.
Let $\alpha_1$, $\alpha_2$, $\alpha_3$ be reals, $v \in \mathcal{P}_2$, $v(\widehat{x}) = \alpha_3\,\widehat{x}^2 + \alpha_2\,\widehat{x} + \alpha_1$ such that $N_1(v) = 0$, $N_2(v) = 0$, $N_3(v) = 0$. Then

$$\left\{\begin{array}{c} \alpha_1 = 0 \\ \alpha_3 + \alpha_2 = 0 \\ \dfrac{1}{4}\,\alpha_3 + \dfrac{1}{2}\,\alpha_2 = 0 \end{array}\right. \iff \left\{\begin{array}{l} \alpha_1 = 0 \\ \alpha_2 = 0 \\ \alpha_3 = 0 \end{array}\right. \implies v = 0\,.$$

Then the mapping $\widehat{\mathcal{P}} \to \mathbb{R}^3$, $v \mapsto (N_1(v)\,, N_2(v)\,, N_3(v))$ is one-by-one, in turn $\widehat{\mathcal{N}}$ is a basis of $\widehat{\mathcal{P}}'$.
Therefore $(\widehat{K}\,, \widehat{\mathcal{P}}\,, \widehat{\mathcal{N}})$ is a finite element.

• **The nodal basis** $(\widehat{\phi}_1\,, \widehat{\phi}_2\,, \widehat{\phi}_3)$ can be computed by using the characterization $\widehat{N}_i(\widehat{\phi}_j) = \delta_{ij}$ the Krönecker symbole. This is equivalent to

$$\left\{\begin{array}{lcl} \widehat{\phi}_1(0) &=& 1 \\ \widehat{\phi}_1(1) &=& 0 \\ \widehat{\phi}_1(1/2) &=& 0 \end{array}\right. \quad ; \quad \left\{\begin{array}{lcl} \widehat{\phi}_2(0) &=& 0 \\ \widehat{\phi}_2(1) &=& 1 \\ \widehat{\phi}_2(1/2) &=& 0 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lcl} \widehat{\phi}_3(0) &=& 0 \\ \widehat{\phi}_3(1) &=& 0 \\ \widehat{\phi}_3(1/2) &=& 1 \end{array}\right.$$

Then

$$\begin{cases} \widehat{\phi}_1(\widehat{x}) = 2\,(\widehat{x}-1)\,(\widehat{x}-\dfrac{1}{2}) \\ \widehat{\phi}_2(\widehat{x}) = 2\,\widehat{x}\,(\widehat{x}-\dfrac{1}{2}) \\ \widehat{\phi}_3(\widehat{x}) = -4\,\widehat{x}\,(\widehat{x}-1) \end{cases} \iff \begin{cases} \widehat{\phi}_1 = 2\,\widehat{\lambda}_1\,(\widehat{\lambda}_1-\dfrac{1}{2})\,, \\ \widehat{\phi}_2 = \widehat{\lambda}_2\,(\widehat{\lambda}_2-\dfrac{1}{2})\,, \\ \widehat{\phi}_3 = 4\,\widehat{\lambda}_1\,\widehat{\lambda}_2\,, \end{cases}$$

where $\widehat{\lambda}_1$, $\widehat{\lambda}_2$ are the barycentric coordinates associated to the nodes $\widehat{a}_1 = 0$, $\widehat{a}_2 = 1$.

**(c)** $\widehat{\mathcal{P}} = \mathcal{P}_3$**,** $\widehat{\mathcal{N}} = \{N_1\,, N_2\,, N_3\,, N_4\}$ **where** $N_1(v) = v(0)$**,** $N_2(v) = v(1)$**,** $N_3(v) = v(1/3)$ **and** $N_4(v) = v(2/3)$**.**

**Answer**
• $\widehat{K} \subset \mathbb{R}$ is a bounded closed set, $\mathring{\widehat{K}} \neq \emptyset$, $\partial\widehat{K}$ is smooth.
• $\dim \mathcal{P}_3 = 4$.
• $\dim \mathcal{N} = 4 = \dim \widehat{\mathcal{P}}'$ the dual of $\mathcal{P}_3$.
Let $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\alpha_4$ be reals, $v \in \mathcal{P}_3$, $v(\widehat{x}) = \alpha_4\,\widehat{x}^3 + \alpha_3\,\widehat{x}^2 + \alpha_2\,\widehat{x} + \alpha_1$ such that $N_1(v) = 0$, $N_2(v) = 0$, $N_3(v) = 0$ and $N_4(v) = 0$. Then $v = 0$ since $v \in \mathcal{P}_3$ owns 4 distinct zeros $0\,, 1/3\,, 1/2\,, 1$. Then $\widehat{\mathcal{N}}$ is a basis of $\widehat{\mathcal{P}}'$.
Therefore $(\widehat{K}\,, \widehat{\mathcal{P}}\,, \widehat{\mathcal{N}})$ is a finite element.

• **The nodal basis** $(\widehat{\phi}_1\,, \widehat{\phi}_2\,, \widehat{\phi}_3\,, \widehat{\phi}_4)$ is given by

$$\begin{cases} \widehat{\phi}_1(\widehat{x}) = \dfrac{9}{2}\,(\widehat{x}-\dfrac{1}{3})\,(\widehat{x}-\dfrac{2}{3})\,(\widehat{x}-1) \\ \widehat{\phi}_2(\widehat{x}) = \dfrac{9}{2}\,\widehat{x}\,(\widehat{x}-\dfrac{1}{3})\,(\widehat{x}-\dfrac{2}{3}) \\ \widehat{\phi}_3(\widehat{x}) = \dfrac{27}{2}\,\widehat{x}\,(\widehat{x}-\dfrac{2}{3})\,(\widehat{x}-1) \\ \widehat{\phi}_4(\widehat{x}) = \dfrac{27}{2}\,\widehat{x}\,(\widehat{x}-\dfrac{1}{3})\,(\widehat{x}-1) \end{cases} \iff \begin{cases} \widehat{\phi}_1 = \dfrac{9}{2}\,\widehat{\lambda}_1\,(\widehat{\lambda}_1-\dfrac{1}{3})\,(\widehat{\lambda}_1-\dfrac{2}{3})\,, \\ \widehat{\phi}_2 = \dfrac{9}{2}\,\widehat{\lambda}_2\,(\widehat{\lambda}_2-\dfrac{1}{3})\,(\widehat{\lambda}_2-\dfrac{2}{3}) \\ \widehat{\phi}_3 = \widehat{\lambda}_1\,\widehat{\lambda}_2\,(\dfrac{27}{2}\widehat{\lambda}_1-\dfrac{9}{2})\,, \\ \widehat{\phi}_4 = \widehat{\lambda}_1\,\widehat{\lambda}_2\,(\dfrac{27}{2}\widehat{\lambda}_2-\dfrac{9}{2})\,, \end{cases}$$

where $\widehat{\lambda}_1$, $\widehat{\lambda}_2$ are the barycentric coordinates associated to the nodes $\widehat{a}_1 = 0$, $\widehat{a}_2 = 1$.

