Université de Nice Sophia-Antipolis
Master MathMods - Finite Elements - 2008/2009

Exercises - Chapter 1 - Chapter 2 (Correction)

Exercise 1.
(a) Let $I =]0,l[$, $l \in \mathbb{R}$. Show that

$$\exists\, C(l) > 0\,,\ \|u\|_{C^0(\bar{I})} \leq C(l)\, \|u\|_{H^1(I)}\,,\ \ \forall u \in \mathcal{D}(\bar{I})\,. \tag{1}$$

Let $u \in \mathcal{D}(\bar{I})$. Let $x\,, y \in \bar{I}$. The fondamental theorem of analysis gives

$$u(x) = u(y) + \int_y^x u'(s)\, ds\,,$$

which leads to

$$|u(x)| \leq |u(y)| + \int_0^l |u'(s)|\, ds\,,$$

By means of Cauchy-Schwarz inequality, one gets

$$|u(x)| \leq |u(y)| + \sqrt{l}\left(\int_0^l |u'(s)|^2\, ds\right)^{\frac{1}{2}}.$$

Integrating the above inequality over $[0,l]$ in the y variable, and applying once again the Cauchy-Schwarz inequality, give

$$l\,|u(x)| \leq \sqrt{l}\left(\int_0^l |u(y)|^2\, dy\right)^{\frac{1}{2}} + l\,\sqrt{l}\left(\int_0^l |u'(s)|^2\, ds\right)^{\frac{1}{2}},$$

or equivalently

$$|u(x)| \leq \max\left(1/\sqrt{l}\,, \sqrt{l}\right)\ \left(\|u\|_{L^2(I)} + \|u'\|_{L^2(I)}\right).$$

By using the the Cauchy-Schwarz inequality in $\mathbb{R}^2$, denoting $C(l) = \sqrt{2}\max\left(1/\sqrt{l}\,, \sqrt{l}\right)$, and taking the supremum on $x \in \bar{I}$, one obtains the result.

Conclude that $H^1(I) \subset C^0(\bar{I})$ in dimension 1.

Let $u \in H^1(I)$. Since $\mathcal{D}(\bar{I})$ is dense in $H^1(I)$ for the norm $\|\ \|_{H^1(I)}$, there exists a sequence $(u_n)_{n\geq 0} \in \mathcal{D}(\bar{I})$ which converges to u in $H^1(I)$ for the norm $\|\ \|_{H^1(I)}$. Then the inequality (1) holds for each fonction u_n:

$$\exists\, C(l) > 0\,,\ \|u_n\|_{C^0(\bar{I})} \leq C(l)\, \|u_n\|_{H^1(I)}\,,\ \ \forall n \geq 0\,. \tag{2}$$

Since $(u_n)_{n\geq 0}$ is a Cauchy sequence in $H^1(I)$, the sequence $(u_n)_{n\geq 0}$ is Cauchy in $C^0(\bar{I})$ for the uniform norm $\|\ \|_{C^0(\bar{I})}$ by the inequality (2). The space $C^0(\bar{I})$ equipped with the norm $\|\ \|_{C^0(\bar{I})}$ is complete: there exists $w \in C^0(\bar{I})$ such that $(u_n)_{n\geq 0}$ converges to w in $C^0(\bar{I})$ for the norm $\|\ \|_{C^0(\bar{I})}$.

On the one hand $(u_n)_{n\geq 0}$ converges to w in $C^0(\bar{I})$ for the norm $\| \, \|_{C^0(\bar{I})}$ implies $\|u_n\|_{C^0(\bar{I})} \to \|w\|_{C^0(\bar{I})}$, and $(u_n)_{n\geq 0}$ converges to u in $H^1(I)$ for the norm $\| \, \|_{H^1(I)}$ leads $\|u_n\|_{H^1(I)} \to \|u\|_{H^1(I)}$.

