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THE SET OF CONCENTRATION FOR SOME HYPERBOLIC MODELS OF CHEMOTAXIS

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Abstract. Chemotaxis models are typically able to develop blow-up in finite times. For some specific models, we obtain some estimates on the set of concentration of cells (defined roughly as the points where the density of cells is infinite with a non vanishing mass). More precisely we consider models without diffusion for which the cells' velocity decreases if the concentration of the chemical attractant becomes too large. We are able to give a lower bound on the Hausdorff dimension of the concentration set, one in the "best" situation where the velocity exactly vanishes for too large concentrations of attractant. This in particular implies that the solution may not form any Dirac mass.

Keywords: Chemotaxis, blow-up , concentration set, angiogenesis.

1. Introduction

The study of cell movement is the center of interest of many researchers. In particular a special interest is given to the study of chemotaxis which is one of the simplest mechanisms of aggregation of biological species as it refers to a situation where organisms move in the direction of high concentrations of a chemical which they secrete.

Chemotaxis plays an important role in many biological processes: Unicellular organisms such as bacteria or amoeba use chemotaxis to avoid harmful substances or to form cell aggregates. A special attention is given to cancer modelling. Many mathematicians are motivated to model angiogenesis which is a morphogenetic process whereby new blood vessels are induced to grow out of a pre-existing vasculature. Indeed in order to keep growing, a tumor emits a signal that will start the angio-

genesis and therefore connect it to blood vessels.
Many rigorous mathematical models for chemotaxis have been studied, the nature of equations used by mathematicians for this goal are various, parabolic models, hyperbolic models and very recently kinetic models were introduced for this aim.

This paper deals with the analysis of a particular assumption for the model, namely that the velocity of cells decreases if the concentration of the chemical attractant $c(t, x)$ exceeds a critical value c_0. This phenomena has a direct consequence on the nature and dimension of the set of concentrations that the cells may form. This study is of course very partial but in our opinion still interesting as it gives an example of models which do not blow up at points (to form Dirac masses like more classical parabolic models) but instead on higher dimensional sets.
This paper is organized as follows, the first section is this introduction, which provides a description of some models used for chemotaxis, together with their motivations and a very brief summary of the obtained results. The second section deals with the presentation of the mathematical model that we consider, which is a specific case of hyperbolic model for chemotaxis where the velocity of the cells decreases after the concentration of the chemical attractant becomes too large. In the third section, we prove the theorems stated in the second one. Finally, the fourth section deals with radially symmetric solutions for the system, where more explicit computations may be performed to test the optimality of the results.

A basic model for chemotaxis was introduced by Patlak [26], and Keller and Segel[16], who considered the following coupled parabolic and elliptic equations

$$\begin{cases} \partial_t \rho = \Delta\rho - \nabla.(\chi\rho\nabla c), \quad x \in \mathbb{R}^d, t > 0, \\ \\ -\Delta c = \rho, \quad x \in \mathbb{R}^d, t > 0, \\ \\ \rho(., t = 0) = \rho_0 \geq 0, \quad x \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

The concentration $c(t, x)$ represents the concentration of the chemoattractant signal. The positive quantity $\rho(t, x)$ corresponds to the density of cells at the position x in the time t.
The positive function χ describes the chemotactic sensitivity, it is the fundamental parameter which characterizes the non linearity of the system. System (1.1) with χ constant is a gradient flow with therefore a preserved energy (see [2], [8] or [14]).

Jäger and Luckhaus [15] obtained a first result on finite time blow-up for this model (1.1) in 1992. since then many results were obtained that prove the global solution existence in time (we refer to [15], [20], [22] or [23]), or the blow up of local solutions (see again [15], [20] or [21], [24]).
For the Keller Segel model, it is proved that blow up can happen even for small initial conditions in the three dimensional case ([15]). For the two dimensional space, blow up may occur or not, depending especially on the size of initial conditions. For the case of one dimensional space, we never have blow up. Modified models have

been developed to prevent the formation of blow-up (see for instance [13]),

Models based on a derive diffusion equations of Patlak, Keller Segel type are suitable to describe many phenomenons on chemotaxis, but this model presents some drawbacks. On one hand, we comment that parabolic models are based on the ensemble average movement of cell populations as a whole and individual movement properties are not taken into account. On the other hand, parabolic models generally lead to pointwise blow-up, consequently the model is not suitable to describe formation of networks, which is the particular case that we are studying. For these reasons, mathematicians have recently been interested by hyperbolic models for chemotaxis (see for example [27] or [5]).

It is clear that the assumptions and parameters which lead to hyperbolic or to parabolic models are different. Hyperbolic models, contrarily to parabolic ones, are based on the individual movement, so parameters are measured by following individual particles. Some numerical results presented by Marrocco [17] shows the presence of networks, his numerical results were comparable to experiments with human endothelial cells and these networks are interpreted by biologists as the beginning of vasculature, so hyperbolic models could be well suited to model angiogenesis.
The third class of models for chemotaxis is kinetic equations. The advantage of this mesoscopic class of models, in comparison with the macroscopic ones, consists in that individual cells movements are incorporated in the equations. The drawback is of course the additional dimensions in the models, especially for efficient numerical simulations. Such kinetic models for the cell density has been applied by Stroock [28], Alt [1] and Othmer, Dunbar et Alt [25].