**(d)** $\widehat{\mathcal{P}} = \mathcal{P}_0$**,** $\widehat{\mathcal{N}} = \{N_1\}$ **where** $N_1(v) = \int_0^1 v(x)\,dx$**.**

**Answer**
• $\widehat{K} \subset \mathbb{R}$ is a bounded closed set, $\mathring{\widehat{K}} \neq \emptyset$, $\partial\widehat{K}$ is smooth.
• $\dim \mathcal{P}_0 = 1$.
• $\dim \mathcal{N} = 1 = \dim \widehat{\mathcal{P}}'$ the dual of $\mathcal{P}_0$.
Let $\alpha_1$ be a real, $v \in \mathcal{P}_0$, $v(\widehat{x}) = \alpha_1$ such that $N_1(v) = 0$. Then $v = 0$ since $0 = \int_0^1 \alpha_1\,dx = \alpha_1$. Then $\widehat{\mathcal{N}}$ is a basis of $\widehat{\mathcal{P}}'$.

Therefore $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ is a finite element.

• **The nodal basis** $\{\widehat{\phi}_1\}$ where
$$\widehat{\phi}_1(\widehat{x}) = 1\,.$$

**(e)** $\widehat{\mathcal{P}} = \mathcal{P}_3$**,** $\widehat{\mathcal{N}} = \{N_1\,,N_2\,,N_3\,,N_4\}$ **where** $N_1(v) = v(0)$**,** $N_2(v) = v(1)$**,** $N_3(v) = v'(0)$ **and** $N_4(v) = v'(1)$**.**

**Answer**
• $\widehat{K} \subset \mathbb{R}$ is a bounded closed set, $\mathring{\widehat{K}} \neq \emptyset$, $\partial\widehat{K}$ is smooth.
• $\dim \mathcal{P}_3 = 4$.
• $\dim \mathcal{N} = 4 = \dim \widehat{\mathcal{P}}'$ the dual of $\mathcal{P}_3$.
Let $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\alpha_4$ be reals, $v \in \mathcal{P}_3$, $v(\widehat{x}) = \alpha_4\,\widehat{x}^3 + \alpha_3\,\widehat{x}^2 + \alpha_2\,\widehat{x} + \alpha_1$ such that $N_1(v) = 0$, $N_2(v) = 0$, $N_3(v) = 0$. Then $v = 0$ since $v \in \mathcal{P}_3$ owns zeros $0\,,1$, the multiplicity order of each zero is 2. Then $\widehat{\mathcal{N}}$ is a basis of $\widehat{\mathcal{P}}'$.
Therefore $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ is a finite element.

• **The nodal basis** $(\widehat{\phi}_1\,,\widehat{\phi}_2\,,\widehat{\phi}_3\,,\widehat{\phi}_4)$ is given by

$$\begin{cases} \widehat{\phi}_1(\widehat{x}) = (2\,\widehat{x}+1)\,(\widehat{x}-1)^2 \\ \widehat{\phi}_2(\widehat{x}) = (-2\,\widehat{x}+3)\,\widehat{x}^2 \\ \widehat{\phi}_3(\widehat{x}) = \widehat{x}\,(\widehat{x}-1)^2 \\ \widehat{\phi}_4(\widehat{x}) = \widehat{x}^2\,(\widehat{x}-1) \end{cases} \iff \begin{cases} \widehat{\phi}_1 = \widehat{\lambda}_1^2\,(3-2\widehat{\lambda}_1)\,, \\ \widehat{\phi}_2 = \widehat{\lambda}_2^2\,(3-2\widehat{\lambda}_2) \\ \widehat{\phi}_3 = \widehat{\lambda}_1^2\,\widehat{\lambda}_2 \\ \widehat{\phi}_4 = \widehat{\lambda}_1\,\widehat{\lambda}_2^2 \end{cases}$$

where $\widehat{\lambda}_1$, $\widehat{\lambda}_2$ are the barycentric coordinates associated to the nodes $\widehat{a}_1 = 0$, $\widehat{a}_2 = 1$.

**Let** $a$**,** $b$ **be reals and** $K = [a\,,b]$**. In each above example where** $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ **is a finite element, give a finite element** $(K\,,\mathcal{P}\,,\mathcal{N})$ **affine equivalent to** $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$**.**

**Answer**
Let $F_K$ be the following mapping $\widehat{K} \to K$, $\widehat{x} \mapsto x = F_K(\widehat{x}) = (b-a)\,\widehat{x} + a$.
The mapping $F_K$ is affine.
One has $F_K(0) = a$, $F_K(1) = b$.
$F_K(\widehat{K}) = K$: $\widehat{K}$ and $K$ are affine equivalent.
Regarding the evaluation of a function $v$ on $K$, one can use the following scheme

$$\widehat{K} \xrightarrow{F_K} K \xrightarrow{v} \mathbb{R}\,,$$

so that
$$\widehat{v}(\widehat{x}) = (v \circ F_K)(\widehat{x}) = v(F_K(\widehat{x})) = v(x)\,.$$
Then $\widehat{v}(\widehat{a}_i) = v(a_i)$ where $F_K(\widehat{a}_i) = a_i$.
From $\widehat{v}(\widehat{x}) = (v \circ F_K)(\widehat{x})$ the derivatives can be computed as follows

$$\widehat{v}'(\widehat{x}) = v'(F_K(\widehat{x})) \cdot F_K'(\widehat{x}) = v'(F_K(\widehat{x})) \cdot (b-a)\,.$$

Then $\widehat{v}'(\widehat{a}_i) = v'(a_i) \cdot (b-a)$ where $F_K(\widehat{a}_i) = a_i$
The barycentric coordinates associated to the edges $a$ and $b$ of $K$ are defined by

$$\lambda(x) = (\widehat{\lambda} \circ F_K^{-1})(x) = \widehat{\lambda}(F_K^{-1}(x)) = \widehat{\lambda}\left(\frac{x-a}{b-a}\right).$$

The above formula uses the scheme $K \overset{F_K{}^{-1}}{\longrightarrow} \widehat{K} \overset{\widehat{\lambda}}{\longrightarrow} \mathbb{R}$.
Then

$$\begin{cases} \lambda_1(x) = 1 - \dfrac{x-a}{b-a}\,, \\ \lambda_2(x) = \dfrac{x-a}{b-a}\,, \end{cases}$$

Therefore
(a) $(K\,,\mathcal{P}\,,\mathcal{N})$ is affine equivalent to $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ where
• $\mathcal{P} = \mathcal{P}_1$, and the nodal basis $(\phi_1\,,\phi_2\,)$ is given by,

$$\begin{cases} \phi_1(x) = \lambda_1(x)\,, \\ \phi_2(x) = \lambda_2(x)\,, \end{cases}$$

where $\lambda_1$, $\lambda_2$ are the barycentric coordinates associated to the nodes $a$, $b$.
• $\mathcal{N} = \{N_1\,, N_2\}$ with $N_1(v) = v(a)$ and $N_2(v) = v(b)$.

