Université de Nice Sophia-Antipolis
Master MathMods - Finite Elements - 2008/2009

Exercises - Chapter 1 Sobolev Spaces - Chapter 2 Variational Formulation of Elliptic Boundary Value Problems

Exercise 1.
(a) Let $I =]0, l[$, $l \in \mathbb{R}$. Show that

$$\exists\, C(l) > 0\,,\ \|u\|_{C^0(\bar{I})} \leq C(l)\, \|u\|_{H^1(I)}\,,\ \ \forall u \in \mathcal{D}(\bar{I})\,.$$

Conclude that $H^1(I) \subset C^0(\bar{I})$ in dimension 1.

(b) Let $\Omega = B(0, 1/2)$ be the ball of radius 1/2 about the origine $(0,0)$ in $\mathbb{R}^2$. Let v be the function defined on Ω by

$$v(x) = \Big|\ln\|x\|\Big|^k\,, k \in \mathbb{R}\,.$$

Study the continuity of v in the neighbourhood of the origine $(0,0)$, and then prove that for $k < 1/2$, $v \in H^1(\Omega)$. Conclude.

Exercise 2.
Let be the following boundary value problem

$$\begin{cases} -u''(x) = f(x) \ \text{ on } \ [0,1]\,, \\ u(0) = 0\ , \\ u'(1) = \alpha\ , \end{cases}$$

where f is a given function of $L^2(0,1)$ and $\alpha \in \mathbb{R}$.

(a) Let $V = \{v \in H^1(0,1), v(0) = 0\}$. Prove that $|v|_{1,\Omega} = \left(\int_\Omega |v'(x)|^2\, dx\right)^{\frac{1}{2}}$, where $\Omega = [0,1]$, is a norm on V and V is a Hilbert Space.

(b) Give the variational problem and show that it has a unique solution.

(c) Recover formally the initial problem.

Exercise 3.
(a) Give the variational formulation of the boundary value problem

$$\begin{cases} -u''(x) + u(x) = f(x) \ \text{ on } \ [0,1]\,, \\ u'(0) = 0\ , \\ u'(1) = 0\ , \end{cases}$$

where f is a given function of $L^2(0,1)$.

(b) Show that the variational problem has a unique solution.

(c) Recover formally the initial problem.

Exercise 4.
Let be the following problem:

$$\begin{aligned}&\text{Find } u \in H_0^1(\Omega) \text{ solution of}\\ &\quad \forall v \in H_0^1(\Omega)\ ,\ \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, dx = \int_\Omega f v \, dx\\ &\text{where } f \text{ is a given function of } L^2(\Omega).\end{aligned}$$

Show that this problem has a unique solution and give the associated initial boundary value problem.