The distributional function $f(t, x, v)$ describes the cells density at the position $x \in \mathbb{R}^d$, which velocity is $v \in V \in \mathbb{R}^n$, at the time $t \geq 0$. It typically satisfies the following linear transport equation

$$\partial_t f + v.\nabla_x f = \int_V (T(c, v, v')f(t, x, v') - T(c, v, v')f(t, x, v))\, dv' = Q(f). \quad (1.2)$$

The turning kernel $T(c, v, v')$ describes the reorientation of cells and may depend on the concentration of the chemo-attractant $c(t, x)$ or on its derivatives. The turning kernel $T(c, v, v')$ may be written under many possible forms ([12]). For the particular form

$$\begin{cases} T(c, v, v') = T_-(c(t, x - \varepsilon v)) + T_+(c(t, x + \varepsilon v')), \\ \\ -\Delta c = \rho(t, x) = \displaystyle\int_V f(t, x, v)\, dv. \end{cases} \quad (1.3)$$

Chalub, Markowich, Perthame and Schmeiser [3] extend the existence theory for the non linear system (1.2), coupled to (1.3); the linear case having been treated by Hillen and Othmer in [12]. In a recent paper, Dolak and Schmeiser [6] study kinetic models for amoebae chemotaxis, incorporating the ability of cells to assess temporal changes of the chemoattractant concentration as well as its spatial variation.

Although diffusion based models and kinetic equations use a different approach, these equations are closely related, because we can obtain a drift-diffusion equation as a macroscopic limit of kinetic model as it has been investigated by Patlak [26], Alt [1] and Hillen and Othmer [12]. Recent papers are interested in the derivation of the Keller Segel equation (1.1) as a macroscopic limit of a kinetic equation. For instance Chalub, Markowich, Perthame and Schmeiser in [3] consider a kinetic model coupled to an equation for the chemo-attractant, and they prove that the Keller Segel equations can be derived rigorously as a scaling limit. The two papers [11] and [12] deal with the moment closure of kinetic equations with stochastic jump velocity to derive hyperbolic equations modelling chemosensitive movement.
Considering the simplest hyperbolic model where the velocity field of the cells depends only on the concentration or its derivative, our aim is to explain what consequences on the shape of the blow-up set has the assumption that this velocity is small when the concentration is too high.

2. Main results

2.1. *The equations*

We consider the general system for a regular, bounded and open domain $\Omega \subset \mathbb{R}^3$ (the proofs would extend without much difficulty to the unbounded case)

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, K(c, \nabla c)) = 0, \quad \forall\, x \in \Omega, \\ -\Delta c = \rho - c, \quad \text{in } \Omega, \\ \rho(t=0, x) = \rho^0(x) \geq 0, \quad \rho^0 \in L^1 \cap L^\infty(\Omega). \end{cases} \tag{2.1}$$

The function $K \in C^1(\mathbb{R}^+ \times \mathbb{R}^3,\ \mathbb{R}^3)$ is the chemotactic sensitivity, it defines how the cells or bacteria are reacting to the chemical signal. The intuition behind the model is that the endothelial cells move outwards, where the signal is higher.
If the domain Ω is bounded, we need to give some boundary conditions. we assume that there is no flux at the boundary for the concentration

$$\frac{\partial c}{\partial \nu} = 0. \tag{2.2}$$

For the density of cells ρ, we assume that no cells are entering the domain, *i.e.*

$$\rho(t, x) = 0 \quad \text{if } K(c(t, x), \nabla c(t, x)) \cdot \nu(x) < 0, \quad \forall x \in \partial\Omega. \tag{2.3}$$

In both cases $\nu = \nu(x)$ is defined on $\partial\Omega$ as the outward normal vector to $\partial\Omega$.

2.2. *Local existence of solutions*

It is difficult to give an existence result (even only local) for an equation with such a general form as (2.1).
A useful assumption for that is that there exists a positive continuous (but not necessarily bounded) function F from $\mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}$ to $\mathbb{R}^+$ such that for any symmetric

matrix m_{ij}

$$-\sum_{i,\,j=1}^{d} \partial_{b_j} K_i(a,b)\, m_{ij} \le F\left(a, b, \sum_{i=1}^{d} m_{ii}\right). \tag{2.4}$$

We can then prove that a solution to (2.1) exists on a finite time interval, according to

Theorem 2.1. *Assume that* (2.4) *holds, there exists* $T > 0$*, two functions* $\rho \ge 0$ *and* $c(t, x)$ *such that*
i) $\rho \in L^\infty([0,\ T],\ L^1 \cap L^\infty(\Omega))$*;*
ii) $c \in L^\infty([0,\ T],\ W^{1,\infty}(\Omega))$*;*
iii) ρ *and* c *are solutions to* (2.1)*, together with* (2.3) *if* Ω *is bounded, in the following weak sense*

$$\begin{aligned} &\int_{\mathbb{R}+}\int_\Omega (\rho \partial_t \phi + \rho\, K(c, \nabla\, c)\cdot \nabla_x \phi)\, dx\, dt \ge -\int_\Omega \phi(0,x)\,\rho^0(x)\, dx,\\ &-\Delta_x c = \rho - c \ \text{ in } \Omega, \quad \nabla_x c\cdot \nu(x) = 0 \ \text{ on } \partial\Omega, \end{aligned} \tag{2.5}$$

for all $\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^3)$ *with* $\phi(t,x) \ge 0$ *on* $\partial\Omega$*.*

Remark 2.2. The boundary condition (2.3) is included in the inequality (2.5). Indeed if ρ, c were classical solutions to (2.1), then multiplying by ϕ and integrating by parts, we would obtain

$$\begin{aligned} \int_{\mathbb{R}+}\int_\Omega (\rho \partial_t \phi + \rho\, K(c, \nabla c)\cdot \nabla_x \phi)\, dx\, dt = &-\int_\Omega \phi(0,x)\,\rho^0(x)\, dx\\ &+\int_{\partial\Omega} \phi(t,x)\,\rho(t,x)\,K(c,\nabla c)\cdot\nu(x)\, d\sigma(x), \end{aligned}$$

and this last term is always non negative as $\phi \ge 0$ on $\partial\Omega$, $\rho \ge 0$ and (2.3).