(b) $(K\,,\mathcal{P}\,,\mathcal{N})$ is affine equivalent to $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ where
• $\mathcal{P} = \mathcal{P}_2$, and the nodal basis $(\phi_1\,,\phi_2\,,\phi_3)$ is given by,

$$\begin{cases} \phi_1 = 2\,\lambda_2\,(\lambda_1 - \dfrac{1}{2})\,, \\ \phi_2 = 2\,\lambda_2\,(\lambda_2 - \dfrac{1}{2})\,, \\ \phi_3 = 4\,\lambda_1\,\lambda_2\,, \end{cases}$$

where $\lambda_1$, $\lambda_2$ are the barycentric coordinates associated to the nodes $a$, $b$.
• $\mathcal{N} = \{N_1\,, N_2\,, N_3\}$ with $N_1(v) = v(a)$, $N_2(v) = v(b)$ and $N_3(v) = v\Big(\dfrac{a+b}{2}\Big)$.
(c) $(K\,,\mathcal{P}\,,\mathcal{N})$ is affine equivalent to $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ where
• $\mathcal{P} = \mathcal{P}_3$, and the nodal basis $(\phi_1\,,\phi_2\,,\phi_3\,,\phi_4)$ is given by,

$$\begin{cases} \phi_1 = \dfrac{9}{2}\lambda_1\,(\lambda_1 - \dfrac{1}{3})\,(\lambda_1 - \dfrac{2}{3})\,, \\ \phi_2 = \dfrac{9}{2}\lambda_2\,(\lambda_2 - \dfrac{1}{3})\,(\lambda_2 - \dfrac{2}{3}) \\ \phi_3 = \lambda_1\,\lambda_2\,(\dfrac{27}{2}\lambda_1 - \dfrac{9}{2})\,, \\ \phi_4 = \lambda_1\,\lambda_2\,(\dfrac{27}{2}\lambda_2 - \dfrac{9}{2})\,, \end{cases}$$

where $\lambda_1$, $\lambda_2$ are the barycentric coordinates associated to the nodes $a$, $b$.
• $\mathcal{N} = \{N_1\,, N_2\,, N_3\,, N_4\}$ with
$N_1(v) = v(a)$, $N_2(v) = v(b)$, $N_3(v) = v\Big(a + \dfrac{1}{3}(b-a)\Big)$ and $N_4(v) = v\Big(a + \dfrac{2}{3}(b-a)\Big)$.

(d) $(K\,,\mathcal{P}\,,\mathcal{N})$ is affine equivalent to $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ where
• $\mathcal{P} = \mathcal{P}_0$, and the nodal basis $\phi_1$ is given by,

$$\phi_1 = 1\,.$$

• $\mathcal{N} = \{N_1\}$ where $N_1(v) = \dfrac{\int_a^b v(x)\,dx}{b-a}$.

(e) $(K\,,\mathcal{P}\,,\mathcal{N})$ is affine equivalent to $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ where
• $\mathcal{P}=\mathcal{P}_3$, and the nodal basis $(\phi_1\,,\phi_2\,,\phi_3\,,\phi_4)$ is given by,

$$\begin{cases} \phi_1 = \lambda_1^2\,(3-2\lambda_1)\,, \\ \phi_2 = \lambda_2^2\,(3-2\lambda_2)\,, \\ \phi_3 = \lambda_1^2\,\lambda_2\,, \\ \phi_4 = \lambda_1\,\lambda_2^2\,, \end{cases}$$

where $\lambda_1$, $\lambda_2$ are the barycentric coordinates associated to the nodes $a$, $b$.
• $\mathcal{N}=\{N_1\,,N_2\,,N_3\,,N_4\}$ with $N_1(v)=v(a)$, $N_2(v)=v(b)$, $N_3(v)=v'(a)\cdot(b-a)$ and $N_4(v)=v'(b)\cdot(b-a)$.

**Exercise 2.**
**Let $\widehat{K}$ be the triangle whose vertices are the points $\widehat{a}_1=(0\,,0)$, $\widehat{a}_2=(1\,,0)$ and $\widehat{a}_3=(0\,,1)$ *i.e.* $\widehat{K}=\{(0\,,0)\,,(1\,,0)\,,(0\,,1)\}$, $\widehat{m}_i$ denote the midpoints of his edges, according to $\widehat{m}_1$ is the midpoint of the edge $(\widehat{a}_2\,,\widehat{a}_3)$, $\widehat{m}_2$ is the midpoint of $(\widehat{a}_3\,,\widehat{a}_1)$, $\widehat{m}_3$ is the midpoint of $(\widehat{a}_1\,,\widehat{a}_2)$.**

**The following triplets $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ are they finite elements? In the favorable case, give the nodal basis of $\widehat{\mathcal{P}}$.**

**(a) $\widehat{\mathcal{P}}=\mathcal{P}_1$, $\widehat{\mathcal{N}}=\{N_1\,,N_2\,,N_3\}$ where $N_i(v)=v(\widehat{a}_i)\,,i=1\,,2\,,3$.**

**Answer**
• $\widehat{K}\subset\mathbb{R}^2$ is a bounded closed set, $\mathring{\widehat{K}}\neq\emptyset$, $\partial\widehat{K}$ is smooth.
• $\dim\mathcal{P}_1=3$.
• $\dim\mathcal{N}=3=\dim\widehat{\mathcal{P}}'$ the dual of $\mathcal{P}_1$.
Let $\alpha_1$, $\alpha_2$, $\alpha_3$ be reals, $v\in\mathcal{P}_1$, $v(\widehat{x}\,,\widehat{y})=\alpha_3\,\widehat{x}+\alpha_2\,\widehat{y}+\alpha_1$ such that $N_1(v)=0$, $N_2(v)=0$, $N_3(v)=0$. Then $\alpha_1=0$, $\alpha_2=0$ and $\alpha_3=0$. Then $\widehat{\mathcal{N}}$ is a basis of $\widehat{\mathcal{P}}'$.
Therefore $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ is a finite element.

• **The nodal basis** $(\widehat{\phi}_1\,,\widehat{\phi}_2\,,\widehat{\phi}_3)$ is given by

$$\begin{cases} \widehat{\phi}_1(\widehat{x}\,,\widehat{y}) = 1-\widehat{x}-\widehat{y} \\ \widehat{\phi}_2(\widehat{x}\,,\widehat{y}) = \widehat{x} \\ \widehat{\phi}_3(\widehat{x}\,,\widehat{y}) = \widehat{y} \end{cases} \iff \begin{cases} \widehat{\phi}_1 = \widehat{\lambda}_1\,, \\ \widehat{\phi}_2 = \widehat{\lambda}_2\,, \\ \widehat{\phi}_3 = \widehat{\lambda}_3\,, \end{cases}$$

where $\widehat{\lambda}_1$, $\widehat{\lambda}_2$ and $\widehat{\lambda}_3$ are the barycentric coordinates associated to the nodes $\widehat{a}_1=(0\,,0)$, $\widehat{a}_2=(1\,,0)$ and $\widehat{a}_3=(0\,,1)$.