On the other hand $(u_n)_{n\geq 0}$ converges to u in $H^1(I)$ for the norm $\| \, \|_{H^1(I)}$, leads to $u_n \to u$ *a.e.*. The sequence $(u_n)_{n\geq 0}$ converges to v in $C^0(\bar{I})$ for the norm $\| \, \|_{C^0(\bar{I})}$, implies $u_n \to w$ in each point of $\bar{I}$. This implies $u = w$ *a.e.*. Then w is **the continuous representant of the class u in $H^1(I)$. Finally $H^1(I) \subset C^0(\bar{I})$ in dimension** 1.

(b) Let $\Omega = B(0, 1/2)$ be the ball of radius $1/2$ about the origine $(0,0)$ in $\mathbb{R}^2$. Let v be the function defined on Ω by

$$v(x) = \left|\ln\|x\|\right|^k, k \in \mathbb{R}\,.$$

Study the continuity of v in the neighbourhood of the origine $(0,0)$, and then prove that for $k < 1/2$, $v \in H^1(\Omega)$. Conclude.

Continuity

$\lim_{x\to(0,0)} \left|\ln\|x\|\right| = +\infty$ implies v is not bounded in the neighbourhood of the origine $(0,0)$ for $k > 0$. Then v is continuous in $\Omega = B(0,1/2)$ for $k \leq 0$.

Regarding the belonging of v to $H^1(\Omega)$

The derivative of v with respect to x_i, $i = 1,2$, is $\dfrac{\partial v}{\partial x_i}(x) = sign(\ln\|x\|)\, k\, \dfrac{x_i}{\|x\|^2} \left|\ln\|x\|\right|^{k-1}$.

By the change of variables from cartesian coordinates to polar coordinates $B(0,1/2) \to]0,1/2[\times]0,2\pi[$, $x = (x_1, x_2) \mapsto (r, \theta)$, one gets

$$\|v\|^2_{H^1(I)} = 2\pi \int_0^{1/2} \left|\ln r\right|^{2k} r\,dr + 2\pi k^2 \int_0^{1/2} \frac{\left|\ln r\right|^{2k-2}}{r^2}\, r\,dr\,.$$

- $\lim_{r\to 0} r = 0$ and for all $\alpha \in \mathbb{R}, \alpha \neq 0$, $\lim_{r\to 0} r(\ln r)^\alpha = 0$, imply the first integral is finite for all $k \in \mathbb{R}$.
- The second integral is finite *if only if* $2k - 2 + 1 < 0$ *i.e.* $k < 1/2$.
- Then for $k < 1/2$, $v \in H^1(\Omega)$.

Finally $k < 1/2$, $v \in H^1(\Omega)$ and v is not continuous in Ω.

Exercise 2.
Let be the following boundary value problem

$$\begin{cases} -u''(x) = f(x) \text{ on } [0,1]\,, \\ u(0) = 0\,, \\ u'(1) = \alpha\,, \end{cases} \tag{3}$$

where f is a given function of $L^2(0,1)$ and $\alpha \in \mathbb{R}$.

(a) Let $V = \{v \in H^1(0,1), v(0) = 0\}$. Prove that $|v|_{1,\Omega} = \left(\int_\Omega |v'(x)|^2\,dx\right)^{\frac{1}{2}}$, where $\Omega = [0,1]$, is a norm on V and V is a Hilbert space.

• The critical point to prove that $|v|_{1,\Omega} = \left(\int_\Omega |v'(x)|^2\,dx\right)^{\frac{1}{2}}$ is a norm, is to show that $|v|_{1,\Omega} = 0$ implies $v = 0$ in $H^1(0,1)$.
Let v to be such a function. Then $v' = 0$ *a.e.* which implies $v = constant$ *a.e.*. In dimension 1, v has a continuous representant, let call it w. Since $v(0) = 0$, one has $w(0) = 0 = constant$ and $w = 0$. Therefore $v = w = 0\,a.e.$.

• *The norm* $|v|_{1,\Omega}$ *is equivalent to the norm* $\|v\|_{H^1(0,1)}$ *on* $H^1(0,1)$.

$|v|_{1,\Omega} \le \left(|v|^2_{1,\Omega} + |v|^2_{L^2(0,1)}\right)^{\frac{1}{2}} = \|v\|_{H^1(0,1)}$.