Remark 2.3. Without (2.4), the result is somewhat weaker : We obtain ρ in $L^\infty([0,\ T],\ L^p(\Omega))$ for any $1 \le p < \infty$ provided that ρ^0 belongs to $L^1 \cap L^\infty(\Omega)$ (and not only to $L^p(\Omega)$ for all $p < \infty$). The proof of that is given after the proof of the previous theorem.

Theorem 2.1 is optimal in the sense that in general it is not possible to obtain solutions for all times, even if the initial data is small. In comparison, for the Keller-Segel model, there are global solutions if a smallness assumption is done on the initial data (as it is exposed in [4] for example). This difference is mainly due to the fact that (2.1) does not exhibit any diffusion and is thus more singular.

If the function K is bounded then we do not need an L^∞ norm on ρ to define a solution. In fact, assume that

$$K \in C^1(\mathbb{R}^+\times\mathbb{R}^3,\ \mathbb{R}^3) \text{ is uniformly bounded.} \tag{2.6}$$

Then we have

Corollary 2.4. *Let* $\rho_n \in L^\infty([0,\ T],\ L^1 \cap L^\infty(\Omega))$ *be a sequence of solutions to* (2.1) *in the sense of* (2.5) *(with* c_n *the solution of the elliptic equation associated with* ρ_n*). Assume* (2.6) *and that the family* $\rho_n(t, .)$ *for* $n \in \mathbb{N}$ *and* $t \in [0,\ T]$ *is equi-integrable, then any weak limits of* ρ_n *and* c_n *is also a solution in the sense of* (2.5)*.*

Remarks.
1. We are not able in general to show the propagation of any L^p norm other than L^1 or L^∞ even for a short time. So in this sense the corollary simply says that if, by chance, we know that we have equi-integrability (coming from a L^p bound for example) then we can define a solution. This would be very different if a viscosity were added.
2. If $\rho \in L^\infty([0,\ T[,\ L^1(\Omega))$, then the solution c to $-\Delta c = \rho - c$ is in $L^\infty([0,\ T[,\ W^{1,p}_{loc}(\Omega))$ with $p \leq 3/2$, through standard results on elliptic equations (see [10] for instance). The function K being bounded and regular, there is no difficulty in defining the product $K(c,\ \nabla c) \times \rho$ and consequently in using (2.5).

2.3. *Properties of the singularities*

The non existence of global solutions leads to the study of the singularities that a solution can develop. Precisely, assume that we have a solution on the maximal time interval $[0,\ T[$. That means that the L^∞ norm of $\rho(t, .)$ is blowing up as $t \to T$. Therefore we may define the blow up set $\mathcal{S}$ as

$$\mathcal{S} = \{a \in \Omega \,|\, \forall r > 0, \quad \sup_{t,\ |x-a|<r} \rho(t, x) = +\infty\},$$

which is the set of all points around which ρ is unbounded. This definition corresponds to the usual one, for the blow up of semilinear heat equations or wave equations for example. The topology and properties of the set for those equations is already well studied (see Giga and Kohn [9], Merle and Zaag [18] and [19], Velázquez [29] and [30], and Zaag [31] among others).

However another set is also of interest here. From (2.5), it is obvious that

$$\|\rho(t, .)\|_{L^1(\Omega)} \leq \|\rho^0\|_{L^1(\Omega)},$$

and therefore that ρ is compact in the weak-* topology of Radon measures. Moreover, provided that K is a bounded function then $\partial_t \rho$ belongs to $L^\infty([0,\ T],\ W^{-1,1}(\Omega))$ and the weak limit of ρ is necessarily unique, which means that we have a unique measure, denoted $\rho(T)$, with

$$\rho(t, x)\, dx \longrightarrow d\rho(T, x), \quad \text{as } t \to T.$$

Consequently we may also consider the set where $\rho(T)$ is concentrated. Precisely decompose $\rho(T)$ into $\tilde{\rho}\, dx + dm_1$ with m_1 orthogonal to the Lebesgue measure in $\mathbb{R}^d$. Then we may define $\mathcal{S}'$ as the support of m_1; It is a set with measure 0 (for Lebesgue measure) but the total mass of the cells located on it is not 0.

Obviously $\mathcal{S}'$ is included in $\mathcal{S}$ but in general the inclusion is strict. From the experimental point of view, what is seen is probably more $\mathcal{S}'$ than $\mathcal{S}$ (but that could be disputed) as it is certainly easier to identify the accumulation of mass. From a mathematical point of view, $\mathcal{S}'$ makes more sense than $\mathcal{S}$ *for this particular problem* because of Corollary 2.4.

We study here the specific case where the velocity of the cells decreases after the concentration of the chemical attractant becomes too large. More specifically we assume that

$$|K(a,b)| \leq \frac{C}{|a|^\beta}, \text{ with } \beta > 1, \text{ if } a > c_0 > 0. \tag{2.7}$$

Theorem 2.5. *Assume that K is bounded and that* (2.7) *holds. Consider a function $\rho(t,x) \in L^\infty([0,\ T[,\ L^1 \cap L^\infty(\Omega))$, solution to* (2.1) *on* $[0,\ T[$ *in the sense of* (2.5)*. Then its weak limit $\rho(T)$ as $t \to T$ (in the sense of* (2.3)*) satisfies*

$$\sup_{x_0} \sup_r \frac{1}{r^\alpha} \int_{B(x_0,r)} d\rho(T,y) < +\infty, \tag{2.8}$$

with $\alpha \leq 1 - \frac{1}{\beta}$.