**(b) $\widehat{\mathcal{P}}=\mathcal{P}_1$, $\widehat{\mathcal{N}}=\{N_1\,,N_2\,,N_3\}$ where $N_i(v)=v(\widehat{m}_i)\,,i=1\,,2\,,3$.**

**Answer**
• $\widehat{K}\subset\mathbb{R}^2$ is a bounded closed set, $\mathring{\widehat{K}}\neq\emptyset$, $\partial\widehat{K}$ is smooth.
• $\dim\mathcal{P}_1=3$.
• $\dim\mathcal{N}=3=\dim\widehat{\mathcal{P}}'$ the dual of $\mathcal{P}_1$.
Let $\alpha_1$, $\alpha_2$, $\alpha_3$ be reals, $v\in\mathcal{P}_1$, $v(\widehat{x}\,,\widehat{y})=\alpha_3\,\widehat{x}+\alpha_2\,\widehat{y}+\alpha_1$ such that $N_1(v)=0$, $N_2(v)=0$, $N_3(v)=0$. Then $\alpha_1=0$, $\alpha_2=0$ and $\alpha_3=0$. Then $\widehat{\mathcal{N}}$ is a basis of $\widehat{\mathcal{P}}'$.
Therefore $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ is a finite element.

• **The nodal basis** $(\widehat{\phi}_1 , \widehat{\phi}_2 , \widehat{\phi}_3)$ is given by

$$\begin{cases} \widehat{\phi}_1(\widehat{x} , \widehat{y}) = 2\widehat{x} + 2\widehat{y} - 1 \\ \widehat{\phi}_2(\widehat{x} , \widehat{y}) = -2\widehat{x} + 1 \\ \widehat{\phi}_3(\widehat{x} , \widehat{y}) = -2\widehat{y} + 1 \end{cases} \iff \begin{cases} \widehat{\phi}_1 = 1 - 2\widehat{\lambda}_1 , \\ \widehat{\phi}_2 = 1 - 2\widehat{\lambda}_2 , \\ \widehat{\phi}_3 = 1 - 2\widehat{\lambda}_3 , \end{cases}$$

where $\widehat{\lambda}_1$, $\widehat{\lambda}_2$ and $\widehat{\lambda}_3$ are the barycentric coordinates associated to the nodes $\widehat{a}_1 = (0 , 0)$, $\widehat{a}_2 = (1 , 0)$ and $\widehat{a}_3 = (0 , 1)$.

**(c)** $\widehat{\mathcal{P}} = \mathcal{P}_2$**,** $\widehat{\mathcal{N}} = \{N_1 , N_2 , N_3 , N_4 , N_5 , N_6\}$ **where** $N_i(v) = v(\widehat{a}_i) , i = 1 , 2 , 3$ **and** $N_i(v) = v(\widehat{m}_{i-3}) , i = 4 , 5 , 6$**.**

**Answer**
• $\widehat{K} \subset \mathbb{R}^2$ is a bounded closed set, $\mathring{\widehat{K}} \neq \emptyset$, $\partial\widehat{K}$ is smooth.
• $\dim \mathcal{P}_2 = 6$.
• $\dim \mathcal{N} = 6 = \dim \widehat{\mathcal{P}}'$ the dual of $\mathcal{P}_2$.
Let $L_1$, $L_2$, $L_3$, $L_4$, $L_5$, $L_6$ be linear functions which define the edges $(\widehat{a}_1 , \widehat{a}_2)$, $(\widehat{a}_2 , \widehat{a}_3)$, $(\widehat{m}_1 , \widehat{m}_2)$, $(\widehat{m}_2 , \widehat{m}_3)$, $(\widehat{m}_3 , \widehat{m}_1)$, respectively, that is $L_i(\widehat{x} , \widehat{y}) = 0$ for $i = 1, \dots, 6$ on the corresponding edge. The functions $L_i$ are

$L_1(\widehat{x} , \widehat{y}) = \widehat{y}$, $L_2(\widehat{x} , \widehat{y}) = 1 - \widehat{x} - \widehat{y}$, $L_3(\widehat{x} , \widehat{y}) = \widehat{x}$, $L_4(\widehat{x} , \widehat{y}) = \dfrac{1}{2} - \widehat{y}$, $L_5(\widehat{x} , \widehat{y}) = \dfrac{1}{2} - \widehat{x} - \widehat{y}$, $L_6(\widehat{x} , \widehat{y}) = \dfrac{1}{2} - \widehat{x}$.

Let $v \in \mathcal{P}_2$ such that $N_1(v) = 0$, $N_2(v) = 0$, $N_3(v) = 0$, $N_4(v) = 0$, $N_5(v) = 0$, $N_6(v) = 0$ *i.e.* $v(\widehat{a}_i) = 0$, $i = 1 , 2 , 3$ and $v(\widehat{m}_i) = 0$, $i = 1 , 2 , 3$. The polynomial $v$ vanishes at two points of edges $(\widehat{a}_2 , \widehat{a}_3)$, $(\widehat{m}_2 , \widehat{m}_3)$. Then there exists a constant $c$ such that $v = c\, L_2\, L_5$. But $v$ vanishes at $\widehat{a}_1$:

$$0 = v(0 , 0) = c\, L_2(0 , 0)\, L_5(0 , 0) \Longrightarrow c = 0 ,$$

since $L_2(0 , 0) \neq 0$ and $L_5(0 , 0) \neq 0$.
Then $v = 0$ and $\widehat{\mathcal{N}}$ is a basis of $\widehat{\mathcal{P}}'$.
Therefore $(\widehat{K} , \widehat{\mathcal{P}} , \widehat{\mathcal{N}})$ is a finite element.

• **The nodal basis** $(\widehat{\phi}_1 , \widehat{\phi}_2 , \widehat{\phi}_3 , \widehat{\phi}_4 , \widehat{\phi}_5 , \widehat{\phi}_6)$ is given by

$$\begin{cases} \widehat{\phi}_1(\widehat{x} , \widehat{y}) = 2\,(\frac{1}{2} - \widehat{x} - \widehat{y})\,(1 - \widehat{x} - \widehat{y}) \\ \widehat{\phi}_2(\widehat{x} , \widehat{y}) = -2\,(\frac{1}{2} - \widehat{x})\,\widehat{x} \\ \widehat{\phi}_3(\widehat{x} , \widehat{y}) = -2\,\widehat{y}\,(\frac{1}{2} - \widehat{y}) \\ \widehat{\phi}_4(\widehat{x} , \widehat{y}) = 4\,\widehat{x}\,\widehat{y} \\ \widehat{\phi}_5(\widehat{x} , \widehat{y}) = 4\,\widehat{y}\,(1 - \widehat{x} - \widehat{y}) \\ \widehat{\phi}_6(\widehat{x} , \widehat{y}) = 4\,\widehat{x}\,(1 - \widehat{x} - \widehat{y}) \end{cases} \iff \begin{cases} \widehat{\phi}_1 = \widehat{\lambda}_1\,(2\widehat{\lambda}_1 - 1) , \\ \widehat{\phi}_2 = \widehat{\lambda}_2\,(2\widehat{\lambda}_2 - 1) , \\ \widehat{\phi}_3 = \widehat{\lambda}_3\,(2\widehat{\lambda}_3 - 1) , \\ \widehat{\phi}_4 = 4\,\widehat{\lambda}_2\,\widehat{\lambda}_3 , \\ \widehat{\phi}_5 = 4\,\widehat{\lambda}_3\,\widehat{\lambda}_1 , \\ \widehat{\phi}_6 = 4\,\widehat{\lambda}_1\,\widehat{\lambda}_2 , \end{cases}$$

where $\widehat{\lambda}_1$, $\widehat{\lambda}_2$ and $\widehat{\lambda}_3$ are the barycentric coordinates associated to the nodes $\widehat{a}_1 = (0 , 0)$, $\widehat{a}_2 = (1 , 0)$ and $\widehat{a}_3 = (0 , 1)$.