The converse inequality is obtained as follows. Let $v \in V$, then $v \in H^1(0,1)$ in dimension 1 and v owns a continuous representant, let call it v. Then one has

$$v(x) = v(0) + \int_0^x v'(y)\,dy\,, \quad \forall x \in [0,1]\,,$$

which implies

$$|v(x)| \le x^{\frac{1}{2}} \left(\int_0^1 |v'(y)|^2\,dy\right)^{\frac{1}{2}}, \quad \forall x \in [0,1]\,,$$

Taking the square of the above inequality and then integrating in x over $(0,1)$, one gets

$$\int_0^1 |v(x)|^2\,dx \le \frac{1}{2} \int_0^1 |v'(y)|^2\,dy\,,$$

which leads to

$$\|v\|^2_{H^1(0,1)} \le \frac{3}{2}\,|v|^2_{1,\Omega}\,. \tag{4}$$

• Let γ be the mapping $\gamma : H^1(0,1) \to \mathbb{R}$, $v \mapsto v(0)$. Since $H^1(0,1) \subset C^0([0,1])$ in dimension 1, the mapping γ is well-defined. The mapping γ is linear and

$$|\gamma(v)| = |v(0)| \le \|v\|_{C^0([0,1])} \le C\,\|v\|_{H^1(0,1)}\,,$$

where $C > 0$. This shows that γ is continuous. Then $V = Ker\,\gamma$ is a closed subset of the Hilbert $H^1(0,1)$. Therefore V is a Hilbert space.

(b) Give the variational problem and show that it has a unique solution.

The variational problem
Let $v \in \mathcal{D}([0,1])$. Multiplying the PDE (3) by v and integrating over $[0,1]$, one gets

$$\int_0^1 -u''(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx\,.$$

Integrating by part the left-hand side, one gets

$$\int_0^1 u'(x)v'(x)\,dx - \left(u'(1)v(1) - u'(0)v(0)\right) = \int_0^1 f(x)v(x)\,dx\,.$$

By taking $v \in V$ et using $u'(1) = \alpha$, one obtains

$$\int_0^1 u'(x)v'(x)\,dx = \alpha\,v(1) + \int_0^1 f(x)v(x)\,dx\,.$$

Let $a : V \times V \to \mathbb{R},\ (u,v) \mapsto \int_0^1 u'(x)v'(x)\,dx,\ L : V \to \mathbb{R},\ v \mapsto \alpha\, v(1) + \int_0^1 f(x)v(x)\,dx$.
The variational problem is the following:

$$\begin{aligned} &\text{Find } u \in V \text{ solution of} \\ &\quad \forall v \in V\ ,\ a(u,v) = L(v)\,. \end{aligned} \tag{5}$$

The solution of the variational problem
- The space V is a Hilbert space.
- The mapping a is a bilinear symmetric, continuous, coercive form on V:

$$\begin{aligned} |a(u,v)| &\le |u|_{1,\Omega}\,|v|_{1,\Omega}\ \ \forall u\,, v \in V\ , \\ |a(v,v)| &= |v|_{1,\Omega}^2\ \ \forall v \in V\ . \end{aligned}$$

- The mapping L is a linear continuous form on V:

$$\begin{aligned} |L(v)| &\le |\alpha||v(1)| + \|f\|_{L^2(0,1)}\|v\|_{L^2(0,1)} \\ &\le |\alpha|\|v\|_{C^0([0,1])} + \|f\|_{L^2(0,1)}\|v\|_{H^1(0,1)} \\ &\le |\alpha|\,C\ \|v\|_{H^1(0,1)} + \|f\|_{L^2(0,1)}\|v\|_{H^1(0,1)} \\ &\le \max(|\alpha|\,C, \|f\|_{L^2(0,1)})\ \|v\|_{H^1(0,1)} \\ &\le \beta|v|_{1,\Omega}\,,\ \ \forall v \in V\,, \end{aligned}$$

with $\beta = \sqrt{\dfrac{3}{2}}\ \max(|\alpha|\,C, \|f\|_{L^2(0,1)})$.
Finally thanks to Lax-Milgram theorem, there exists a unique solution u for the above variational problem.

