Master MathMods
Finite element method - Implementation in `Scilab`

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The aim of these exercises is to create a first program which implements the finite element method. We will treat only the on-dimensional case but we can consider also uniform meshes.

1 The steps of a finite element code

A finite element code is composed of the following steps :

Pre-treatment : Read the data of the problem : the mesh, right hand sides, boundary conditions, physical parameters, etc...

Assembling : Build the linear system (use the discrete variational formulation to compute matrix coefficients and right hand side).

Solution : Use an adapted resolution method for the linear system in function of the properties of the matrix (symmetry, sparsity , etc...)

Post-treatment : Generate an exploitable information by visualisation software (or functions) from the solution of the linear system.

2 Model problem

We will discretize the following problem :

$$\begin{cases} -u''(x) + cu(x) = f(x) & x \in [0, L] \\ \text{Boundary conditions at } x = 0 \text{ and } x = L \end{cases} \tag{1}$$

where $f(x)$ is a given function and $c > 0$ a real constant. We will treat different types of boundary conditions, which will allow us to see the different techniques associated to these conditions.

Suppose that we already obtained the variation formulation

$$\text{Find } u \in V \text{ such that } a(u, v) = L(v),\ \forall v \in V$$

and we proved the existence and unicity of the solution. We will use the Lagrange P_1 finite element method. We consider a partition of the domain $0 = x_0 < x_1 < x_2 < \ldots < x_N = L$. We will look for locally affine solutions of (1), that is :

$$u_h = \sum_{i=0}^{N} u_i \psi_i(x) \tag{2}$$

where $\psi_i(x)$ are the basis functions which verify $\psi_i(x) = 0$ if $x \notin [x_{i-1}, x_{i+1}]$, $\psi_i(x_i) = 1$ and $\psi_i(x)$ is affine on $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$.

3 Assembling the sparse linear system

Each test function can be expressed as a linear combination of the basis functions, that is :

$$v(x) = \sum_{i=0}^{N} v_j \psi_i(x)$$

The equality $a(u, v) = Lv$ is true for each v, iif it is verified for all ψ_i $(a(u, \psi_i) = L\psi_i)$. This gives :

$$\sum_{j=0}^{N} u_j a(\psi_j, \psi_i) = L\psi_i$$

where u_j are the unknowns of the problem. We can write this in matrix form $A\boldsymbol{u} = \boldsymbol{b}$ where

$$A_{i,j} = a(\psi_j, \psi_i),\ b_i = \int_0^L f(x)\psi_i(x)\, dx$$

The right hand side is often computed by a quadrature rule, for example a two-point Gauss quadrature :

$$\int_a^b f(x)\, dx \approx \frac{b-a}{2}(f(\xi_0) + f(\xi_1)) \tag{3}$$

where :

$$\xi_0 = \frac{a+b}{2} - \frac{\sqrt{3}}{6}(b-a) \text{ and } \xi_1 = \frac{a+b}{2} + \frac{\sqrt{3}}{6}(b-a)$$

which is exact for polynoms of degree 3.
Linear systems issued from the discretization of differential equations by the finite difference or finite element methods are sparse, that is have a big quantity of zeros. For the P_1 finite element method, the basis functions ψ_i and ψ_j have non-disjoint supports iif there is an element T_k such that $x_i \in T_k$ and $x_j \in T_k$. This property is the origin of the sparse matrices.

Moreover, in the one-dimensional case, given the mesh points x_i and x_j, there exist and element T such that $x_i \in T$ and $x_j \in T$ only if $j = i$, $j = i-1$ or $j = i+1$. We have that $A_{i,j} = 0$ if $|i - j| > 1$. This implies that the matrix of the discrete system is tridiagonal. The symmetry of the bilinear form $a(.,.)$ implies that A is symmetric. The matrix has thus the following form :

$$A = \begin{bmatrix} d_0 & s_0 & 0 & \dots & 0 \\ s_0 & d_1 & s_1 & \dots & 0 \\ 0 & s_1 & d_2 & \dots & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & \dots & d_{n-1} \end{bmatrix}$$

For the computation of the right hand side a Gauss quadrature is used.