**(d)** $\widehat{\mathcal{P}} = \mathcal{P}_0$**,** $\widehat{\mathcal{N}} = \{N_1\}$ **where** $N_1(v) = \dfrac{\int_{\widehat{K}} v(\widehat{x} , \widehat{y})\, d\widehat{x}\, d\widehat{y}}{|\widehat{K}|}$**.**

**Answer**

• $\widehat{K} \subset \mathbb{R}$ is a bounded closed set, $\overset{\circ}{\widehat{K}} \neq \emptyset$, $\partial\widehat{K}$ is smooth.
• $\dim \mathcal{P}_0 = 1$.
• $\dim \mathcal{N} = 1 = \dim \widehat{\mathcal{P}}'$ the dual of $\mathcal{P}_0$.
Let $\alpha_1$ be a real, $v \in \mathcal{P}_0$, $v(\widehat{x}\,,\widehat{y}) = \alpha_1$ such that $N_1(v) = 0$. Then $v = 0$ since $0 = \int_0^1 \alpha_1\, d\widehat{x}\, d\widehat{y} = \alpha_1$. Then $\widehat{\mathcal{N}}$ is a basis of $\widehat{\mathcal{P}}'$.
Therefore $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ is a finite element.

• **The nodal basis** $\{\widehat{\phi}_1\}$ where

$$\widehat{\phi}_1(\widehat{x}\,,\widehat{y}) = 1\,.$$

**(e)** $\widehat{\mathcal{P}} = \mathcal{P}_2$**,** $\widehat{\mathcal{N}} = \{N_1\,,N_2\,,N_3\,,N_4\,,N_5\,,N_6\}$ **where** $N_i(v) = v(\widehat{a}_i)\,, i = 1\,,2\,,3$ **and** $N_i(v) = \boldsymbol{\nabla}\, v(\widehat{m}_{i-3}) \cdot \widehat{m}_{i-3}\widehat{a}_{i-3}\,, i = 4\,,5\,,6$**, with** $\boldsymbol{\nabla}$ **the gradient operator,** $\widehat{m}_{i-3}\widehat{a}_{i-3}$ **the vector whose ends are** $\widehat{m}_{i-3}\,,\widehat{a}_{i-3}$ **for** $i = 4\,,5\,,6$**.**

**Answer**
• $\widehat{K} \subset \mathbb{R}^2$ is a bounded closed set, $\overset{\circ}{\widehat{K}} \neq \emptyset$, $\partial\widehat{K}$ is smooth.
• $\dim \mathcal{P}_2 = 6$.
• $\dim \mathcal{N} = 6 = \dim \widehat{\mathcal{P}}'$ the dual of $\mathcal{P}_2$.
Let $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$, $\alpha_5$, $\alpha_6$ be reals, $v \in \mathcal{P}_2$, $v(\widehat{x}\,,\widehat{y}) = \alpha_6\widehat{x}^2 + \alpha_5\widehat{y}^2 + \alpha_4\widehat{x}\widehat{y} + \alpha_3\,\widehat{x} + \alpha_2\,\widehat{y} + \alpha_1$ such that $N_1(v) = 0$, $N_2(v) = 0$, $N_3(v) = 0$, $N_4(v) = 0$, $N_5(v) = 0$, $N_6(v) = 0$. Then $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\alpha_4 = 0$, $\alpha_5 = 0$ and $\alpha_6 = 0$. Then $\widehat{\mathcal{N}}$ is a basis of $\widehat{\mathcal{P}}'$.
Therefore $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ is a finite element.

• **The nodal basis** $(\widehat{\phi}_1\,,\widehat{\phi}_2\,,\widehat{\phi}_3\,,\widehat{\phi}_4\,,\widehat{\phi}_5\,,\widehat{\phi}_6)$ is given by

$$\begin{cases} \widehat{\phi}_1(\widehat{x}\,,\widehat{y}) = \frac{1}{2}\widehat{x}^2 + \frac{1}{2}\widehat{y}^2 + 2\widehat{x}\,\widehat{y} - \frac{3}{2}\widehat{x} - \frac{3}{2}\widehat{y} + 1 \\ \widehat{\phi}_2(\widehat{x}\,,\widehat{y}) = \frac{1}{2}\widehat{x}^2 - \widehat{y}^2 - \widehat{x}\,\widehat{y} + \frac{1}{2}\widehat{x} + \widehat{y} \\ \widehat{\phi}_3(\widehat{x}\,,\widehat{y}) = -\widehat{x}^2 + \frac{1}{2}\widehat{y}^2 - \widehat{x}\,\widehat{y} + \widehat{x} + \frac{1}{2}\widehat{y} \\ \widehat{\phi}_4(\widehat{x}\,,\widehat{y}) = -\widehat{x}^2 - \widehat{y}^2 - 2\widehat{x}\,\widehat{y} + \widehat{x} + \widehat{y} \\ \widehat{\phi}_5(\widehat{x}\,,\widehat{y}) = -\widehat{x}^2 + \widehat{x} \\ \widehat{\phi}_6(\widehat{x}\,,\widehat{y}) = -\widehat{y}^2 + \widehat{y} \end{cases} \iff \begin{cases} \widehat{\phi}_1 = \frac{1}{2}(\widehat{\lambda}_1^2 + 2\,\widehat{\lambda}_2\widehat{\lambda}_3 + \widehat{\lambda}_1)\,, \\ \widehat{\phi}_2 = \frac{1}{2}(\widehat{\lambda}_2^2 + 2\,\widehat{\lambda}_3\widehat{\lambda}_1 + \widehat{\lambda}_2)\,, \\ \widehat{\phi}_3 = \frac{1}{2}(\widehat{\lambda}_3^2 + 2\,\widehat{\lambda}_1\widehat{\lambda}_2 + \widehat{\lambda}_3)\,, \\ \widehat{\phi}_4 = -\widehat{\lambda}_1^2 + \widehat{\lambda}_1\,, \\ \widehat{\phi}_5 = -\widehat{\lambda}_2^2 + \widehat{\lambda}_2\,, \\ \widehat{\phi}_6 = -\widehat{\lambda}_3^2 + \widehat{\lambda}_3\,, \end{cases}$$

where $\widehat{\lambda}_1$, $\widehat{\lambda}_2$ and $\widehat{\lambda}_3$ are the barycentric coordinates associated to the nodes $\widehat{a}_1 = (0\,,0)$, $\widehat{a}_2 = (1\,,0)$ and $\widehat{a}_3 = (0\,,1)$.

**Let** $a_1$**,** $a_2$**,** $a_3$ **be points in** $\mathbb{R}^2$ **and** $K$ **the triangle whose vertices are the points whose vertices are** $a_1$**,** $a_2$**,** $a_3$**:** $K = \{a_1\,,a_2\,,a_3\}$**. In each above example where** $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ **is a finite element, give a finite element** $(K\,,\mathcal{P}\,,\mathcal{N})$ **affine equivalent to** $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$**.**

**Answer**
Let $F_K$ be the following mapping $\widehat{K} \to K$, $(\widehat{x}\,,\widehat{y}) \mapsto (x\,,y) = F_K(\widehat{x}\,,\widehat{y})$,

$$F_K(\widehat{x}\,,\widehat{y}) = A_K \begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} + b = \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

where $a_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $a_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, $a_3 = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$.