(c) Recover formally the initial problem.

Recovering the PDE
Let u be the solution of the variational problem (5).
Let one takes $v \in \mathcal{D}(]0,1[) \subset V$ in the variational problem (5).

$$\int_0^1 u'(x)v'(x)\,dx = \alpha\, v(1) + \int_0^1 f(x)v(x)\,dx\,.$$

Since $v \in \mathcal{D}(]0,1[)$, one has $v(1) = 0$ and

$$\int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx\,. \tag{6}$$

One has also $f \in L^2(0,1) \subset \mathcal{D}'(]0,1[)$, $u' \in L^2(0,1) \subset \mathcal{D}'(]0,1[)$. Then (6) rewrites as

$$\begin{aligned} &\langle u',\, v'\rangle_{\mathcal{D}'\,\mathcal{D}(]0,1[)} = \langle f,\, v\rangle_{\mathcal{D}',\mathcal{D}(]0,1[)}\,, \\ \text{or}\ \ &\langle -u'',\, v\rangle_{\mathcal{D}'\,\mathcal{D}(]0,1[)} = \langle f,\, v\rangle_{\mathcal{D}'\,\mathcal{D}(]0,1[)}\,, \\ \text{or}\ \ &\langle -u'' - f,\, v\rangle_{\mathcal{D}'\,\mathcal{D}(]0,1[)} = 0\,. \end{aligned}$$

Then the PDE is

$$-u'' - f = 0\ \text{ in }\ \mathcal{D}'(]0,1[)\,.$$

Since $f \in L^2(0,1)$, the above PDE turns into

$$-u'' = f\ \text{ in }\ L^2(0,1)\ \ a.e.\,. \tag{7}$$

Recovering the boundary conditions
Let one takes $v \in V$, multiplying (7) by v and integrating one gets

$$\int_0^1 -u''(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx\,.$$

Then integrating by part and using $v(0) = 0$, one obtains

$$\int_0^1 u'(x)v'(x)\,dx - u'(1)v(1) = \int_0^1 f(x)v(x)\,dx\,.$$

The comparison of the above equation with the variational problem (5) gives $u'(1) = \alpha$.
Obviously one has $u(0) = 0$ since $u \in V$.
Therefore the boundary value problem is

$$\begin{cases} -u'' = f \quad a.e. \ \text{ on } \ [0,1]\,, \\ u(0) = 0\ , \\ u'(1) = \alpha\ , \end{cases} \tag{8}$$

where f is a given function of $L^2(0,1)$ and $\alpha \in \mathbb{R}$.

Exercise 3.
(a) Give the variational formulation of the boundary value problem

$$\begin{cases} -u''(x) + u(x) = f(x) \ \text{ on } \ [0,1]\,, \\ u'(0) = 0\ , \\ u'(1) = 0\ , \end{cases}$$

where f is a given function of $L^2(0,1)$.

Let $V = H^1(0,1)$.
Let $a : V \times V \to \mathbb{R},\ (u,v) \mapsto \int_0^1 u'(x)v'(x)\,dx + \int_0^1 u(x)v(x)\,dx,\ L : V \to \mathbb{R},\ v \mapsto \int_0^1 f(x)v(x)\,dx$.
The variational problem of the PDE (3) is the following:

$$\begin{aligned} &\text{Find } u \in V \text{ solution of} \\ &\quad \forall v \in V\ ,\ a(u,v) = L(v)\,. \end{aligned} \tag{9}$$

(b) Show that the variational problem has a unique solution.