4 Solution of the sparse linear system

Once the matrix and the right hand side are assembled, we need to solve the linear system in order to find the approximate solution. We can use for example the Cholesky factorization $A = LL^t$ where L is a triangular inferior matrix. We find the solution of $A\boldsymbol{u} = \boldsymbol{b}$ by successively solving the following systems :

$$L\boldsymbol{y} = \boldsymbol{b},\ L^t\boldsymbol{u} = \boldsymbol{y}.$$

Since A is symmetric on can use also a Conjugate Gradient method.

5 Examples of different boundary value problems

5.1 The homogeneous Neumann problem

The simplest case is the homogeneous Neumann problem :

$$\begin{cases} -u''(x) + cu(x) = f(x) \quad x \in [0, L] \\ u'(0) = 0,\ u'(L) = 0. \end{cases} \tag{4}$$

The variational formulation is :

$$\begin{cases} \text{Find } u \in H^1(0, L) : \\ a(u, v) = Lv \quad \forall v \in H^1(0, L) \end{cases} \tag{5}$$

where :

$$\begin{aligned} a(u, v) &= \int_0^L u'(x)v'(x)\ dx + c\int_0^L u(x)v(x)\ dx \\ Lv &= \int_0^L f(x)v(x)\ dx \end{aligned}$$

In order to test the program on needs to compute the approximate solution of a problem whose analytic solution is known. Consider the following problem :

$$\begin{cases} -u''(x) + u(x) = 20\cos(3x) \quad x \in [0, 2\pi] \\ u'(0) = 0 \\ u'(2\pi) = 0. \end{cases} \tag{6}$$

It is easy to verify that :

$$u(x) = 2\cos(3x)$$

is a solution. One can check now if the values u_i are good approximations of $2\cos(x_i)$ and what is the error. In order to study the error we will trace the norm $L^2(0, 2\pi)$ of the error in function of the discretization step h in logarithmic scale and one has to find a straight line of slope 2 which is a consequence of the theoretical results :

$$\|u - u_h\|_{L^2(0,2\pi)} \leq Ch^2.$$

5.2 The non-homogeneous Neumann problem

We look for the solution of :

$$\begin{cases} -u''(x) + cu(x) = f(x) \quad \forall x \in (0, L) \\ u'(0) = a, \quad u'(L) = b \end{cases} \tag{7}$$

where the function $f : (0, L) \to \mathbb{R}$ and the constants a, b, c and L are given. By multiplying the equation by a test function and integrating by parts we get :

$$\begin{aligned} &-\int_0^L u''(x)v(x)\ dx + c\int_0^L u(x)v(x)\ dx = \int_0^L f(x)v(x)\ dx \\ &\int_0^L u'(x)v'(x)\ dx - u'(L)v(L) + u'(0)v(0) + c\int_0^L u(x)v(x)\ dx = \int_0^L f(x)v(x)\ dx \\ &\int_0^L u'(x)v'(x)\ dx + c\int_0^L u(x)v(x)\ dx = \int_0^L f(x)v(x)\ dx + bv(L) - av(0) \end{aligned}$$

The variational formulation is :

$$\begin{cases} \text{Find } u \in H^1(0, L) : \\ a(u, v) = Lv \quad \forall v \in H^1(0, L) \end{cases} \tag{8}$$

where :

$$a(u,v) = \int_0^L u'(x)v'(x)\,dx + c\int_0^L u(x)v(x)\,dx$$
$$Lv = \int_0^L f(x)v(x)\,dx + bv(L) - av(0)$$

In the discrete formulation this translates by adding in the right hand side two extra terms : $-a\psi_i(0)$ and $b\psi_i(L)$. Note that :

$$\psi_i(0) = 0 \quad \forall i \neq 0 \quad \psi_0(0) = 1 \text{ and } \psi_i(L) = 0 \quad \forall i \neq N \quad \psi_N(L) = 1$$

that means, when assembling $\boldsymbol{b}$ we have :

$$b_0 = \int_0^L f(x)\psi_0(x)\,dx - a$$
$$b_N = \int_0^L f(x)\psi_N(x)\,dx + b$$

all the other terms are inchanged w.r.t. the homogeneous Neumann problem.
For testing we look for the solution of :