If the triangle $K = \{a_1 \,, a_2 \,, a_3\}$ is non-degenerate the mapping $F_K$ is invertible. One has $\det(A_K) = 2\,Area(K)$.
The barycentric coordinates $\lambda_i : K \to \mathbb{R}$ associated to the edges $a_i$ of $K$ can be computed by using the scheme $K \overset{F_K{}^{-1}}{\longrightarrow} \widehat{K} \overset{\widehat{\lambda}}{\longrightarrow} \mathbb{R}$. Then the barycentric coordinates are defined by

$$\lambda(x\,,y) = (\widehat{\lambda} \circ F_K^{-1})(x\,,y) = \widehat{\lambda}(F_K^{-1}(x\,,y)) = \widehat{\lambda}(\widehat{x}\,,\widehat{y})\,.$$

From $\begin{pmatrix} x \\ y \end{pmatrix} = A_K \begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, one gets $\begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} = A_K^{-1} \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix}$.
Since $A_K^{-1} = \dfrac{1}{\det A_K} \begin{pmatrix} y_3 - y_1 & -(x_3 - x_1) \\ -(y_2 - y_1) & x_2 - x_1 \end{pmatrix}$, one gets

$$\lambda_2(x\,,y) = \widehat{\lambda}_1(\widehat{x}\,,\widehat{y}) = \widehat{x} = \frac{1}{\det A_K}\Big[(y_3 - y_1)(x - x_1) - (x_3 - x_1)(y - y_1)\Big]\,,$$

$$\lambda_3(x\,,y) = \widehat{\lambda}_2(\widehat{x}\,,\widehat{y}) = \widehat{y} = \frac{1}{\det A_K}\Big[-(y_2 - y_1)(x - x_1) + (x_2 - x_1)(y - y_1)\Big]\,,$$

$$\lambda_1(x\,,y) = 1 - \widehat{x} - \widehat{y} = \frac{1}{\det A_K}\Big[(y_2 - y_3)(x - x_2) - (x_2 - x_3)(y - y_2)\Big]\,.$$

Straightforward computations lead to the desired affine equivalents finite elements.

**Exercise 3.**
**Let** $\mathcal{Q}_k = \left\{ \sum_j c_j\, p_j(x)\, q_j(y) \;\; \textbf{such that} \;\; p_j\,, q_j \;\; \textbf{are polynomials of degrees} \;\; \leq k \right\}$**.**

**Let $\widehat{K}$ be the squarre whose vertices are the points $\widehat{a}_1 = (0\,,0)$, $\widehat{a}_2 = (1\,,0)$, $\widehat{a}_3 = (1\,,1)$ and $\widehat{a}_4 = (0\,,1)$, *i.e.* $\widehat{K} = \{(0\,,0)\,,(1\,,0)\,,(1\,,1)\,,(0\,,1)\}$. The midpoints of the edges of this squarre are denoted by $\widehat{a}_i$, $i = 5\,,6\,,7\,,8$, according to $\widehat{a}_5$ is the midpoint of the edge $(\widehat{a}_1\,,\widehat{a}_2)$, $\widehat{a}_6$ is the midpoint of $(\widehat{a}_2\,,\widehat{a}_3)$, $\widehat{a}_7$ is the midpoint of the edge $(\widehat{a}_3\,,\widehat{a}_4)$, $\widehat{a}_8$ is the midpoint of $(\widehat{a}_4\,,\widehat{a}_1)$. The center of the squarre is denoted by $\widehat{a}_9$.**

**(a) Let $\widehat{\mathcal{P}} = \mathcal{Q}_1$, $\widehat{\mathcal{N}} = \{N_1\,,\ldots,N_4\}$ where $N_i(v) = v(\widehat{a}_i)\,, i = 1\,,2\,,3\,,4$. Show that $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ is a finite element.**

**Answer**
Let us recall that the dimension of the space $\mathcal{Q}_k$ is

$$\dim\, \mathcal{Q}_k = (\dim\, \mathcal{P}_k)^2 = (k+1)^2\,.$$

where $\mathcal{P}_k$ denote the set of all polynomials of degree less than or equal to $k$ in one variable.
- $\widehat{K} \subset \mathbb{R}^2$ is a bounded closed set, $\mathring{\widehat{K}} \neq \emptyset$, $\partial\widehat{K}$ is smooth.
- $\dim\, \mathcal{Q}_1 = (1+1)^2 = 4$.
- $\dim\, \mathcal{N} = 4 = \dim\, \widehat{\mathcal{P}}'$ the dual of $\mathcal{Q}_1$.

Let $L_1$, $L_2$, $L_3$, $L_4$ be linear functions in one variable which define edges of the squarre, $(\widehat{a}_1\,,\widehat{a}_2)$, $(\widehat{a}_2\,,\widehat{a}_3)$, $(\widehat{a}_3\,,\widehat{a}_4)$, $(\widehat{a}_4\,,\widehat{a}_1)$, respectively,
$L_1(\widehat{x}\,,\widehat{y}) = \widehat{y}$, $L_2(\widehat{x}\,,\widehat{y}) = 1 - \widehat{x}$, $L_3(\widehat{x}\,,\widehat{y}) = 1 - \widehat{y}$, $L_4(\widehat{x}\,,\widehat{y}) = \widehat{x}$.
Let $v \in \mathcal{Q}_1$ such that $N_1(v) = 0$, $N_2(v) = 0$, $N_3(v) = 0$, $N_4(v) = 0$ *i.e.* $v(\widehat{a}_i) = 0$, $i = 1\,,2\,,3\,,4$. The polynomial $v$ vanishes at two points $\widehat{a}_1\,,\widehat{a}_2$ of the edge $(\widehat{a}_1\,,\widehat{a}_2)$. The polynomial $v$ vanishes also at two points $\widehat{a}_2\,,\widehat{a}_3$ of the edge $(\widehat{a}_2\,,\widehat{a}_3)$. Then there exists a constant $c$ such that $v = c\,L_1\,L_2$. But $v$ vanishes at $\widehat{a}_4$:

$$0 = v(0\,,1) = cL_1(0\,,1)L_2(0\,,1) \Longrightarrow c = 0\,,$$

since $L_1(0\,,1) \neq 0$ and $L_2(0\,,1) \neq 0$.
Then $v = 0$ and $\widehat{\mathcal{N}}$ is a basis of $\widehat{\mathcal{P}}'$.
Therefore $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ is a finite element.