Remarks.

1. The solution does not need to belong to L^∞, considering Cor. 2.4, any space ensuring equi-integrability in L^1 would be fine.
2. This result gives a topological information on $\mathcal{S}'$, it roughly states that $\mathcal{S}'$ is at least of dimension α. For instance, $\mathcal{S}'$ cannot contain any Dirac mass. More precisely, the theorem gives a bound on what is called the α dimensional upper density of $\rho(T,.)$ (see [7]) and this implies that if $\omega \subset \Omega$ has positive measure for $\rho(T,.)$ ($\rho(T,\omega) > 0$) then its α dimensional Hausdorff measure is positive.
3. The same remark as in Corollary 2.4 applies here.
4. If K is compactly supported, then it is possible to choose $\alpha = 1$.
5. The limit case $\alpha = 1$ is directly connected to the singularity in $1/|x|$ of the fundamental solution of the laplacian in dimension 3. For a generic dimension d, the exponent would be $d - 2$ instead.

3. Proof of Theorem 2.1 and Corollary 2.4

3.1. *Proof of Theorem 2.1*

We consider (ρ_n, c_n) a sequence of solutions to (2.1) with initial data $\rho_n(t = 0)$ converging strongly toward ρ^0, then we show that there exists a time T, such that the limits (ρ, c) satisfy the condition $i)$ and $ii)$ and (ρ, c) is a solution of (2.1). The existence result 2.1 would follow by a standard approximation procedure. Let us start by proving some uniform bounds on $(\rho_n,\ c_n)$. On one hand

$$\rho_n \geq 0, \quad \|\rho_n(t,.)\|_{L^1(\Omega)} \leq \|\rho_n(t=0)\|_{L^1(\Omega)} = \|\rho^0\|_{L^1(\Omega)}.$$

On the other hand, (2.1) may be written as

$$\partial_t\rho_n+K(c_n,\nabla c_n).\nabla_x\rho_n=-\rho_n\sum_{i,j}\partial_{b_j}K_i(c_n,\nabla c_n)\times\partial_{x_i}(\partial_{x_j}c_n)-\rho_n\sum_i\partial_a K_i(c_n,\nabla c_n)\times\partial_i c_n.$$

So

$$\partial_t\rho_n+K(c_n,\nabla c_n).\nabla_x\rho_n\leq\rho_n F(c_n,\nabla c_n,\Delta c_n)-\rho_n K'.\nabla c_n,$$

and consequently we obtain

$$\frac{d}{dt}\|\rho_n(t,.)\|_{L^\infty(\Omega)}\leq\tilde F(\|\rho_n\|_\infty),\tag{3.1}$$

because we have the following inequalities (for Ω bounded)

$$\|\Delta c_n\|_\infty\leq C\|\rho_n\|_\infty,$$

and

$$\|\nabla c_n\|_\infty,\quad\|c_n\|_\infty\leq\|\rho_n\|_\infty.$$

Thanks to Gronwall Lemma, (3.1) implies that there exists a time T and a constant C depending only on $\|\rho^0\|_{L^\infty}$ for which, uniformly in n,

$$\|\rho_n\|_{L^\infty([0,\ T]\times\Omega)}\leq C.$$

As a consequence we may extract subsequences of ρ_n and c_n such that (still denoted the same) we have in the corresponding weak topology

$$\rho_n\rightarrow\rho\quad\text{in } L^\infty([0,\ T]\times\Omega).$$

$$c_n\rightarrow c\quad\text{in } W^{1,\infty}([0,\ T]\times\Omega).$$

Finally, it comes

$$K(c_n,\nabla c_n)\longrightarrow K(c,\nabla c),\quad\text{in } L^\infty,$$

and we may pass in the limit in the equation (2.1).

3.2. *Without assumption* (2.4)

Instead of (2.4), assume we only have the much weaker property that $K\in C^1(\mathbb{R}\times\mathbb{R}^3)$.

Following the previous proof, it is enough to show that there exists bounds M_p and a time T such that for any $p<\infty$, any $t<T$ and any n

$$\int_\Omega|\rho_n(t,x)|^p\,dx\leq M_p.\tag{3.2}$$

Now using the equation we have that

$$\begin{aligned}\frac{d}{dt}\int_\Omega|\rho_n(t,x)|^p\,dx\leq&-p\int_\Omega\rho_n^p\sum_{i,j}\partial_{b_j}K_i(c_n,\nabla c_n)\times\partial_{x_i}(\partial_{x_j}c_n)\,dx\\&-p\int_\Omega\rho_n^p\sum_i\partial_a K_i(c_n,\nabla c_n)\times\partial_i c_n\,dx.\end{aligned}$$

Now in dimension 3 by Sobolev embedding the L^∞ norm of c_n and ∇c_n are dominated for instance by the L^4 norm of ρ_n. Therefore using Hölder inequality, we deduce that there exists a function F such that

$$\frac{d}{dt}\int_\Omega |\rho_n(t,x)|^p\,dx \le F(\|\rho_n\|_{L^4}^4)\,\|\rho_n\|_{L^{p+1}}^{p/(p+1)}\ \|c_n\|_{W^{2,p+1}}.$$