$$\begin{cases} -u''(x) + cu(x) = 0 \quad \forall x \in (0,L) \\ u'(0) = a, \quad u'(L) = b \end{cases} \tag{9}$$

for arbitrary $c > 0$, $L > 0$, a and b. One can easily verify that the function :

$$u(x) = c_1 e^{\sqrt{c}x} + c_2 e^{-\sqrt{c}x}$$

is solution of the problem, where the constants are given by :

$$c_1 = -\frac{a - be^{\sqrt{c}L}}{\left(e^{2\sqrt{c}L} - 1\right)\sqrt{c}},\ c_2 = -\frac{e^{\sqrt{c}L}\left(-b + ae^{\sqrt{c}L}\right)}{\left(e^{2\sqrt{c}L} - 1\right)\sqrt{c}}$$

One can test different values of c, a, b and L and trace the $L^2(0,L)$ error.

5.3 The homogeneous Dirichlet problem

We look for the solution of the problem :

$$\begin{cases} \text{Find } u \in H_0^1(0,L) : \\ a(u,v) = Lv \quad \forall v \in H_0^1(0,L) \end{cases} \tag{10}$$

where $a(.,.)$ et L are defined like in the homogeneous Neumann case.
If one considers the Lagrange P_1 approximation of this problem, on can express $u_h \in V_h$ as a linear combination of the same basis functions as in the Neumann case except ψ_0 and ψ_N, which gives :

$$u_h = \sum_{i=1}^{N-1} u_i \psi_i(x) \tag{11}$$

The linear system has les unknowns than the previous case (u_0 and u_N are fixed). Note that the integrals are always takes on $(0,L)$, which means that we still have to integrate over the elements $(0,x_1)$ and (x_{N-1},L) even if u_0 and u_N are not unknowns.

In this case there is also a little post-treatment to do : one needs to add the two boundary values before generating the visualisation data, in order to trace the entire solution which includes $u_h(0)$ and $u_h(L)$.

For the testing, consider the problem :

$$\begin{cases} -u''(x) + \frac{8\pi^2}{3}u(x) = 20\pi^2\sin(2\pi x) \quad \forall x \in (0,1) \\ u(0) = 0, \quad u(1) = 0 \end{cases} \tag{12}$$

with the exact solution :

$$u(x) = 3\sin(2\pi x).$$

5.4 The non-homogeneous Dirichlet problem

We will solve the following boundary value problem :

$$
\begin{cases} -u''(x) + cu(x) = f(x) \quad \forall x \in (0, L) \\ u(0) = a, \quad u(L) = b \end{cases} \tag{13}
$$

Let $w(x)$ a function of $H^1(0, L)$ such that $w(0) = a$ and $w(L) = b$. The weak formulation consists in finding a function $u_1 \in H_0^1(0, L)$ such that $u(x) = u_1(x) + w(x)$. Multiplying by a test function and integrating by parts we get :

$$
\int_0^L (u_1(x) + w(x))' v'(x)\, dx + c \int_0^L (u_1(x) + w(x)) v(x)\, dx = \int_0^L f(x) v(x)\, dx
$$

$$
\int_0^L u_1'(x) v'(x)\, dx + c \int_0^L u_1(x) v(x)\, dx =
$$

$$
\int_0^L f(x) v(x)\, dx - \int_0^L w'(x)' v'(x)\, dx + c \int_0^L w(x) v(x)\, dx
$$

which gives the following weak formulation :

$$
\begin{cases} \text{Find } u_1 \in H_0^1(0, L) : \\ a(u_1, v) = Lv \quad \forall v \in H_0^1(0, L) \end{cases} \tag{14}
$$

where $a(.,.)$ is defined like in the homogeneous Dirichlet case and :

$$
Lv = \int_0^L f(x) v(x)\, dx - \int_0^L w'(x) v'(x)\, dx - c \int_0^L w(x) v(x)\, dx.
$$

We have the choice of the function w. In practice we have to use the simplest w possible in ordre to simplify the border integrals of the right-hand side. In the discrete formulation one can take :

$$
w(x) = a\psi_0(x) + b\psi_N(x)
$$

which verifies the conditions and is supported in $[0, x_1] \cup [x_{N-1}, L]$.