• **The nodal basis** $(\widehat{\phi}_1\,,\widehat{\phi}_2\,,\widehat{\phi}_3\,,\widehat{\phi}_4)$ is given by

$$\left\{\begin{array}{ccccc} \widehat{\phi}_1(\widehat{x}\,,\widehat{y}) & = & (1-\widehat{x})\,(1-\widehat{y}) & = & \widehat{\lambda}_1(\widehat{x}\,,\widehat{y}) \\ \widehat{\phi}_2(\widehat{x}\,,\widehat{y}) & = & \widehat{x}\,(1-\widehat{y}) & = & \widehat{\lambda}_2(\widehat{x}\,,\widehat{y}) \\ \widehat{\phi}_3(\widehat{x}\,,\widehat{y}) & = & \widehat{x}\,\widehat{y} & = & \widehat{\lambda}_3(\widehat{x}\,,\widehat{y}) \\ \widehat{\phi}_4(\widehat{x}\,,\widehat{y}) & = & (1-\widehat{x})\,\widehat{y} & = & \widehat{\lambda}_2(\widehat{x}\,,\widehat{y}) \end{array}\right. \quad\Longleftrightarrow\quad \left\{\begin{array}{l} \widehat{\phi}_1 = \widehat{\lambda}_1\,, \\ \widehat{\phi}_2 = \widehat{\lambda}_2\,, \\ \widehat{\phi}_3 = \widehat{\lambda}_3\,, \\ \widehat{\phi}_4 = \widehat{\lambda}_4\,, \end{array}\right.$$

where $\widehat{\lambda}_1$, $\widehat{\lambda}_2$, $\widehat{\lambda}_3$ and $\widehat{\lambda}_4$ are introduced for commodity.

**(b) Let $\widehat{\mathcal{P}} = \mathcal{Q}_2$, $\widehat{\mathcal{N}} = \{N_1\,,...\,,N_9\}$ where $N_i(v) = v(\widehat{a}_i)\,,i = 1\,,....\,,9$. Show that $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ is a finite element.**

• $\widehat{K} \subset \mathbb{R}^2$ is a bounded closed set, $\mathring{\widehat{K}} \neq \emptyset$, $\partial\widehat{K}$ is smooth.
• $\dim\,\mathcal{Q}_2 = (2+1)^2 = 9$.
• $\dim\,\mathcal{N} = 4 = \dim\,\widehat{\mathcal{P}}'$ the dual of $\mathcal{Q}_2$.
Let $L_1$, $L_2$, $L_3$, $L_4$, $L_5$, $L_6$ be linear functions in one variable which define the edges, $(\widehat{a}_1\,,\widehat{a}_2)$, $(\widehat{a}_2\,,\widehat{a}_3)$, $(\widehat{a}_3\,,\widehat{a}_4)$, $(\widehat{a}_4\,,\widehat{a}_1)$, $(\widehat{a}_5\,,\widehat{a}_7)$, $(\widehat{a}_8\,,\widehat{a}_6)$, respectively,
$L_1(\widehat{x}\,,\widehat{y}) = \widehat{y}$, $L_2(\widehat{x}\,,\widehat{y}) = 1-\widehat{x}$, $L_3(\widehat{x}\,,\widehat{y}) = 1-\widehat{y}$, $L_4(\widehat{x}\,,\widehat{y}) = \widehat{x}$, $L_5(\widehat{x}\,,\widehat{y}) = \dfrac{1}{2}-\widehat{x}$, $L_6(\widehat{x}\,,\widehat{y}) = \dfrac{1}{2}-\widehat{y}$.
Let $v \in \mathcal{Q}_1$ such that $N_1(v) = 0$, $N_2(v) = 0$, $N_3(v) = 0$, $N_4(v) = 0$, $N_5(v) = 0$, $N_6(v) = 0$, $N_7(v) = 0$, $N_8(v) = 0$, $N_9(v) = 0$ *i.e.* $v(\widehat{a}_i) = 0$, $i = 1\,,...\,,9$. The polynomial $v$ vanishes at two points of edges $(\widehat{a}_1\,,\widehat{a}_2)$, $(\widehat{a}_2\,,\widehat{a}_3)$, $(\widehat{a}_5\,,\widehat{a}_7)$, $(\widehat{a}_8\,,\widehat{a}_6)$. Then there exists a constant $c$ such that $v = c\,L_1\,L_2\,L_5\,L_6$. But $v$ vanishes at $\widehat{a}_4$:

$$0 = v(0\,,1) = c\,L_1(0\,,1)\,L_2(0\,,1)\,L_5(0\,,1)\,L_6(0\,,1) \Longrightarrow c = 0\,,$$

since $L_1(0\,,1) \neq 0$, $L_2(0\,,1) \neq 0$, $L_5(0\,,1) \neq 0$ and $L_6(0\,,1) \neq 0$.
Then $v = 0$ and $\widehat{\mathcal{N}}$ is a basis of $\widehat{\mathcal{P}}'$.
Therefore $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ is a finite element.

• **The nodal basis** $(\widehat{\phi}_1\,,\widehat{\phi}_2\,,\widehat{\phi}_3\,,\widehat{\phi}_4\,,\widehat{\phi}_5\,,\widehat{\phi}_6\,,\widehat{\phi}_7\,,\widehat{\phi}_8\,,\widehat{\phi}_9)$ is given by

$$\left\{\begin{array}{l} \widehat{\phi}_1(\widehat{x}\,,\widehat{y}) = (1-\widehat{x})\,(1-\widehat{y})\,(1-2\widehat{x})\,(1-2\widehat{y}) \\ \widehat{\phi}_2(\widehat{x}\,,\widehat{y}) = -\widehat{x}\,(1-\widehat{y})\,(1-2\widehat{x})\,(1-2\widehat{y}) \\ \widehat{\phi}_3(\widehat{x}\,,\widehat{y}) = \widehat{x}\,\widehat{y}\,(1-2\widehat{x})\,(1-2\widehat{y}) \\ \widehat{\phi}_4(\widehat{x}\,,\widehat{y}) = -(1-\widehat{x})\,\widehat{y}\,(1-2\widehat{x})\,(1-2\widehat{y}) \\ \widehat{\phi}_5(\widehat{x}\,,\widehat{y}) = 4\widehat{x}\,(1-\widehat{x})\,(1-\widehat{y})(1-2\widehat{y}) \\ \widehat{\phi}_6(\widehat{x}\,,\widehat{y}) = -4\widehat{x}\,\widehat{y}\,(1-2\widehat{x})(1-\widehat{y}) \\ \widehat{\phi}_7(\widehat{x}\,,\widehat{y}) = -4\widehat{x}\,\widehat{y}\,(1-\widehat{x})(1-2\widehat{y}) \\ \widehat{\phi}_8(\widehat{x}\,,\widehat{y}) = 4\,\widehat{y}\,(1-\widehat{x})\,(1-2\widehat{x})\,(1-\widehat{y}) \\ \widehat{\phi}_9(\widehat{x}\,,\widehat{y}) = 16\,\widehat{x}\,\widehat{y}\,(1-\widehat{x})(1-\widehat{y}) \end{array}\right. \quad\Longleftrightarrow\quad \left\{\begin{array}{l} \widehat{\phi}_1 = \widehat{\lambda}_1\,(\widehat{\lambda}_1-\widehat{\lambda}_2+\widehat{\lambda}_3-\widehat{\lambda}_4)\,, \\ \widehat{\phi}_2 = -\widehat{\lambda}_2\,(\widehat{\lambda}_1-\widehat{\lambda}_2+\widehat{\lambda}_3-\widehat{\lambda}_4)\,, \\ \widehat{\phi}_3 = \widehat{\lambda}_3\,(\widehat{\lambda}_1-\widehat{\lambda}_2+\widehat{\lambda}_3-\widehat{\lambda}_4)\,, \\ \widehat{\phi}_4 = -\widehat{\lambda}_4\,(\widehat{\lambda}_1-\widehat{\lambda}_2+\widehat{\lambda}_3-\widehat{\lambda}_4)\,, \\ \widehat{\phi}_5 = 4\,\widehat{\lambda}_1\,(\widehat{\lambda}_2-\widehat{\lambda}_3)\,, \\ \widehat{\phi}_6 = 4\,\widehat{\lambda}_2\,(\widehat{\lambda}_3-\widehat{\lambda}_4)\,, \\ \widehat{\phi}_7 = 4\,\widehat{\lambda}_3\,(\widehat{\lambda}_4-\widehat{\lambda}_1)\,, \\ \widehat{\phi}_8 = 4\,\widehat{\lambda}_4\,(\widehat{\lambda}_1-\widehat{\lambda}_2)\,, \\ \widehat{\phi}_9 = 16\,\widehat{\lambda}_1\,\widehat{\lambda}_3 = 16\,\widehat{\lambda}_2\widehat{\lambda}_4\,, \end{array}\right.$$