The solution of the variational problem
- The space $V = H^1(0,1)$ is a Hilbert space.
- The mapping a is a bilinear symmetric, continuous, coercive form on V:

$$\begin{aligned} |a(u,v)| &\le \|u'\|_{L^2(0,1)}\,\|v'\|_{L^2(0,1)} + \|u\|_{L^2(0,1)}\,\|v\|_{L^2(0,1)}\,, \text{ by Cauchy-Schwarz inequality in } L^2(0,1) \\ &\le \left(\|u\|^2_{L^2(0,1)} + \|u'\|^2_{L^2(0,1)}\right)^{1/2} \left(\|u\|^2_{L^2(0,1)} + \|u'\|^2_{L^2(0,1)}\right)^{1/2}, \text{ by Cauchy-Schwarz inequality in } \mathbb{R}^2 \\ &= \|u\|_{H^1(0,1)}\,\|v\|_{H^1(0,1)} \quad \forall u\,, v \in V\ , \\ |a(v,v)| &= \|v\|^2_{H^1(0,1)} \quad \forall v \in V\ . \end{aligned}$$

• The mapping L is a linear continuous form on V:

$$|L(v)| \leq \|f\|_{L^2(0,1)} \|v\|_{L^2(0,1)} \leq \|f\|_{L^2(0,1)} \|v\|_{H^1(0,1)}, \quad \forall v \in V \,.$$

By Lax-Milgram theorem, there exists a unique solution u for the above variational problem.

(c) Recover formally the initial problem.

Recovering the PDE
Let u be the solution of the variational problem (9).
Let one takes $v \in \mathcal{D}(]0,1[) \subset V$ in the variational problem (9).
Since $f \in L^2(0,1) \subset \mathcal{D}'(]0,1[)$, $u' \in L^2(0,1) \subset \mathcal{D}'(]0,1[)$, the variational problem (9) turns to

$$\begin{aligned} &\langle u', v' \rangle_{\mathcal{D}'\,\mathcal{D}(]0,1[)} + \langle u, v \rangle_{\mathcal{D}'\,\mathcal{D}(]0,1[)} = \langle f, v \rangle_{\mathcal{D}'\,\mathcal{D}(]0,1[)} \,, \\ \text{or} \quad &\langle -u'', v \rangle_{\mathcal{D}'\,\mathcal{D}(]0,1[)} + \langle u, v \rangle_{\mathcal{D}'\,\mathcal{D}(]0,1[)} = \langle f, v \rangle_{\mathcal{D}'\,\mathcal{D}(]0,1[)} \,, \\ \text{or} \quad &\langle -u'' + u - f, v \rangle_{\mathcal{D}'\,\mathcal{D}(]0,1[)} = 0 \,. \end{aligned}$$

Then the PDE is

$$-u'' + u - f = 0 \;\text{ in }\; \mathcal{D}'(]0,1[) \,.$$

Since $f \in L^2(0,1)$ and $u \in L^2(0,1)$, one has $u'' \in L^2(0,1)$ and the above PDE turns into

$$-u'' + u = f \;\text{ in }\; L^2(0,1) \;\; a.e. \,. \tag{10}$$

Recovering the boundary conditions
Let one takes $v \in V = H^1(0,1)$, multiplying (9) by v and integrating one gets

$$\int_0^1 -u''(x) v'(x) \, dx = \int_0^1 f(x) v(x) \, dx \,.$$

Then integrating by part gives

$$\int_0^1 u'(x) v'(x) \, dx - \Big(u'(1) v(1) - u'(0) v(0) \Big) = \int_0^1 f(x) v(x) \, dx \,.$$

• Then the comparison of the above equation with the variational problem (9) gives $u'(0) = 0$ when taking $v \in V = H^1(0,1)$ with $v(1) = 0$ and $v(0) \neq 0$.
• Then when taking $v \in V = H^1(0,1)$ with $v(0) \neq 0$, one gets $u'(0) = 0$.

Therefore the boundary value problem is

$$\begin{cases} -u'' + u = f \;\; a.e. \;\text{ on }\; [0,1] \,, \\ u'(0) = 0 \,, \\ u'(1) = 0 \,, \end{cases} \tag{11}$$

where f is a given function of $L^2(0,1)$.

Exercise 4.
Let Ω be a bounded subset of $\mathbb{R}^n$ where $n \geq 1$, with regular boundary $\partial\Omega$.
Let be the following problem:

Find $u \in H^1_0(\Omega)$ solution of

$$\forall v \in H^1_0(\Omega) \,, \; \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, dx = \int_\Omega f v \, dx \tag{12}$$

where f is a given function of $L^2(\Omega)$.