Denote

$$\alpha_p(t) = \int_\Omega |\rho_n(t,x)|^p\,dx.$$

As the $W^{2,p+1}$ norm of c_n is dominated by the L^{p+1} norm of ρ_n, we obtain

$$\alpha_p' \le F(\alpha_4)\,\alpha_{p+1},$$

with the initial condition

$$\alpha_p(0) \le \|\rho^0\|_{L^1}\,\|\rho^0\|_{L^\infty}^{p-1}.$$

Define

$$N_n(t) = \sup_p \sup_{s\le t} (\alpha_p(s))^{1/p}.$$

Then $N_n(0) \le \max(1,\ \|\rho^0\|_{L^1})\ \max(1,\ \|\rho^0\|_{L^\infty})$, $N_n(t)$ is well defined as the ρ_n are regular and as such $\sup_{s\le t}\|\rho_n(s,.)\|_{L^\infty} < \infty$ for any n. Moreover

$$\frac{dN_n}{dt} \le F(N_n^4)\,N_n^2.$$

Consequently, thanks to Gronwall lemma, there exists T and M depending only on $\|\rho^0\|_{L^1\cap L^\infty}$ such that for any $t < T$,

$$N_n(t) \le M.$$

Taking $M_p = M^p$, we satisfy (3.2).

3.3. *Proof of Corollary 2.4*

If $\rho_n \in L^1$ uniformly, then $c_n \in W^{1,p}(\Omega)$ uniformly for all $1 \le p \le p_0$ with $p_0 = 3/2$ in three dimensions (p_0 is such that $W^{2,1} \hookrightarrow W^{1,p_0}$ according Sobolev embedings). Therefore as the embedding is compact for $p < p_0$ we have

$$\nabla c_n, c_n \longrightarrow \nabla c,\ c, \quad \text{in}\ \ L^p(\Omega),$$

and as K is bounded

$$K(c_n, \nabla c_n) \longrightarrow K(c, \nabla c), \quad \text{in}\ \ L^p(\Omega).$$

It remains to prove that $\rho_n K(c_n,\nabla c_n) \rightharpoonup \rho K(c,\nabla c)$. For all $\varphi \in L^\infty([0,T]\times\Omega)$, we have

$$\begin{aligned}
&\int \varphi(K(c_n,\nabla c_n)\rho_n - K(c,\nabla c)\rho)\\
&= \int \varphi\Big(K(c_n,\nabla c_n)\rho_n - K(c,\nabla c)\rho_n + K(c,\nabla c)\rho_n - K(c,\nabla c)\rho\Big)\\
&= \int \rho_n\varphi\Big(K(c_n,\nabla c_n) - K(c,\nabla c)\Big) + \int \varphi K(c,\nabla c)(\rho_n - \rho).
\end{aligned}$$

The term $\int \rho_n \varphi\Big(K(c_n, \nabla c_n) - K(c, \nabla c)\Big)$ satisfies

$$
\begin{aligned}
&|\int \rho_n \varphi\Big(K(c_n, \nabla c_n) - K(c, \nabla c)\Big)| \\
&= |\int_{\{\rho_n \le M\}} \rho_n \varphi\Big(K(c_n, \nabla c_n) - K(c, \nabla c)\Big)| + |\int_{\{\rho_n > M\}} \rho_n \varphi\Big(K(c_n, \nabla c_n) - K(c, \nabla c)\Big)| \\
&\le M\,C\|K(c_n, \nabla c_n) - K(c, \nabla c)\|_{L^1} + C\,\|\int_{\{\rho_n > M\}} \rho_n\|_{L^\infty(L^1)}.
\end{aligned}
$$

As ρ_n is equi-integrable, the term $\| \int_{\{\rho_n > M\}} \rho_n\|_{L^\infty(L^1)}$ converges toward 0 as $M \to \infty$, uniformly in n. This is enough to conclude.

For the term $\int \varphi K(c, \nabla c)(\rho_n - \rho)$, the sequence ρ_n is equi-integrable and therefore converges weakly in L^1. As $K(c, \nabla c)$ is bounded, this term converges also toward 0. From

$$K(c_n, \nabla c_n)\rho_n \rightharpoonup K(c, \nabla c)\rho,$$

we may pass to the limit in the equation and extend the time of existence.

4. Proof of Theorem 2.5

By contradiction, let us assume that there exist a point x_0, a sequence r_n converging to zero and another η_n converging to $+\infty$ when n goes to infinity such that

$$\frac{1}{{r_n}^{\alpha}} \int_{B(x_0, r_n)} \rho(t, x)\, dx \ge \eta_n, \tag{4.1}$$

with $\alpha \le 1 - \frac{1}{\beta}$. For simplicity we assume that $x_0 = 0$.
We denote $V(t, x) = K(c(t, x), \nabla c(t, x))$ and define the characteristics as usual by

$$\begin{cases} \dot X(t, s, x) = V(t, X(t, s, x)), \\ X(s, s, x) = x. \end{cases} \tag{4.2}$$

We will simply write $X(t, x)$ for $X(t, 0, x)$. We recall that K is bounded and therefore V as well, so $X(t, s, x)$ is lipshitz in time. Consequently, X has a limit when t converges to T which is the time of blow up.
For all n, we denote

$$B_n = \{y \,|\, |X(T, y)| \le r_n\} \ne \emptyset.$$

We also define $R_n(t) = \sup_{y \in B_n} |X(t, y)|$, so we have

$$R_n(t) \le v(T - t) + r_n,$$

where $v = \sup |K|$ is the maximal velocity of displacement of cells along the characteristics curves (we remind that for all y, $|X(T, y) - X(t, y)| \le v(T - t)$).
We introduce the time t_n such that for all $t \ge t_n$

$$\begin{cases} R_n(t) \le 2r_n, \\ R_n(t_n) = 2r_n. \end{cases}$$

Therefore

$$T - t_n \geq \frac{r_n}{v}.$$

Now, for all $t \geq t_n$, for all $|x| \leq R_n(t)$, we have

$$\begin{aligned} c(t,x) &\geq \int_{|y| \leq R_n(t)} \frac{1}{|x-y|} \rho(t,y)\, dy, \\ &\geq \frac{1}{2R_n(t)} \int_{|y| \leq R_n(t)} \rho(t,y)\, dy, \\ &\geq \frac{1}{2R_n(t)} \int_{\{y \mid X(0,t,y) \in B_n\}} \rho(t,y)\, dy, \\ &= \frac{1}{2R_n(t)} \int_{|z| \leq r_n} \rho(t,z)\, dz. \end{aligned}$$