where $\widehat{\lambda}_1$, $\widehat{\lambda}_2$, $\widehat{\lambda}_3$ and $\widehat{\lambda}_4$ are already introduced in (a).

**Exercise 4.**

**(a) Given a finite element $(K, \mathcal{P}, \mathcal{N})$, let the set $\{\phi_i : 1 \le i \le k\} \cap \mathcal{P}$ be the basis dual of $\mathcal{N}$. If $v$ is a function for which all $N_i \in \mathcal{N}$, $i = 1, ..., k$, are defined, then we defined the local interpolant by**

$$\mathcal{I}_K v = \sum_{i=1}^{k} N_i(v)\phi_i \,.$$

**Prove that the local interpolant is linear.**

**Answer**

Let $v$ be a function for which all $N_i \in \mathcal{N}$, $i = 1, ..., k$, are defined. Then, $v \mapsto N_i(v)$ is linear for all $N_i \in \mathcal{N}$, $i = 1, ..., k$. Therefore the local interpolant $\mathcal{I}_K$ is linear for such function $v$.

**(b) Let $T_1$ be the triangle whose vertices are $a_1 = (0\,,0)$, $a_2 = (1\,,0)$ and $a_3 = (0\,,1)$, $T_2$ be the triangle whose vertices are $a_2 = (1\,,0)$, $a_4 = (1\,,1)$ and $a_3 = (0\,,1)$. Following finite elements are considered:**

**$(T_1\,,\mathcal{P}_1\,,\mathcal{N}_1)$ with $\mathcal{N}_1 = \{N_1\,,N_2\,,N_3\}$ where $N_i(v) = v(a_i)\,, i = 1\,,2\,,3$ ;**

**$(T_2\,,\mathcal{P}_1\,,\mathcal{N}_2)$ with $\mathcal{N}_2 = \{N_4\,,N_5\,,N_6\}$ where $N_4(v) = v(a_2)\,, N_5(v) = v(a_4)\,, N_6(v) = v(a_3)$.**

**Finally let $f$ and $g$ be functions in $\mathbb{R}^2$: $f(x,y) = e^{xy}$ and $g(x,y) = \sin(\pi(x+y)/2)$.**

**Compute the local interpolations $\mathcal{I}_K f$ and $\mathcal{I}_K g$ where $K = T_1\,,T_2$.**

**Answer**

The nodal basis of the finite element $(T_1\,,\mathcal{P}_1\,,\mathcal{N}_1)$ is $\{\phi_1\,,\phi_2\,,\phi_3\}$ defined by
$\phi_1(x,y) = 1 - x - y\,, \phi_2(x,y) = x\,, \phi_3(x,y) = y$.

Then

$$\begin{aligned}\mathcal{I}_K f &= N_1(f)(1-x-y) + N_2(f)x + N_3(f)y \\ &= f(0,0) \times (1-x-y) + f(1,0) \times x + f(0,1) \times y \\ &= 1 \times (1-x-y) + 1 \times x + 1 \times y \\ &= 1\,,\end{aligned}$$

$$\begin{aligned}\mathcal{I}_K g &= N_1(g)(1-x-y) + N_2(g)x + N_3(g)y \\ &= g(0,0) \times (1-x-y) + g(1,0) \times x + g(0,1) \times y \\ &= 0 \times (1-x-y) + 1 \times x + 1 \times y \\ &= x + y\,.\end{aligned}$$

The nodal basis of the finite element $(T_2\,,\mathcal{P}_1\,,\mathcal{N}_2)$ is $\{\phi_4\,,\phi_5\,,\phi_6\}$ defined by
$\phi_4(x,y) = 1 - y\,, \phi_5(x,y) = x + y - 1\,, \phi_6(x,y) = 1 - x$.

Then

$$\begin{aligned}\mathcal{I}_K f &= N_4(f)(1-y) + N_5(f)(x+y-1) + N_6(f)(1-x) \\ &= f(1,0) \times (1-y) + f(1,1) \times (x+y-1) + f(0,1) \times (1-x) \\ &= 1 \times (1-y) + e \times (x+y-1) + 1 \times (1-x) \\ &= (e-1)(x+y) + 2 - e\,,\end{aligned}$$

$$\begin{aligned}\mathcal{I}_K g &= N_4(g)(1-y) + N_5(g)(x+y-1) + N_6(g)(1-x) \\ &= g(1,0) \times (1-y) + g(1,1) \times (x+y-1) + g(0,1) \times (1-x) \\ &= 1 \times (1-y) + 0 \times (x+y-1) + 1 \times (1-x) \\ &= 2 - x - y\,.\end{aligned}$$

**Exercise 5.**
**Let $\mathcal{P}_k^n$ denote the space of polynomials of degree $\leq\ k$ in $n$ variables.**
**Prove that dim $(\mathcal{P}_k^n) = \begin{pmatrix} n+k \\ k \end{pmatrix}$, where the latter is the binomial coefficient.**

**Answer**
The dimension of the space of polynomials in $n$ variables being exactly of degree $l \in \mathbb{N}$ is the number of combination with repetition of $l$ objets within a set of $n$ objects, that is $\begin{pmatrix} n+l-1 \\ l \end{pmatrix}$.
Then the dimension of the space of polynomials of degree $\leq\ k$ in $n$ variables is

$$\dim \mathcal{P}_k^n = \sum_{l=0}^{k} \begin{pmatrix} n+l-1 \\ l \end{pmatrix} = \begin{pmatrix} n+k \\ k \end{pmatrix} = \frac{(n+k)!}{k!\ n!}\,, \text{thanks to Pascal's triangle}\,.$$

When $\mathbb{R}^n$ denotes the physical space the above formula turns into

$$\dim \mathcal{P}_k^n = \begin{pmatrix} n+k \\ k \end{pmatrix} = \begin{cases} k+1 & \text{if} \quad n = 1\,, \\ \frac{1}{2}(k+1)(k+2) & \text{if} \quad n = 2\,, \\ \frac{1}{6}(k+1)(k+2)(k+3) & \text{if} \quad n = 3\,. \end{cases}$$