Show that this problem has a unique solution and give the associated initial boundary value problem.

The solution of the variational problem

Let $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R},\ (u,v) \mapsto \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, dx,\ L : H_0^1(\Omega) \to \mathbb{R},\ v \mapsto \int_\Omega f v \, dx.$

- *The space $V = H_0^1(\Omega)$ is a Hilbert space.*

The *trace* mapping $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega),\ u \mapsto u|_{\partial\Omega}$ is linear continuous. Its kernel $Ker\,\gamma_0 = H_0^1(\Omega)$ is a closed subset of the Hilbert space $H^1(\Omega)$. Therefore $H_0^1(\Omega)$ is a Hilbert space.

Thanks to Poincaré theorem, the mapping $u \mapsto |u|_{1,\Omega} = \left(\int_\Omega |\boldsymbol{\nabla} u(x)|^2 \, dx \right)^{1/2}$ is a norm equivalent to the norm $\| \, \|_{H^1(\Omega)}$ on $H_0^1(\Omega)$.

- The mapping a is a bilinear symmetric, continuous, coercive form on $H_0^1(\Omega)$:

$$
\begin{aligned}
&|a(u,v)| \le \|\boldsymbol{\nabla} u\|_{L^2(\Omega)} \, \|\boldsymbol{\nabla} v\|_{L^2(\Omega)} = |u|_{1,\Omega} \, |u|_{1,\Omega} \, , \quad \forall u \, , v \in H_0^1(\Omega) \, , \\
&a(v,v) = \|v\|_{1,\Omega}^2 \quad \forall v \in H_0^1(\Omega) \, .
\end{aligned}
$$

- The mapping L is a linear continuous form on $H_0^1(\Omega)$:

$$
|L(v)| \le \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)} \, \|v\|_{H^1(\Omega)} \le C \, \|f\|_{L^2(\Omega)} \, |v|_{1,\Omega}, \quad \forall v \in H_0^1(\Omega) \, .
$$

By Lax-Milgram theorem, there exists a unique solution u for the above variational problem.

Recovering the PDE
Let u be the solution of the variational problem (12).
Let one takes $v \in \mathcal{D}(\Omega) \subset H_0^1(\Omega)$ in the variational problem (12).
Since $f \in L^2(\Omega) \subset \mathcal{D}'(\Omega)$, $u' \in L^2(\Omega) \subset \mathcal{D}'(\Omega)$, the variational problem (12) turns to

$$
\begin{aligned}
&\langle u', \, v' \rangle_{\mathcal{D}' \, \mathcal{D}(\Omega)} = \langle f, \, v \rangle_{\mathcal{D}' \, \mathcal{D}(\Omega)} \, , \\
\text{or} \quad &\langle -u'', \, v \rangle_{\mathcal{D}' \, \mathcal{D}(\Omega)} = \langle f, \, v \rangle_{\mathcal{D}' \, \mathcal{D}(\Omega)} \, , \\
\text{or} \quad &\langle -u'' - f, \, v \rangle_{\mathcal{D}' \, \mathcal{D}(\Omega)} = 0 \, .
\end{aligned}
$$

Then the PDE is

$$
-u'' - f = 0 \ \text{ in } \ \mathcal{D}'(\Omega) \, .
$$

Since $f \in L^2(\Omega)$ and $u \in L^2(\Omega)$, one has $u'' \in L^2(\Omega)$ and the above PDE turns into

$$
-u'' = f \ \text{ in } \ L^2(\Omega) \ \ a.e.. \tag{13}
$$

Recovering the boundary conditions
The solution of the the variational problem (12) $u \in H_0^1(\Omega)$, then $u = 0$ *on* $\partial\Omega$.

Therefore the boundary value problem is

$$
\begin{cases}
-u'' = f \ \ a.e. \ \text{ in } \ \Omega \, , \\
u = 0 \ \text{ on } \ \partial\Omega \, ,
\end{cases} \tag{14}
$$

where f is a given function of $L^2(\Omega)$.