In this case we have $N - 1$ degrees of freedom ($\{x_1, \ldots x_{N-1}\}$) and the boundary condition modifies the coordinates of the right hand side as follows :

$$
b_1 = \int_0^L f(x) \psi_1(x)\, dx - a \int_{x_0}^{x_1} \psi_0'(x) \psi_1'(x)\, dx - ca \int_{x_0}^{x_1} \psi_0(x) \psi_1(x)\, dx
$$

et :

$$
b_{N-1} = \int_0^L f(x) \psi_{N-1}(x)\, dx - b \int_{x_{N-1}}^{x_N} \psi_N'(x) \psi_{N-1}'(x)\, dx -
$$

$$
cb \int_{x_{N-1}}^{x_N} \psi_N(x) \psi_{N-1}(x)\, dx
$$

We will test the program by solving the equation :

$$
\begin{cases} -u''(x) + cu(x) = 0 \quad \forall x \in (0, 1) \\ u(0) = 0, \quad u(1) = 1 \end{cases} \tag{15}
$$

whose exact solution is :

$$
u(x) = \frac{1}{e^{\sqrt{c}} - e^{-\sqrt{c}}} \left(e^{\sqrt{c}x} - e^{-\sqrt{c}x} \right).
$$

5.5 The Robin problem

In this section we will treat the equation :

$$\begin{cases} -u''(x) + cu(x) = f(x) \quad \forall x \in (0, L) \\ u'(0) + a_1 u(0) = b_1 \\ u'(L) + a_2 u(L) = b_2, \end{cases} \tag{16}$$

Multiplying by a test function and integration over $(0, L)$, gives :

$$\int_0^L u'(x)v'(x)\,dx - u'(L)v(L) + u'(0)v(0) + c\int_0^L u(x)v(x)\,dx = \int_0^L f(x)v(x)\,dx$$

$$\int_0^L u'(x)v'(x)\,dx - (b_2 - a_2 u(L))v(L) + (b_1 - a_1 u(0))v(0) + c\int_0^L u(x)v(x)\,dx = \int_0^L f(x)v(x)\,dx$$

The variational formulation becomes :

$$\begin{cases} \text{Find } u \in H^1(0, L) : \\ a(u, v) = Lv \quad \forall v \in H^1(0, L) \end{cases} \tag{17}$$

where :

$$a(u, v) = \int_0^L u'(x)v'(x)\,dx + c\int_0^L u(x)v(x)\,dx + a_2 u(L)v(L) - a_1 u(0)v(0)$$

$$Lv = \int_0^L f(x)v(x)\,dx + b_2 v(L) - b_1 v(0).$$

For the discrete formulation one considers the same Galerkin approximation as in the Neumann case. The elements of the matrix and of the right hand side are given by :

$$A_{i,j} = \int_0^L \psi_j'(x)\psi_i'(x)\,dx + c\int_0^L \psi_j(x)\psi_i(x)\,dx + a_2\psi_j(L)\psi_i(L) - a_1\psi_j(0)\psi_i(0)$$

$$b_i = \int_0^L f(x)\psi_i(x)\,dx + b_2\psi_i(L) - b_1\psi_i(0),$$

where we used the properties of the basis functions

$$\psi_j(0) = 0 \quad \forall j \neq 0,\ \psi_j(L) = 0 \quad \forall j \neq N.$$

If we denote by $\bar{A}$ the matrix of the homogeneous Neumann problem and $\bar{\boldsymbol{b}}$ the right hand side of the same problem we get :

$$A_{0,0} = \bar{A}_{0,0} - a_1$$
$$A_{N,N} = \bar{A}_{N,N} + a_2$$
$$b_0 = \bar{b}_0 - b_1$$
$$b_N = \bar{b}_N + b_2$$

and in all the other cases :

$$A_{i,j} = \bar{A}_{i,j}$$
$$b_i = \bar{b}_i.$$

In order to test the problem we can take :

$$\begin{cases} -u''(x) + u(x) = 0 \quad \forall x \in (0, 1) \\ u'(0) + u(0) = 2 \\ u'(1) + 2u(1) = 1 \end{cases} \tag{18}$$

which has the exact solution :

$$u(x) = e^x + (e - 3e^2)e^{-x}$$