Using (4.1), we know

$$\int_{|z| \leq r_n} \rho(t,z)\ dz \geq \eta_n\, r_n^\alpha.$$

Thus, for $t \geq t_n$ and $|x| \leq R_n(t)$

$$\begin{aligned} c(t,x) &\geq \frac{\eta_n}{2R_n(t)} r_n^\alpha, \\ &\geq \frac{\eta_n}{2^{\alpha+1}} R_n^{\alpha-1}(t). \end{aligned}$$

However we have for all $y \in B_n, \forall t \geq t_n, |X(t,y)| \leq R_n(t)$, and so in particular, we deduce that

$$c(t, X(t,y)) \geq \frac{\eta_n}{2^{\alpha+1}} R_n^{\alpha-1}(t).$$

As $\alpha < 1$ and $R_n \to 0$, this last quantity converges to $+\infty$ and thanks to (2.7), we obtain

$$|K(c(t, X(t,y))) \leq C \left(\frac{\eta_n}{2^{\alpha+1}} R_n^{\alpha-1}(t) \right)^{-\beta}.$$

On the other hand, the characteristic equation (4.2) implies that

$$X(T,y) - X(t,y) = \int_t^T K(c(s, X(s,y)),\, \nabla c(s, X(s,y)))\, ds.$$

Consequently for $y \in B_n$

$$|X(t,y) - X(T,y)| \leq C\, 2^{\beta(\alpha+1)}\, \eta_n^{-\beta} \int_t^T R_n^{\beta(1-\alpha)}(s)\ ds.$$

Therefore, for all $y \in B_n$,

$$|X(t,y)| \leq r_n + C\, 2^{\beta(\alpha+1)}\, \eta_n^{-\beta} \int_t^T R_n^{\beta(1-\alpha)}(s)\ ds. \tag{4.3}$$

Now, for all positive time t, by the definition of $R_n(t)$ there exists y_m such that $|X(t, y_m)|$ converges to $R_n(t)$ as m goes to infinity. Writing (4.3) for y_m and taking the limit, we get

$$R_n(t) \leq r_n + C\,2^{\beta(\alpha+1)}\,\eta_n^{-\beta} \int_t^T R_n^{\beta(1-\alpha)}(s)\,ds.$$

Applying Gronwall lemma and since $\beta(1-\alpha) \geq 1$, we finally obtain that

$$R_n(t) \leq r_n\, e^{\tilde{C}\,\eta_n^{-\beta}\,(T-t)},$$

where $\tilde{C} = C\,2^{\beta(\alpha+1)}$. Taking this for $t = t_n$,

$$2r_n \leq r_n\, e^{\tilde{C}\,\eta_n^{-\beta}\,(T-t_n)} \leq r_n\, e^{\tilde{C}\,\eta_n^{-\beta}\,T},$$

or

$$e^{\tilde{C}\,\eta_n^{-\beta}\,T} \geq 2.$$

As η_n converges toward $+\infty$, this cannot hold true for all n thus a contradiction and the theorem is proved.

5. Control through the gradient of the concentration: The radially symmetric case

Let us consider a simplified situation where the velocity of a cell does not depend at all on the concentration $c(t,x)$, namely

$$\begin{cases} \partial_t \rho(t,x) + \nabla_x(\rho(t,x) K(\nabla c(t,x))) = 0, \\ -\Delta c(t,x) = \rho(t,x). \end{cases} \tag{5.1}$$

Given our previous result, one may wonder if it is not possible to control the set of concentration of the solution also by assuming that $K(\nabla c)$ decreases fast enough when $|\nabla c| \to +\infty$, instead of $|K(c,\nabla c)| \leq C\,|c|^{-\beta}$. Of course this would be more general but moreover as $\nabla c = C\,\dfrac{x}{|x|^3} \star \rho$ is more singular, the control on the dimension of the set would be more precise.

We cannot say much for the general case so let us consider the simple case of positive radially symmetric solution ρ of the system. This is quite similar to many studies for the classical Keller-Segel model (where $K(\xi) = \xi$) in a slightly more complicated setting.

The concentration of chemical substance is consequently a radially symmetric function, and we have

$$\begin{cases} c(r) = c(0) - \displaystyle\int_0^r (s - \frac{s^2}{r})\rho(t,s)\,ds, \\ c(0) = \displaystyle\int_0^{+\infty} s\rho\,ds. \end{cases}$$

We recall that as ρ has a bounded mass, the integral $\displaystyle\int_0^\infty r^2\,\rho(t,r)\,dr$ is bounded.

From the formula for c, we get

$$\partial_r c(t,r) = -\frac{1}{r^2}\int_0^r s^2\rho(t,s)\,ds.$$

And finally, in order to have radially symmetric solutions we take the function K of the form

$$K(\nabla c) = -\phi(|\partial_r c|)\times\frac{x}{|x|} = -\phi(-\partial_r c)\times\frac{x}{|x|}, \tag{5.2}$$

because $|\partial_r c| = -\partial_r c$. The function $\phi(\xi)$ should satisfy $\phi(0) = 0$ (for $K(0)$ to be defined), $\phi \geq 0$ (in order to have concentrations and not dispersion). Now let us assume in the spirit of what we did before that for $\beta > 1/2$

$$\phi(\xi) \leq \frac{C}{1+\xi^\beta}. \tag{5.3}$$

Then we have

Theorem 5.1. *Consider* $\rho \in L^\infty([0,\ T[,\ L^1\cap L^\infty(\mathbb{R}^3))$*, non negative, radially symmetric and solution to* (5.1) *with* K *defined by* (5.2) *and with* (5.3)*. Then for any* $\alpha \leq 2-\frac{1}{\beta}$*, we have that*

$$\sup_{\eta>0}\frac{1}{\eta^\alpha}\int_0^\eta \rho(T,s)\,s^2\,ds < \infty.$$

Proof. The characteristic curves are defined by

$$\partial_t R(t,r) = -\phi(-\partial_r c(t,R(t,r))), \quad R(0,r) = r.$$

If we denote $m(t,r) = \displaystyle\int_0^r s^2\rho(t,s)\,ds$, then we can write

$$-\partial_r c(t,r) = \frac{m(t,r)}{r^2},$$

but $m(t,R(t,r)) = m^0(r)$ because of mass conservation, which implies

$$\partial_t R(t,r) = -\phi(\frac{m^0(r)}{R^2(t,r)}).$$

Then

$$\left(1+\frac{(m^0)^\beta}{R^{2\beta}}\right)\partial_t R \geq -C.$$

This can easily be solved and it gives

$$\frac{(m^0(r))^\beta}{R^{2\beta-1}(t,r)} - (2\beta-1)\,R \leq \tilde{C}\,t + \frac{(m^0(r))^\beta}{r^{2\beta-1}} - (2\beta-1)\,r.$$

If $\rho^0 \in L^\infty$ then $m^0(r)$ behaves like r^3 and $(m^0)^\beta\,r^{1-2\beta} \leq C\,r$. As $R(t,r) \leq r$, we find for some constant C (depending on β and $\|\rho^0\|_{L^\infty}$)

$$R(t,r) \geq C\frac{(m^0(r))^{\beta/(2\beta-1)}}{(t+r)^{1/(2\beta-1)}}. \tag{5.4}$$

Now for $\alpha > 0$, let us estimate

$$\sup_{\eta>0} \frac{1}{\eta^\alpha} \int_0^\eta \rho(t,s)\, s^2\, ds = \sup_{\eta>0} \frac{m(t,\eta)}{\eta^\alpha} = \sup_{r>0} \frac{m^0(r)}{R^\alpha(t,r)}.$$

Thanks to (5.4), we deduce that

$$\sup_{\eta>0} \frac{1}{\eta^\alpha} \int_0^\eta \rho(t,s)\, s^2\, ds < \infty, \quad \text{if} \quad \alpha \le 2 - \frac{1}{\beta}. \tag{5.5}$$

This is the equivalent of Theorem 2.5 with a better estimate for α. However, as we have already said, we do not know how to prove this without radial symmetry. □

Before concluding this section, we point out that if K (or ϕ here) does not decay fast enough, Dirac masses may occur. More precisely let us simply assume, still in the case of radially symmetric functions, that

$$\int_0^{\xi_0} \frac{1}{\phi(\xi^{-2})}\, d\xi < \infty, \quad \forall\, \xi_0 > 0. \tag{5.6}$$

This is true if for example $\phi(\xi) \ge C\, \xi^\beta$ for large ξ with $\beta < 1/2$. We recall that we also have $\phi(0) = 0$ and we assume that

$$\phi'(0) > 0. \tag{5.7}$$

We will only consider initial data $\rho^0(r)$ which are bounded from below around $r = 0$. Then we can choose carefully ρ^0 and T_0 such that the solution $\rho(t,r)$ blows up and forms a Dirac mass at $t = T_0$. Because of mass conservation, this means that there exists $\bar r > 0$ such that $m(0, R_0) = m$ and $R(T,r) = 0$ for all $r \le \bar r$.

We introduce the following changes of variables

1. We choose $R_0 \le \bar r$ such that $\dfrac{r}{\sqrt{m^0(r)}}$ is a bijective and decreasing function from $[0,\ R_0]$ to $[U_0,\ +\infty[$, and we denote $\chi(u)$ the inverse of $\dfrac{r}{\sqrt{m^0(r)}}$. This is always possible as ρ^0 is assumed to be bounded from below around $r = 0$ and
$$\frac{d}{dr}\left(\frac{r}{\sqrt{m^0(r)}}\right) = \frac{1}{\sqrt{m^0(r)}}\ \left(1 - \frac{r\,\rho^0(r)}{m^0(r)}\right).$$
2. We define
$$U(t,u) = \frac{1}{\sqrt{M(u)}} R(\sqrt{M(u)}\, t, \sqrt{M(u)}\, u),$$
denoting $M(u) = m^0(\chi(u))$. The definition of U leads to
$$R(t,r) = \sqrt{m^0(r)} U(\frac{t}{\sqrt{m^0(r)}}, \frac{r}{\sqrt{m^0(r)}}).$$

We evidently have

$$\partial_t U = \frac{1}{\sqrt{M(u)}} \sqrt{M(u)} \partial_t R(\sqrt{M} t, \sqrt{M} u),$$
$$= -\phi\Big(\frac{m^0(\sqrt{M(u)}\, u)}{R^2(\sqrt{M}\, t, \sqrt{M}\, u)}\Big).$$

But we know that $\dfrac{\chi(u)}{\sqrt{m^0(\chi(u))}} = u$, so

$$\chi(u) = u\sqrt{m^0(\chi(u))} = u\sqrt{M(u)},$$

and consequently

$$m^0(\sqrt{M(u)}u) = m^0(\chi(u)) = M(u).$$

Therefore we simply have

$$\partial_t U = -\phi\left(\frac{1}{U^2}\right) = -F(U).$$

Introduce $g(\xi)$ such that $g'(\xi) = \dfrac{1}{F(\xi)}$, and $g(0) = 0$. This is possible only because of (5.6). Notice also that as $\phi(0) = 0$, we typically have that $\phi(\xi^{-2})$ behaves like ξ^{-2} for large ξ. Therefore that $g(\xi)$ converges to $+\infty$ as $\xi \to +\infty$. As

$$\frac{\partial_t U}{F(U)} = -1,$$

we find

$$g(U(t,u)) = -t + g(u).$$

We remark that g is an increasing function, and consequently that it is a bijective function, so

$$R(t,r) = \sqrt{m^0(r)} g^{-1}\left(-\frac{t}{\sqrt{m^0(r)}} + g\left(\frac{r}{\sqrt{m^0(r)}}\right)\right). \tag{5.8}$$

We want to have T_0 and ρ^0 such that for all positive $r \leq R_0$, $R(T_0, r) = 0$. We define $\alpha(r) = \sqrt{m^0(r)}$ and $\beta(r) = (m^0(r))^{-1/2}$, it has to satisfy that

$$\beta T_0 = g(\beta r). \tag{5.9}$$

On the other hand, as $g'(\xi)$ behaves like ξ^2, there exists ξ_0 such that g is strictly convex after ξ_0. Choosing R_0 small enough, for all $r < R_0$ there exists a unique $\beta(r) > \xi_0\, r^{-1}$ solution of (5.9). Moreover $g(\xi)$ behaves like ξ^3 and consequently this $\beta(r)$ behaves like $r^{-3/2}$. We now have defined β (and thus α and ρ^0) such that $R(T_0, r) = 0$. This is not enough however as we did not prove that some blow-up did not occur before T_0. What we have proved is that if there exists a regular solution until T_0 then this solution forms a Dirac mass at T_0. To prove that the solution is regular until T_0 we need to control the characteristics and more precisely to show that they are lipschitz continuous. As we work with radially symmetric solutions, this reduces to showing that $\partial_r R(t,r) > 0$ for all $r < R_0$ and $t < T_0$.

Let us write the equation for α and differentiate in r to get

$$\alpha'\Big(g'\left(\frac{r}{\alpha}\right)\frac{r}{\alpha} - g(\frac{r}{\alpha})\Big) = g'(\frac{r}{\alpha}). \tag{5.10}$$

Differentiating (5.8), we have

$$\partial_r R(t,r) = \alpha' g^{-1}(-\frac{t}{\alpha} + g(\frac{r}{\alpha})) + \frac{\alpha[\frac{t}{\alpha^2}\alpha' + g'(\frac{r}{\alpha})(\frac{1}{\alpha} - r\frac{\alpha'}{\alpha^2})]}{g'(g^{-1}(-\frac{t}{\alpha} + g(\frac{r}{\alpha})))},$$

so, using (5.10), we obtain

$$\partial_r R(t,r) = \alpha'\frac{R}{\alpha} + F\left(\frac{R}{\alpha}\right)\left[t\frac{\alpha'}{\alpha} - \alpha' g\left(\frac{r}{\alpha}\right)\right].$$

On the other hand $g(\frac{r}{\alpha}) = \frac{T_0}{\alpha}$, consequently

$$\partial_r R(t,r) = \alpha'\left[\frac{R}{\alpha} + F\left(\frac{R}{\alpha}\right)\frac{t - T_0}{\alpha}\right].$$

Finally, we obtain

$$\partial_r R = 0 \Leftrightarrow \frac{R}{\alpha} = F\left(\frac{R}{\alpha}\right)\frac{T_0 - t}{\alpha}.$$

Consider the equation $\xi = \gamma F(\xi)$, where γ is a given real number. The function $F(\xi) = \phi(\xi^{-2})$ is increasing in $[0,\ \xi_1]$ (for ξ_1 small enough). So there exists a unique solution ξ_0 in $[\xi_1, +\infty[$ if and only if $\xi_1 < \gamma F(\xi_1)$.

On the other hand $R(t,r)$ is decreasing in time thus $R/\alpha \le r/\alpha$. This last function (also equal to $\beta\, r$) may be chosen as small as we wish (we recall that the value of $\beta\, r$ at $r = 0$ may be chosen freely as long as it does not vanish). Therefore $\partial_r R \neq 0$ for all r, for all $t < T_0$, if and only if for all $t < T_0$, for all $r \in [0,\ R_0]$, $R(t,r) > F\left(\frac{R(t,r)}{\alpha(r)}\right)(T_0 - t)$.

Since

$$\partial_t R(t,r) = -F(\frac{R(t,r)}{\alpha(r)}),$$

we know that

$$R(t,r) - R(T_0,r) = \int_t^{T_0} F(\frac{R(s,r)}{\alpha(r)})\,ds.$$

Now we remind that $R(T_0,r) = 0$ and if $s > T$, $R(s,r) < R(t,r)$ so

$$F\left(\frac{R(s,r)}{\alpha(r)}\right) > F\left(\frac{R(t,r)}{\alpha(r)}\right).$$

Finally we conclude that for all t, for all r we have

$$R(t,r) > F\left(\frac{R(t,r)}{\alpha(r)}\right)(T_0 - t).$$

Consequently if $t < T_0$, we have

$$\partial_r R > 0,$$

and no blow up may occur before T_0.

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